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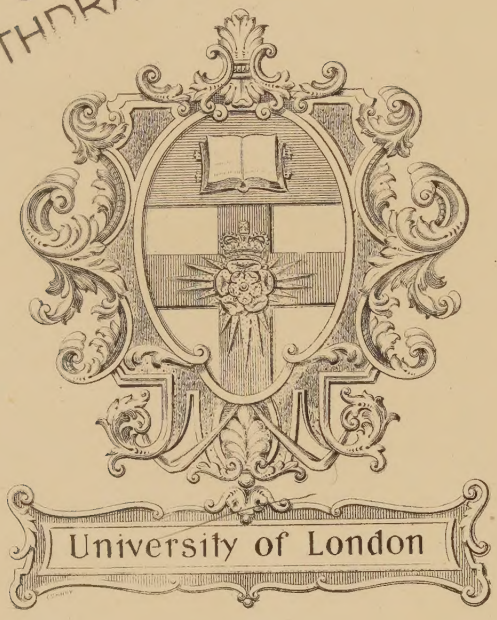

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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

SOME GENERAL SOLUTIONS OF ST. VENANT'S TORSION AND FLEXURE PROBLEM (II)

By Rosa M. Morris.

[Received 8 August, 1941.—Read 16 April, 1942.]

1. In the first paper of this series—referred to subsequently as (I)*—a complete solution to the flexure problem of St. Venant was given for the general and inclusive class of prism whose cross section boundary can be described analytically as the curve $\eta = 0$ in the net defined by a conformal transformation of the "internal" type†. At the end of that paper it was pointed out that there are prisms whose cross-section equations cannot be simply expressedin this form, and since these include certain simple cases—the elliptic prism, for example—for which solutions of the relevant problems are known, it seemed desirable to try and extend the methods of our analysis to cover also these cases. After discussing with

* *Proc. London Math. Soc.* (2), 46 (1940), 81–98.

† By an "internal" transformation we mean a transformation of the type

$$z = \sum_{n=0}^{\infty} a_n w^n$$

transforming the interior of a curve in the z-plane to the interior of the circle $|w| = 1$ The external transformations are of the type

$$z = w^{-1} \sum_{n=0}^{\infty} a_n w^n \quad (a_0 \neq 0),$$

transforming the exterior of a curve on to the interior of the same circle. We use these formulae with $w = \exp i\zeta$.

me several unsuccessful attempts to overcome the difficulties involved in such extension, Prof. Livens pointed out that a variation of a method used by Wilton* in a particular problem of a different kind enables the solution to be obtained for another general class of prism which includes all these outstanding cases and a number of others of some interest and practical importance. The present paper is devoted to a detailed development of this solution.

2. On examination it is found that, for all the apparently intractable cases, more or less simple external transformation formulae are available and that these all possess two essential characteristics:

(*a*) They exhibit the given boundary as the equipotentials or level surfaces of the edges of one or more unclosed cuts† in the plane. For example, the ellipse given by $\eta = 0$ in the net defined by

$$z = 2c \cos(\zeta + i\alpha), \tag{2.1}$$

is the equipotential $\eta = \alpha$ in the net defined by

$$z = 2c \cos\zeta, \tag{2.2}$$

of which the fundamental curve $\eta = 0$ ($\eta = -\alpha$ in the net defined by (2.1)) is the edge of the straight cut joining the points $z = \pm 2c$. The essence of this condition in the general case is that the whole of the physical space inside the cross section curve in the z-plane corresponds in the transformation to a finite rectangle ($0 < \xi < 2\pi$, $0 > \eta > -\alpha$) in the ζ-plane.

(*b*) The particular type of transformation involved ensures that in every case the singular points of the transformation are all on the fundamental curve (the open cut) and are therefore on or inside the given curve.

The first of these conditions enables us to dispense with the previous questions of convergence of the series expansions at $\eta = +\infty$ and to

* J. R. Wilton, "On the solution of certain problems of two-dimensional Physics", *Phil. Mag.* (6), 30 (1915), 761–779. The solutions of some of the internal problems given in this paper are not, however, free from ambiguity. The author failed to realise that he was dealing with problems on conformal transformations and that in consequence it was necessary to secure a unique one-one correspondence between the physical space and the points in the part of the Riemann surface defined by the transformation which cover it.

† That is, cuts which do not enclose an area of the plane.

concentrate our attention on their behaviour at the singular points of the transformation, the corners of the interior cut or cuts.

3. We formulate therefore the solution to the general tension and flexure problem for a uniform solid prism whose cross section is the curve $\eta = 0$ in the net defined by the external type transformation

$$z = z(\zeta) = e^{-i\zeta}\left(\sum_{n=0}^{\infty} a_n e^{ni\zeta}\right) \quad (a_0 \neq 0) \tag{3.1}$$

in which the inner curve $\eta = -a$ is a curve of zero area (the edges of an unclosed cut) on which all the singular points of the transformation lie.

This formula includes as special cases the following curves:

(i) The ellipse $\eta = 0$ in

$$z = ce^{-i\zeta}(e^{-ia} + e^{2i\zeta + ia}).$$

(ii) The level surfaces of the circular arc $\eta = 0$ subtending an angle 4β at the centre, the transformation for which is

$$\frac{z - ae^{2i\beta}}{z - ae^{-2i\beta}} = \left\{\frac{e^{-i\zeta} - ie^{i\beta}}{e^{-i\zeta} + ie^{-i\beta}}\right\}^2.$$

(iii) The four-bladed "propeller"

$$z = c\sqrt{\{\cos 2(\zeta + ia)\}},$$

for which the level surface $\eta = -a$ is the cross cut joining the pairs of points $z = \pm c$, $z = \pm ic$.

(iv) The more general n-bladed "propeller"

$$z = c\{\cos \tfrac{1}{2}n(\zeta + ia)\}^{2/n},$$

for which the level surface $\eta = -a$ is an n-bladed cut of equal branches joining pairs of points

$$z = c\{\cos k\pi\}^{2/n} \quad [k = 0, 1, \ldots, (n-1)].$$

(v) The unsymmetrical "propeller"

$$z = c\sqrt{\{\cos 2(\zeta + ia) - \cos 2\beta\}},$$

for which the level surface $\eta = -a$ is the cross cut of unequal branches joining the pairs of points

$$z = \pm c\sqrt{(1 - \cos 2\beta)}, \quad z = \pm ic\sqrt{(1 + \cos 2\beta)}.$$

(vi) The more general case

$$z = c[\cos \tfrac{1}{2}n(\zeta + i\alpha) - \cos \tfrac{1}{2}n\beta]^{2/n},$$

for which the level surface $\eta = -\alpha$ is an n-bladed cut of alternate long and short branches joining pairs of points

$$z = c[\cos k\pi - \cos \tfrac{1}{2}n\beta]^{2/n} \quad [k = 0, 1, \ldots, (n-1)].$$

In addition there is the "keel and rudder" surface

$$z = Ae^{-i(\zeta+i\alpha)}\{e^{i(\zeta+i\alpha)} - e^{-i\epsilon}\}^{1+k}\{e^{i(\zeta+i\alpha)} - e^{i\epsilon}\}^{1-k},$$

the cut for which consists of two straight lines meeting at an angle.

The approach to the problem is exactly as in (I), and we shall therefore take up the present account at the stage where it is necessary to depart from the previous procedure, adopting the same notation and assumptions where they are relevant, and quoting the formulae which we wish to use for the present analysis.

It is assumed, as in (I), that the constant term a_1 in the formula (3.1) for z has been adjusted so that the origin of z is the centroid of the cross-sectional area under consideration. The method of adjustment is exactly similar to that already indicated in (I), but owing to the different form of equation (3.1) the formulae for Sz_G in terms of its coefficients are slightly different. We have

$$S = -\tfrac{1}{2}i\int_c \bar{z}\,dz = +\tfrac{1}{2}i\int_0^{2\pi} \bar{z}(\zeta)\,z'(\zeta)\,d\zeta$$

$$= -\pi \sum_{n=0}^{\infty} (n-1)\,a_n \bar{a}_n,$$

$$Sz_G = -\tfrac{1}{2}i\int_c z\bar{z}\,dz = +\tfrac{1}{2}i\int_0^{2\pi} z\bar{z}z'\,d\zeta$$

$$= -\pi\Big[-a_0 b_1 + \sum_{n=1}^{\infty} n a_{n+1}\bar{b}_n\Big].$$

We notice here, too, that in converting the integrals from the z-plane to the ζ-plane there is a change of sign, which arises from the fact that the convention which we have adopted that $\eta = +\infty$ corresponds to $z = +\infty$ makes the positive directions in the z-plane and the w-plane opposite, with this particular transformation.

4. In (I) it is shown that the solution of the simple flexure problem depends on the determination of four potential functions ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} which, with their associated conjugate functions ψ_{11}, ψ_{12}, ψ_{21}, ψ_{22}, are

harmonic and regular throughout the interior of the cross-section curve of the prism and satisfy the following conditions respectively on its boundary.

$$\frac{d}{dx}(\phi_{11}-\tfrac{1}{3}x^3)=0, \quad \frac{d}{dx}(\phi_{22}-\tfrac{1}{3}y^3)=0, \tag{4.1}$$

and

$$\psi_{12}=\tfrac{1}{3}x^3+\text{constant}, \quad \psi_{21}=\tfrac{1}{3}y^3+\text{constant}. \tag{4.2}$$

Still continuing as in (I), we find that the first two functions ϕ_{11}, ϕ_{22} are the real parts of the complex potentials Ω_{11}, Ω_{22} respectively which, interpreted in terms of ζ, satisfy, at the boundary, the following equations

$$\Omega'_{11}=F'_{11}(\zeta)+\tfrac{1}{8}\{z(\zeta)+\bar{z}(\zeta)\}^2\{z'(\zeta)-\bar{z}'(\zeta)\}, \tag{4.3}$$

$$\Omega'_{22}=F'_{22}(\zeta)+\frac{i}{8}\{z(\zeta)-\bar{z}(\zeta)\}^2\{z'(\zeta)+\bar{z}'(\zeta)\}, \tag{4.4}$$

bars over a quantity denoting as usual the conjugate complex quantity and dashes denoting differentiation with respect to ζ.

In these formulae the functions F_{11}, F_{22} are real functions of their arguments which have now to be chosen to secure the regularity of the potentials throughout the interior of the cylinder. Regularity in this case requires that the gradients of the potentials should not become infinite anywhere inside. But $d\Omega/dz=\Omega'/z'$, so that infinities in the gradients of Ω will occur, if at all, at the points where $z'=0$, that is, at the singular points of the transformation. To ensure then that there are no such infinities we must make Ω' vanish whenever z' vanishes and to the same order at least. This we do, following Wilton, by choosing the F-functions so that Ω' is proportional to z'. In the case of Ω_{11} given by (4.3) one set of terms already has z' as a factor; to make the remainder, namely

$$F'_{11}-\tfrac{1}{8}(z+\bar{z})^2\bar{z}',$$

proportional to z' with F_{11} real, we write

$$F'_{11}=\tfrac{1}{8}(z+\bar{z})^2(\bar{z}'+z'),$$

so that

$$F_{11}=\tfrac{1}{24}(z+\bar{z})^3.$$

Similarly in the second potential Ω_{22}, we shall secure all the conditions of regularity by writing

$$F_{22}=\frac{i}{24}(z-\bar{z})^3.$$

Thus finally we have

$$(4.5) \qquad \Omega_{11} = \tfrac{1}{24}\{z(\zeta)+\bar{z}(\zeta)\}^3+\tfrac{1}{8}\int^{\zeta}\{z(t)+\bar{z}(t)\}^2\{z'(t)-\bar{z}'(t)\}\,dt,$$

and

$$(4.6) \qquad \Omega_{22} = \frac{i}{24}\{z(\zeta)-\bar{z}(\zeta)\}^3+\frac{i}{8}\int^{\zeta}\{z(t)-\bar{z}(t)\}^2\{z'(t)+\bar{z}'(t)\}\,dt.$$

5. The remaining two flexure potentials are slightly more involved. Again, as in (I), the conditions at the boundary of the prism (the curve $\eta = 0$) are satisfied by the appropriate parts of the complex potentials Ω_{12}, Ω_{21} which are given by the formulae

$$(5.1) \qquad \Omega_{12} = \frac{i}{24}\{z(\zeta)+\bar{z}(\zeta)\}^3+F_{12}(\zeta),$$

$$(5.2) \qquad \Omega_{21} = -\tfrac{1}{24}\{z(\zeta)-\bar{z}(\zeta)\}^3+F_{21}(\zeta),$$

where the notation is as before, F_{12} and F_{21} being real functions which have again to be chosen to secure the regularity conditions. Taking Ω_{12} first, we have

$$\Omega'_{12} = \frac{i}{8}\{z(\zeta)+\bar{z}(\zeta)\}^2\{z'(\zeta)+\bar{z}'(\zeta)\}+F'_{12}(\zeta),$$

and this must vanish where $z' = 0$. This means that

$$F'_{12}+\frac{i}{8}\{z+\bar{z}\}^2\bar{z}',$$

must be proportional to z'. Since F_{12} is real, we could secure this by taking as before

$$(5.3) \qquad F'_{12} = \frac{i}{8}\{z+z'\}^2\{z'-\bar{z}'\},$$

but now this form has to be supplemented by another term of the type $kz'\bar{z}'$. In the simplest cases where the connectivity of the space between the level surfaces $\eta = 0$ and $\eta = -a$ is of order two, and these are the only cases we have so far considered, integration of (5.3) introduces in general a cyclic term $\alpha\zeta$, α being the constant term of the integral. But the presence of such a term in Ω would imply a cyclic character in the stresses in the flexed beam, which is of course physically impossible. The additional term has therefore to be introduced to secure the cancelling of

this term in Ω_{12}. Thus finally we must take

$$F_{12} = \frac{i}{8}\int^{\zeta}[\{z(t)+\bar{z}(t)\}^2\{z'(t)-\bar{z}'(t)\}+K_{12}z'(t)\bar{z}'(t)]\,dt,$$

while similarly

$$F_{21} = -\tfrac{1}{8}\int^{\zeta}[\{z(t)-\bar{z}(t)\}^2\{z'(t)+\bar{z}'(t)\}+K_{21}z'(t)\bar{z}'(t)]\,dt.$$

Hence therefore

$$\Omega_{12} = \frac{i}{24}\{z(\zeta)+\bar{z}(\zeta)\}^3+\frac{i}{8}\int^{\zeta}[\{z(t)+\bar{z}(t)\}^2\{z'(t)-\bar{z}'(t)\}+K_{12}z'(t)\bar{z}'(t)]\,dt,$$

$$\Omega_{21} = -\tfrac{1}{24}\{z(\zeta)-\bar{z}(\zeta)\}^3-\tfrac{1}{8}\int^{\zeta}[\{z(t)-\bar{z}(t)\}^2\{z'(t)+\bar{z}'(t)\}+K_{21}z'(t)\bar{z}'(t)]\,dt,$$

the coefficients K_{12}, K_{21} being chosen to secure the non-cyclic character of the solution.

This completes the determination of the four flexure potentials of Stevenson's analysis.

6. Again, repetition of forms suggests the advisability at this stage of returning to the complete potentials of the problem. These are defined in (I) by the relations

$$\Omega_1 = (1+\sigma)\,\Omega_{11}-\sigma\Omega_{21},$$

$$\Omega_2 = (1+\sigma)\,\Omega_{22}+\sigma\Omega_{12}.$$

On inserting the above expansions we have

$$\Omega_1 = \tfrac{1}{24}(1+\sigma)\{z(\zeta)+\bar{z}(\zeta)\}^3+\tfrac{1}{8}(1+\sigma)\int^{\zeta}\{z(t)+\bar{z}(t)\}^2\{z'(t)-\bar{z}'(t)\}\,dt$$

$$+\frac{\sigma}{24}\{z(\zeta)-\bar{z}(\zeta)\}^3+\frac{\sigma}{8}\int^{\zeta}[\{z(t)-\bar{z}(t)\}^2\{z'(t)+\bar{z}'(t)\}+K_{21}z'(t)\bar{z}'(t)\}]\,dt.$$

In the usual way this can be rearranged in the more appropriate form

$$\tfrac{1}{24}(1+2\sigma)\{z^3(\zeta)+3z(\zeta)\bar{z}^2(\zeta)\}+\tfrac{1}{2}(1+\sigma)\int^{\zeta}z(t)\bar{z}(t)\{z'(t)-\bar{z}'(t)\}\,dt$$

$$-\tfrac{1}{2}\sigma\int^{\zeta}z(t)\bar{z}(t)\{z'(t)+\bar{z}'(t)\}\,dt+\tfrac{1}{8}\sigma\int^{\zeta}K_{21}z'(t)\bar{z}(t)\,dt.$$

We have similarly for Ω_2 the expression

$$\frac{i}{24}(1+2\sigma)\{z^3(\zeta)+3z(\zeta)\bar{z}^2(\zeta)\}-\frac{i}{2}(1+\sigma)\int^{\zeta} z(t)\bar{z}(t)\{z'(t)+\bar{z}'(t)\}\,dt$$

$$+\tfrac{1}{2}i\sigma\int^{\zeta} z(t)\bar{z}(t)\{z'(t)-\bar{z}'(t)\}\,dt+\tfrac{1}{8}\sigma\int^{\zeta} K_{12}z'(t)\bar{z}'(t)\,dt.$$

7. We have seen in (I) that, in order to calculate the position of the centre of flexure, we require the moment integral $\iint(xZ_y-yZ_x)\,dx\,dy$, which was there found to be the real part of

$$(7.1)\quad \tfrac{1}{2}\mu\int_c z\bar{z}[K_1\,d\Omega_1'+K_2\,d\Omega_2']-\frac{\mu}{16}(K_1-iK_2)(1-2\sigma)\int_c z^2\bar{z}^2\,dz,$$

where

$$\Omega_1'=\Omega_1-\tfrac{1}{12}(1+2\sigma)z^3,$$

$$\Omega_1'=\Omega_1-\frac{i}{12}(1+2\sigma)z^3.$$

In this case then

$$\Omega_1'=\tfrac{1}{4}(1+2\sigma)z\bar{z}^2+\tfrac{1}{2}(1+\sigma)\int^{\zeta} z\bar{z}(z'-\bar{z}')\,dt$$

$$-\tfrac{1}{2}\sigma\int^{\zeta} z\bar{z}\{z'+\bar{z}'\}\,dt+\frac{\sigma}{8}\int K_{21}z'\bar{z}'\,dt,$$

and

$$\Omega_2'=\frac{i}{4}(1+2\sigma)z\bar{z}^2-\frac{i}{2}(1+\sigma)\int^{\zeta} z\bar{z}(z'+\bar{z}')\,dt$$

$$+\frac{i\sigma}{2}\int^{\zeta} z\bar{z}(z'-\bar{z}')\,dt+\frac{\sigma}{8}\int^{\zeta} K_{12}z'\bar{z}'\,dt.$$

By substituting these expressions in the above moment integral it may be possible in certain simple cases to complete the solution without resort to the series expansion for z, but by expressing z as a series expansion it is possible to express the moment integral explicitly in terms of the coefficients in such an expansion.

Taking then our general case when

$$z=\sum_{n=0}^{\infty} a_n e^{(n-1)i\zeta},$$

we introduce a notation similar to that of (I) and we have

$$z\bar{z}=b_0+\sum_{n=1}^{\infty}(b_n e^{ni\zeta}+\bar{b}_n e^{-ni\zeta}).$$

Also

$$z^2\bar{z} = \left\{b_0 + \sum_{n=1}^{\infty}(b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta})\right\}\left\{\sum_{n=0}^{\infty} a_n e^{(n-1)i\zeta}\right\}$$

$$= \sum_{r=-\infty}^{+\infty} f_r e^{ri\zeta},$$

where

$$f_0 = a_0 b_1 + a_1 \bar{b}_0 + a_2 \bar{b}_1 + a_3 \bar{b}_2 + \ldots,$$

$$f_r = a_0 b_{r+1} + a_1 b_r + \ldots a_{r+1} b_0 + a_{r+2}\bar{b}_1 + \ldots \quad (r > 0),$$

$$f_{-r} = a_0 \bar{b}_{r-1} + a_1 \bar{b}_r + \ldots \quad (r > 0),$$

and then

$$\bar{z}^2 z = \sum_{r=-\infty}^{+\infty} \bar{f}_r e^{-ri\zeta} = \sum_{r=-\infty}^{+\infty} \bar{f}_{-r} e^{ri\zeta}.$$

Similarly we write

$$z\bar{z}z' = \left\{b_0 + \sum_{n=1}^{\infty}(b_n e^{ni\zeta} + \bar{b}_n e^{ni\zeta})\right\}\left\{\sum_{n=0}^{\infty}(n-1)\,i a_n e^{(n-1)i\zeta}\right\}$$

$$= i\left\{g_0 + \sum_{r=-\infty}^{+\infty} r g_r e^{ri\zeta}\right\},$$

where

$$g_0 = -b_1 a_0 + \bar{b}_1 a_2 + 2\bar{b}_2 a_3 + \ldots,$$

$$rg_r = -a_0 b_{r+1} + a_2 b_{r-1} + 2a_3 b_{r-2} + \ldots + r a_{r+1} b_0 + (r+1)\,a_{r+2}\bar{b}_1 + \ldots, \quad (r > 0),$$

$$-rg_{-r} = -a_0 \bar{b}_{r-1} + a_2 \bar{b}_{r+1} + 2a_3 \bar{b}_{r+2} + \ldots \quad (r > 0),$$

so that

$$z\bar{z}\bar{z}' = -i\left\{\bar{g}_0 + \sum_{r=-\infty}^{+\infty} r\bar{g}_r e^{-ri\zeta}\right\},$$

and

$$\int^{\zeta} z\bar{z}\{z' \mp \bar{z}'\}\,dt = \int^{\zeta} i\left\{g_0 \pm \bar{g}_0 + \sum_{r=-\infty}^{+\infty} r g_r e^{ri\zeta} \pm r\bar{g}_r e^{-ri\zeta}\right\} d\zeta$$

$$= i(g_0 \pm \bar{g}_0)\,\zeta + \sum_{r=-\infty}^{+\infty}(g_r \mp \bar{g}_{-r})\,e^{ri\zeta}.$$

But, with our choice of origin, we see from our formula for z_G that

$$g_0 = \bar{g}_0 = 0$$

and as a result there is no cyclic term in either Ω_1' or Ω_2' and we can therefore take

$$K_{12} = K_{21} = 0.$$

Thus altogether

$$\Omega_1' = \tfrac{1}{4}(1+2\sigma) \sum_{r=-\infty}^{+\infty} \bar{f}_{-r} e^{ri\zeta} + \tfrac{1}{2}(1+\sigma) \sum_{r=-\infty}^{+\infty} (g_r - \bar{g}_{-r}) e^{ri\zeta} - \tfrac{1}{2}\sigma \sum_{r=-\infty}^{+\infty} (g_r + \bar{g}_{-r}) e^{ri\zeta}$$

$$= \sum_{r=-\infty}^{+\infty} \Omega_{r_1} e^{ri\zeta} = \sum_{r=1}^{\infty} (\Omega_{r_1} e^{ri\zeta} + \Omega_{-r_1} e^{-ri\zeta}),$$

where

$$\Omega_{r_1} = \tfrac{1}{4}(1+2\sigma)\bar{f}_{-r} + \tfrac{1}{2}(g_r - \bar{g}_{-r}) - \sigma\bar{g}_{-r},$$

$$\Omega_{-r_1} = \tfrac{1}{4}(1+2\sigma)\bar{f}_r + \tfrac{1}{2}(g_{-r} - \bar{g}_r) - \sigma\bar{g}_r.$$

Similarly

$$\Omega_2' = \frac{i}{4}(1+2\sigma) \sum_{r=-\infty}^{+\infty} (\bar{f}_{-r} e^{ri\zeta}) - \frac{i}{2}(1+\sigma) \sum_{r=-\infty}^{+\infty} (g_r + \bar{g}_{-r}) e^{ri\zeta}$$

$$+ \frac{i}{2}\sigma \sum_{r=-\infty}^{+\infty} (g_r - \bar{g}_{-r}) e^{ri\zeta}$$

$$= \sum_{r=-\infty}^{+\infty} \Omega_{r_2} e^{ri\zeta} = \sum_{r=1}^{\infty} (\Omega_{r_2} e^{ri\zeta} + \Omega_{-r_2} e^{-ri\zeta}),$$

where

$$\Omega_{r_2} = \frac{i}{4}(1+2\sigma)\bar{f}_{-r} - \frac{i}{2}(g_r + \bar{g}_{-r}) - 2\sigma\bar{g}_{-r},$$

$$\Omega_{-r_2} = \frac{i}{4}(1+2\sigma)\bar{f}_r - \frac{i}{2}(g_{-r} + \bar{g}_r) - i\sigma\bar{g}_r.$$

8. Using these values in the integral

$$\tfrac{1}{2}\mu \int_\sigma z\bar{z}(K_1 d\Omega_1' + K_2 d\Omega_2'),$$

we have

$$d\Omega_1' = \left\{ \sum_{r=1}^{\infty} ri\Omega_{r_1} e^{ri\zeta} - \sum_{r=1}^{\infty} ri\Omega_{-r_1} e^{-ri\zeta} \right\} d\zeta,$$

$$d\Omega_2' = \left\{ \sum_{r=1}^{\infty} ri\Omega_{r_2} e^{ri\zeta} - \sum_{r=1}^{\infty} ri\Omega_{-r_2} e^{-ri\zeta} \right\} d\zeta.$$

Thus

$$\tfrac{1}{2}\mu\int_c z\bar{z}\,d\Omega_1' = -\tfrac{1}{2}\mu\int_0^{2\pi}\left\{b_0+\sum_1^\infty(b_n e^{ni\zeta}+\bar{b}_n e^{-ni\zeta})\right\}$$
$$\times\left\{\sum_{n=1}^\infty ni\Omega_{n_1}e^{ni\zeta}-\sum_{n=1}^\infty ni\Omega_{-n_1}e^{-ni\zeta}\right\}d\zeta$$
$$= -\pi\mu i\sum_{n=1}^\infty\left\{n\bar{b}_n\Omega_{n_1}-nb_n\Omega_{-n_1}\right\},$$

and

$$\tfrac{1}{2}\mu\int_c z\bar{z}\,d\Omega_2' = -\pi\mu i\sum_{n=1}^\infty n\{\bar{b}_n\Omega_{n_2}-b_n\Omega_{-n_2}\},$$

so that altogether the first integral in (7.1) is

$$-\pi\mu i\sum_{n=1}^\infty n[\bar{b}_n(K_1\Omega_{n_1}+K_2\Omega_{n_2})-b_n(K_1\Omega_{-n_1}+K_2\Omega_{-n_2})].$$

To calculate the second integral in (7.1) we use the expansion

$$z\bar{z} = b_0+\sum_{n=1}^\infty(b_n e^{ni\zeta}+b_n e^{-ni\zeta}),$$

as well as

$$z\bar{z}z' = i\sum_{n=1}^\infty n(g_n e^{ni\zeta}-g_{-n}e^{-ni\zeta}),$$

giving

$$\frac{\mu}{16}\int_e z^2\bar{z}^2\,dz = -\frac{\pi\mu i}{8}\sum_{n=1}^\infty n(\bar{b}_n g_n-b_n g_{-n}).$$

Thus, if x_0, y_0 is the position of the centre of flexure, and F_1, F_2 are the components of the force, we have $F_2x_0-F_1y_0$ as the real part of

$$-\pi\mu i\sum_{n=1}^\infty n\{\bar{b}_n(K_1\Omega_{n_1}+K_2\Omega_{n_2})-b_n(K_1\Omega_{-n_1}+K_2\Omega_{-n_2})\},$$
$$-\frac{\pi\mu i}{8}(K_1-iK_2)(1-2\sigma)\sum_{n=1}^\infty n(\bar{b}_n g_n-b_n g_{-n}).$$

Inserting the usual values of K_1 and K_2 from (3.9) of (I), we get x_0 as the real part of

$$-\pi\mu i\sum_{n=1}^\infty\left[n\bar{b}_n\left\{\frac{-H\Omega_{n_1}}{E(AB-H^2)}+\frac{A\Omega_{n_2}}{E(AB-H^2)}\right\}\right.$$
$$\left.-\frac{nb_n}{E(AB-H^2)}\{-H\Omega_{-n_1}+A\Omega_{-n_2}\}\right]$$
$$+\tfrac{1}{8}\pi\mu\frac{(A-iH)(1-2\sigma)}{E(AB-H^2)}\sum_{n=1}^\infty n(\bar{b}_n g_n-b_n g_{-n}),$$

which is

$$-\pi\mu i E^{-1}(AB-H^2)^{-1}\sum_{n=1}^{\infty}[n\bar{b}_n\{A\Omega_{n_2}-H\Omega_{n_1}\}-nb_n\{A\Omega_{-n_2}-H\Omega_{-n_1}\}]$$

$$+\tfrac{1}{8}\pi\mu E^{-1}(AB-H^2)^{-1}(A-iH)(1-2\sigma)\sum_{n=1}^{\infty}n(\bar{b}_n g_n-b_n g_{-n}).$$

Similarly y_0 is the real part of

$$-\pi\mu i E^{-1}(AB-H^2)^{-1}\sum_{n=1}^{\infty}[n\bar{b}_n(H\Omega_{n_2}-B\Omega_{n_1})+nb_n(B\Omega_{-n_1}-H\Omega_{-n_2})]$$

$$-\tfrac{1}{8}\pi\mu i E^{-1}(AB-H^2)^{-1}(B+iH)(1-2\sigma)\sum_{1}^{\infty}n(\bar{b}_n g_n-b_n g_{-n}),$$

where A, B and H are the usual moments and product of inertia respectively which, with the formula (3.1) for the transformation, are given by

$$(8.1)\qquad A+B=\tfrac{1}{2}i\int_c z^2\bar{z}\,dz$$

$$=\tfrac{1}{2}\pi\left[-\bar{a}_0 g_{-1}+\sum_{n=2}^{\infty}(n-1)g_{n-1}\bar{a}_n\right],$$

and

$$(8.2)\qquad B-A-2iH=-\tfrac{1}{4}i\int_c z\bar{z}^2\,dz$$

$$=\pi\left[-g_1 a_0+\sum_{n=2}^{\infty}(n-1)g_{-(n-1)}a_n\right].$$

For principal axes, for which $H=0$, x_0 is the real part of

$$-\pi\mu i E^{-1}B^{-1}\sum_{n=1}^{\infty}n(\bar{b}_n\Omega_{n_2}-b_n\Omega_{-n_2})+\tfrac{1}{8}\pi\mu E^{-1}B^{-1}(1-2\sigma)\sum_{n=1}^{\infty}n(\bar{b}_n g_n-b_n g_{-n})$$

and y_0 is the real part of

$$\pi\mu i E^{-1}A^{-1}\sum_{1}^{\infty}n(\bar{b}_n\Omega_{n_1}-b_n\Omega_{-n_1})-\tfrac{1}{8}\pi\mu i E^{-1}A^{-1}(1-2\sigma)\sum_{1}^{\infty}n(\bar{b}_n g_n-b_n g_{-n}).$$

9. The simplest example of the type of cylinder we are considering is of course the elliptic cylinder, but it is obvious that the centre of flexure for such symmetrical cylinders is at the centroid and the general flexure problem is then solved immediately when we have the solution of the simple torsion problem. The details of this solution for the particular

type of cylinder are given elsewhere*, but I will briefly summarise the results here. Again, as in (8.2) of (I), the complex potential Ω_3 assumes the form

$$\Omega_3 = \tfrac{1}{2}iz(\zeta)\bar{z}(\zeta)+F_3(\zeta),$$

where, as usual, $F_3(\zeta)$ is a real function of its argument introduced to secure regularity of Ω_3 and its derivatives in the space considered. Pursuing then methods analogous to those developed in § 4 above, we find that

$$\Omega_3' = \tfrac{1}{2}i\{z'\bar{z}+z\bar{z}'\}+F'(\zeta),$$

must be proportional to z' and we must therefore take

$$F'(\zeta) = -\tfrac{1}{2}i\{z\bar{z}'-z'\bar{z}\}+kz'\bar{z}',$$

where k is real and determined as usual so as to eliminate any cyclic term that may arise. With this restriction on k the complex function Ω_3 is given by

$$\Omega_3 = \tfrac{1}{2}iz(\zeta)\bar{z}(\zeta)-\tfrac{1}{2}i\int^{\zeta}\{z(t)\bar{z}'(t)-z'(t)\bar{z}(t)+2ikz'(t)\bar{z}'(t)\}\,dt.$$

The torsion potential is thus completely determined and then, as in (I), the torsion constant Γ is the real part of the integral

$$\tfrac{1}{4}\tau\mu\int_c\left\{2\frac{d}{dz}\,\Omega_3-i\bar{z}\right\}z\bar{z}\,dz.$$

With the transformation

$$z = \sum_{n=0}^{\infty} a_n e^{(n-1)i\zeta},$$

we find that the complex potential is given by

$$\Omega_3 = \tfrac{1}{2}i\left\{b_0+\sum_{n=1}^{\infty}(b_n e^{ni\zeta}+\bar{b}_n e^{-ni\zeta})\right\}$$

$$-\int^{\zeta}\left[c_0+\bar{c}_0+\sum_{n=1}^{\infty}\{(c_n+\bar{c}_{-n})e^{ni\zeta}+(\bar{c}_n+c_{-n})e^{-ni\zeta}\}\right]d\zeta$$

$$+k\int^{\zeta}\left\{d_0+\sum_{n=1}^{\infty}(d_n e^{ni\zeta}+\bar{d}_n e^{-ni\zeta})\right\}d\zeta,$$

* "The internal problems of two-dimensional potential theory", *Math. Annalen*, 117 (1939), 31–38.

where

$$c_n = \sum_{r=0}^{\infty} (n+r-1)\, a_{n+r}\bar{a}_r,$$

$$c_{-n} = \sum_{r=0}^{\infty} (r-1)\, \bar{a}_{n+r} a_r,$$

and $$d_n = \sum_{r=0}^{\infty} (n+r-1)(r-1)\, a_{n+r}\bar{a}_r.$$

To eliminate the cyclic term, we must take

$$2kd_0 = c_0+\bar{c}_0,$$

and then

$$\Omega_3 = \tfrac{1}{2}i\left\{b_0+\sum_{n=1}^{\infty}(b_n e^{ni\zeta}+\bar{b}_n e^{-ni\zeta})\right\}$$

$$+\tfrac{1}{2}i\left\{\sum_{n=1}^{\infty}\frac{1}{n}(c_n+\bar{c}_{-n})e^{ni\zeta}-\sum_{n=1}^{\infty}\frac{1}{n}(\bar{c}_n+c_{-n})e^{-ni\zeta}\right\}$$

$$-i\left\{\frac{(c_0+\bar{c}_0)}{2d_0}\right\}\sum_{n=1}^{\infty}\frac{1}{n}(d_n e^{ni\zeta}-\bar{d}_n e^{-ni\zeta}).$$

Inserting this in the integral for the torsion couple we get

$$\Gamma = \tfrac{1}{2}\pi\tau\mu\left[b_0 c_0+\sum_{n=1}^{\infty}\left\{-b_n\bar{c}_n-\bar{b}_n\bar{c}_{-n}+\left(\frac{c_0+\bar{c}_0}{d_0}\right)(b_n\bar{d}_n+\bar{b}_n d_n)\right\}\right].$$

For the elliptic cylinder whose transformation is

$$z = c\cos(\zeta+i\alpha),$$

we have immediately

$$a_0 = \tfrac{1}{2}ce^{\alpha}; \qquad a_1 = 0; \quad a_2 = \tfrac{1}{2}ce^{-\alpha};$$

$$b_0 = \tfrac{1}{2}c^2\cosh 2\alpha; \quad b_1 = 0; \quad b_2 = \tfrac{1}{4}c^2;$$

$$c_0 = \tfrac{1}{2}c^2\sinh 2\alpha; \quad c_1 = 0; \quad c_2 = -\tfrac{1}{4}c^2;$$

$$c_{-1} = 0; \quad c_{-2} = -\tfrac{1}{4}c^2;$$

$$d_0 = \tfrac{1}{2}c^2\cosh 2\alpha; \quad d_1 = 0; \quad d_2 = -\tfrac{1}{4}c^2;$$

so that in this case

$$\Omega = \text{const.} + \tfrac{1}{4}ic^2\{\cosh 2(\zeta+i\alpha)\}/\cosh 2\alpha,$$

and the couple is

$$\tfrac{1}{8}\pi\tau\mu c^4\{\sinh^3 2\alpha\}/\cosh 2\alpha.$$

In terms of z these results are

$$\Omega = \tfrac{1}{2}i\,\frac{(a^2-b^2)}{(a^2+b^2)}\,z^2,$$

and

$$\Gamma = \frac{\pi\tau\mu a^3 b^3}{(a^2+b^2)},$$

where a, b have their usual values.

10. As an example of the method used for cylinders which are not bi-symmetrical we have to take a more complex boundary curve. Such a one is the curve whose internal cut is a semi-circular arc. The expansion for such a boundary curve is found by putting $(\zeta+i\alpha)$ in place of ζ in case (v) with $\beta = \frac{1}{4}\pi$. The transformation is then

$$\frac{z-ai}{z+ai} = \left\{\frac{e^{-i(\zeta+i\alpha)} - i(1+i)\sqrt{2}}{e^{-i(\zeta+i\alpha)} + i(1-i)\sqrt{2}}\right\}^2,$$

or

$$z = \sum_{n=0}^{\infty} a_n e^{(n-1)i\zeta},$$

where

$$a_0 = \frac{ae^{\alpha}}{\sqrt{2}}; \quad n>0,\ a_n = \frac{a(-)^{n-1}e^{-(n-1)\alpha}}{(\sqrt{2})^{n+1}}.$$

We have first, however, to transfer the origin to the centroid of the area. Using the relations of § 3, and noticing that

$$b_0 = \frac{a^2 e^{2\alpha}}{2-e^{-2\alpha}}; \quad n>0,\ b_n = \frac{a^2(-)^n e^{-n\alpha}(1-e^{2\alpha})}{(\sqrt{2})^n(2-e^{-2\alpha})},$$

we have

$$s = \frac{2\pi a^2(e^{2\alpha}-1)}{(2-e^{-2\alpha})^2}, \qquad Sz_G = \frac{\pi a^3(e^{2\alpha}-1)(4-2e^{-2\alpha}+e^{-4\alpha})}{2(2-e^{-2\alpha})^3},$$

which gives

$$z_G = \frac{a}{2} + \frac{ae^{-4\alpha}}{4(2-e^{-2\alpha})}.$$

When we change the origin to this point, the transformation equation becomes

$$z = \sum_{n=0}^{\infty} a_n e^{(n-1)i\zeta},$$

where now

$$a_0 = \frac{ae^{a}}{\sqrt{2}}; \quad a_1 = \frac{-ae^{-4a}}{4(2-e^{-2a})}; \quad n>1, \ a_n = \frac{a(-)^{n-1}e^{-(n-1)a}}{(\sqrt{2})^{n+1}},$$

and then

$$b_0 = a^2\{32e^{2a}-32+16e^{-2a}-4e^{-4a}+e^{-8a}\};$$

$$b_1 = -\frac{a^2e^{-3a}(4-e^{-2a})}{8\sqrt{2}(2-e^{-2a})};$$

and, for $n>1$,

$$b_n = \frac{a^2(-)^{n-1}}{(\sqrt{2})^{n+6}}e^{-na}(1+2e^{2a})(2-e^{-2a}).$$

The additional coefficients required in the analysis are

$$f_1 = -\frac{a^3e^{-a}}{16\sqrt{2}}\{8e^{2a}-e^{-2a}+e^{-4a}\};$$

and, for $r>1$,

$$f_r = \frac{ra^3(-)^{r-1}}{16(\sqrt{2})^r}e^{-ra}(4e^{2a}-e^{-2a})$$

$$+\frac{a^3(-)^re^{-ra}}{16(\sqrt{2})^r(2-e^{-2a})^2}\{64e^{2a}-64+16e^{-2a}+8e^{-4a}-5e^{-6a}+e^{-8a}\};$$

$$f_{-1} = \frac{a^3e^{a}}{32\sqrt{2}(2-e^{-2a})^2}\{64e^{2a}-64+48e^{-2a}-16e^{-4a}-4e^{-6a}+8e^{-8a}-e^{-10a}\};$$

$$f_{-2} = \frac{a^3e^{-2a}}{64(2-e^{-2a})}\{-24+8e^{-2a}+2e^{-4a}-e^{-6a}\};$$

$r>2$,

$$f_{-r} = \frac{a^3(-)^re^{-ra}}{32(\sqrt{2})^r}\{16e^{4a}-8+e^{-4a}\};$$

$$g_1 = \frac{a^3e^{-a}}{32\sqrt{2}(2-e^{-2a})^2}\{8e^{-2a}-12e^{-4a}+4e^{-6a}-e^{-8a}\};$$

$r > 1$,

$$rg_r = \frac{a^3(-)^r e^{-ra}}{32(\sqrt{2})^r(2-e^{-2a})^2}\{r^2(16e^{2a}-16+4e^{-4a}-e^{-6a}) + r(-16e^{2a}+16+24e^{-2a}-16e^{-4a}+e^{-6a}+e^{-8a}) - 32e^{-2a}+24e^{-4a}-4e^{-6a}\};$$

$$g_{-1} = \frac{a^3 e^{-a}}{16\sqrt{2}(2-e^{-2a})^2}\{32e^{4a}-24e^{2a}+16-6e^{-2a}+e^{-6a}\};$$

$$-2g_{-2} = \frac{2a^3 e^{-2a}(1-e^{-2a})}{16(2-e^{-2a})};$$

$r > 2$,

$$-rg_{-r} = -\frac{a^3(-)^r e^{-ra}}{4(\sqrt{2})^r(2-e^{-2a})}(4e^{4a}-2e^{2a}+e^{-2a}).$$

Since the values of all the coefficients are real, the formula (8.2) for $B-A-2iH$ shows immediately that $H=0$, and the axes are therefore principal axes. With the above values, we have then

$$B-A = \frac{\pi a^4}{64(2-e^{-2a})^2}\{32e^{2a}-24+16e^{-2a}+6e^{-4a}+2e^{-6a}+2e^{-8a}\},$$

$$A+B = \pi a^4\left[\frac{1}{256(2-e^{-2a})^2}\{-128e^{4a}+96e^{2a}-64+72e^{-2a} + 20e^{-4a}-12e^{-6a}+5e^{-8a}\} + \frac{(24+e^{-6a})}{256(2-e^{-2a})^3}+\frac{e^{-12a}}{128(2-e^{-2a})^4}\right].$$

The cylinder is uni-symmetrical and it is immediately obvious that y_0 for this case is zero, as may also be seen from the formula for y_0 when the coefficients are all real. The formula for x_0 involves B only, which is given by

$$B = \frac{\pi a^4}{256(2-e^{-2a})^4}\{-512e^{4a}+1408e^{2a}-1616+2324e^{-2a}-1288e^{-4a} + 688e^{-6a}+151e^{-8a}-56e^{-10a}+15e^{-12a}\}.$$

The coefficients being all real, the formula for x_0 reduces to

$$x_0 = -\tfrac{1}{4}\pi\mu E^{-1}B^{-1}(1+2\sigma)\left[b_1(f_{-1}-f_1)+2b_2(f_{-2}-f_2)+\sum_{n=3}^{\infty} nb_n(f_{-n}-f_n)\right] + \tfrac{1}{8}\pi\mu E^{-1}B^{-1}(1+6\sigma)\left[b_1(g_{-1}-g_1)+2b_2(g_{-2}-g_2)+\sum_{n=3}^{\infty} nb_n(g_{-n}-g_n)\right],$$

which, when we substitute the previous values for the coefficients, becomes

$$-\frac{\pi\mu E^{-1}B^{-1}a^5}{16\times 64(2-e^{-2\alpha})^3}\{(-64+22e^{-2\alpha}+50e^{-4\alpha}+16e^{-8\alpha}-32e^{-10\alpha}+3e^{-14\alpha})$$
$$+2\sigma(320-768e^{-2\alpha}+752e^{-4\alpha}-384e^{-6\alpha}$$
$$+956e^{-8\alpha}-96e^{-10\alpha}+9e^{-14\alpha})\}.$$

The torsional constant for this type of cylinder is

$$\Gamma=\frac{\pi\tau\mu a^4}{16(2-e^{-2\alpha})^3}[16(1-e^{2\alpha})(2-e^{-2\alpha})(32e^{2\alpha}-32+16e^{-2\alpha}-4e^{-4\alpha}+e^{-8\alpha})$$
$$-(2e^{-2\alpha}-5e^{-4\alpha}+3e^{-6\alpha}-e^{-8\alpha})]$$
$$+\frac{\pi\tau\mu a^4(1-e^{2\alpha})(8-8e^{-2\alpha}+6e^{-4\alpha}-2e^{-6\alpha}-e^{-8\alpha})}{2(4e^{2\alpha}-6+2e^{-2\alpha}-e^{-4\alpha})(2-e^{-2\alpha})^3},$$

and with this the solution is complete.

For the cylinder whose cross-section is a thin semi-circular arc we can approximate to the result by putting $\alpha=0$ when the above formulae reduce to

$$x_0=-\frac{\pi a(1578\sigma-5)}{16\times 557(1+\sigma)},$$

$$\Gamma=\pi\tau\mu a^4/16.$$

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SUBSPACES OF A FINSLER SPACE

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1. *Introduction.* The theory of spaces immersed in a Finsler space has been treated by Cartan (1*, IX, X, XI), Berwald (2), Haimovici (3), Hombu (4), and Wegener (5). The formal apparatus of the "D-symbolik" of Van der Waerden and Bortolotti, which has been so extensively used in the treatment of subspaces of a Riemannian space, has been extended by Hombu, together with the equations of Frenet, Gauss, Codazzi and Kühne.

In this paper the treatment will be less formal than that of Hombu, and we shall concern ourselves with some special problems such as deformation, the second fundamental form, and minimal subspaces.

We consider the point submanifold as given parametrically by the set of equations

$$x^\lambda = x^\lambda(u^1, u^2, \ldots, u^m)\dagger. \tag{1}$$

We shall denote a Finsler space for brevity by F_n, and hence a subspace subordinated to it by F_m $(m < n)$. The fundamental metric quadratic form in F_n is given by

$$ds^2 = g_{\lambda\mu}(x, x')\,dx^\lambda\,dx^\mu, \tag{2}$$

where $x'^\lambda = dx^\lambda/d\tau$ denotes the line-element which is called the "element of support", the τ being any parameter.

If we take the case in which the element of support is tangential to F_m, it will, regarded as a vector of F_m, have components given by

$$x'^\lambda = B_a{}^\lambda u'^a, \tag{3}$$

* These numbers refer to the list of papers at the end.

† Greek indices run from 1 to n; Latin indices from 1 to m except p, q, r which run from $m+1$ to n.

where

$$B_a^{\ \lambda} = \frac{\partial x^\lambda}{\partial u^a}. \tag{4}$$

The metric coefficients in F_m will be $'g_{ab}(u, u')$, which are related to those of F_n by the equations

$$'g_{ab} = g_{\lambda\mu} B_a^{\ \lambda} B_b^{\ \mu} \equiv g_{\lambda\mu} B_{ab}^{\lambda\mu}. \tag{5}$$

With this system $'g_{ab}$, provided that their determinant $'g$ does not vanish, there is associated the reciprocal system $'g^{bc}$ defined by

$$'g^{ab}\,'g_{bc} = \delta_c^{\ a}, \tag{6}$$

so that, as in Riemannian geometry, we may introduce

$$B_\lambda^{\ a} = 'g^{ad} g_{\lambda\mu} B_d^{\ \mu} \tag{7}$$

satisfying

$$B_\lambda^{\ a} B_b^{\ \lambda} = \delta_b^{\ a}. \tag{8}$$

If we now differentiate (5) with respect to u', since $B_a^{\ \lambda}$ is independent of the element of support, we shall have the following relation between the torsion tensors of F_m and F_n:

$$'C_{abc} = C_{\lambda\mu\nu} B_{abc}^{\lambda\mu\nu} = \overline{C}_{abc} \tag{9}$$

where the bar denotes a quantity obtained from the corresponding quantity in F_n "by projection" (by contraction with one of the "projection factors" $B_a^{\ \lambda}$ or $B_\mu^{\ b}$).

Having now introduced the $'g_{ab}$ and $'C_{abc}$ for the subspace, by applying conventions A–E of Cartan (**1**, p. 10), we can deduce a Euclidean connection in the subspace, which we shall call the *intrinsic* connection, and we shall denote its parameters by $'\Gamma_{abc}$, with

$$'\Gamma_{abc} = '\gamma_{abc} + 'S_{abc}, \tag{10}$$

where $'\gamma_{abc}$ are the 3-index symbols of Christoffel formed from $'g_{ab}$. These $'\Gamma_{abc}$ may be expressed entirely in terms of the $'\gamma_{abc}$ and of the $'C_{abc}$ (or of $'A_{abc} = L\,'C_{abc}$) by using equations (17) and (19) of Cartan (*loc. cit.*). For we have from (19), written in a slightly different form,

$$'\Gamma_{0j}^h = '\gamma_{0j}^h - 'A_d^{\ h}\,'\gamma_{00}^d, \tag{11}$$

and from (17) it follows that

$$'S_{abc} = 'A_{acd}\,'\Gamma_{0b}^d - 'A_{bcd}\,'\Gamma_{0a}^d \equiv 2\,'A_{[a|cd}\,'\Gamma_{0|b]}^d. \tag{12}$$

When we use (11) this becomes

$$'S_{abc} = 2\,'A_{[a|ch}\,'\gamma^{h}_{0|b]} - 2\,'A_{[a|ch|}\,'A^{h}_{b]d}\,'\gamma^{d}_{00},$$

and when we insert this value in (10) the $'\Gamma_{abc}$ is completely determined in terms of A and γ and their contractions with the unit element l.

The *induced* connection parameters are obtained by defining the absolute differential of a vector which is tangent to F_m as the projection, on the tangent plane of the F_m, of the absolute differential of that vector, regarded as a vector of F_n.

This gives, when we write $B^{\nu}_{bc} = \partial_b B_c{}^{\nu} \equiv \partial B_c{}^{\nu}/\partial u^b$,

$$\bar{\Gamma}^{a}_{bc} = B_{\nu}{}^{a}(B^{\nu}_{bc} + \Gamma^{\nu}_{\mu\omega} B^{\mu\omega}_{bc} + A^{\nu}_{\mu\omega} B^{\omega}_{0c} B_b{}^{\mu}).$$

If we denote the corresponding covariant differentials by D' and $\bar{D}$, then both $D'\,'g_{ab}$ and $\bar{D}\,'g_{ab}$ vanish, so that both connections are Euclidean. The D' and $\bar{D}$ differentials of a tensor coincide only for a differentiation in *any* direction provided the torsion tensor $A^{\nu}_{\mu\omega}$ vanishes (4, § 2). They always coincide, however, for a differentiation in the direction of the line-element itself.

2. *The projection factors as functions of the element of support.* The subspace F_m will be said to be "spanned" in the F_n provided there is associated with the tangent m-plane of F_m at every point an $(n-m)$-direction which is normal to it with respect to the metric of the surrounding space F_n. This normal $(n-m)$-direction is specified by the m *covariant* vectors $B^{(a)}_{\lambda}$, where (a) means that the index (a) is considered as an ordinal index. If, therefore, we choose in this $(n-m)$-direction a set of $(n-m)$ contravariant vectors $C_p{}^{\lambda}$ $(p = m+1, \ldots, n)$ which are mutually orthogonal (as vectors of F_n), and also normal to F_m, i.e. to all the contravariant vectors $B^{\lambda}_{(a)}$, we can say that the orientation determined by the particular vector $B^{(a)}_{\lambda}$ contains the directions of all the vectors $B_b{}^{\lambda}$ $(b = 1, \ldots, m)$ and $C_p{}^{\lambda}$ $(p = m+1, \ldots, n)$ except the direction $B^{\lambda}_{(a)}$. We introduce also the covariant span vectors $C_{\lambda}{}^{p}$, and we can then write down the following relations as subsisting between the various projection factors:

$$(13)\quad \begin{cases} (a)\ B_a{}^{\lambda} B_{\lambda}{}^{b} = \delta_a{}^{b}, & (b)\ B_{\lambda}{}^{a} C_p{}^{\lambda} = 0, \\ (c)\ C_{\lambda}{}^{p} B_a{}^{\lambda} = 0, & (d)\ C_p{}^{\lambda} C_{\lambda}{}^{q} = \delta_p{}^{q}, \\ (e)\ B_a{}^{\lambda} B_{\mu}{}^{a} = B_{\mu}{}^{\lambda}, & (f)\ C_p{}^{\lambda} C_{\mu}{}^{p} = C_{\mu}{}^{\lambda}, \\ (g)\ B_{\mu}{}^{\lambda} + C_{\mu}{}^{\lambda} = \delta_{\mu}{}^{\lambda}. \end{cases}$$

We can expect that, with the exception of $B_a{}^\lambda$, these projection factors will depend on the element of support. We therefore consider the effect of applying to them the operator $L\frac{\partial}{\partial u'^c}$ which, following Cartan, we write $\|_c$.

Starting from the fact that $B_a{}^\lambda$ does not involve the element of support, and that the $B_\lambda{}^a\|_c$ can be calculated from the expression (7) for $B_\lambda{}^a$ in terms of the metric coefficients, we have

$$(14)\qquad (a)\quad B_a{}^\lambda\|_c = 0, \qquad (b)\quad B_\lambda{}^b\|_c = 2B_\rho{}^b B_c{}^\tau C_\lambda{}^\sigma A^\rho_{\sigma\tau},$$

the result (14b) being obtained from (5), (6), (9) and (14a).

On using (13) (e) and (f), we obtain further

$$(15)\qquad B_\lambda{}^\mu\|_c = B_b{}^\mu B_\lambda{}^b\|_c = 2B_\rho{}^\mu B_c{}^\tau C_\lambda{}^\sigma A^\rho_{\sigma\tau},$$

and from (13) (g), we have

$$(16)\qquad C_\lambda{}^\mu\|_c = -B_\lambda{}^\mu\|_c.$$

Again, from (13) (c), on using (14), we have $B_a{}^\lambda C_\lambda{}^p\|_c = 0$, so that $C_\lambda{}^p\|_c = C_\lambda{}^r Q^p_{rc}$, where Q^p_{rc} have to be determined. The equations (13) do not provide us with enough information to determine the Q^p_{rc}, so they may be taken arbitrarily. The simplest assumption to introduce is to take them all to be vanishing, so that $C_\lambda{}^p\|_c = 0$. This is justified geometrically by the fact that the orientation determined by the $(n-m)$ covariant vectors $C_\lambda{}^p$ is precisely that of the m-vector $B^{\lambda_1}_{[1}\dots B^{\lambda_m}_{m]}$ the individual vectors of which do not depend on the element of support. Having made this assumption with regard to $C_\lambda{}^p$ and taking account of (15) and (16), we have finally

$$(17)\qquad C_p{}^\lambda\|_c = -2B_\rho{}^\lambda B_c{}^\tau C_p{}^\sigma A^\rho_{\sigma\tau}.$$

3. *Normal curvature.* The extremal curves in F_n are given by the equations (1, p. 17)

$$\frac{d^2x^\lambda}{ds^2} + 2G^\lambda\left(x, \frac{dx}{ds}\right) = 0.$$

On using

$$(18)\qquad \frac{dx^\lambda}{ds} = B_a{}^\lambda\frac{du^a}{ds}, \qquad \frac{d^2x^\lambda}{ds^2} = B_a{}^\lambda\frac{d^2u^a}{ds^2} + B^\lambda_{ab}\frac{du^a}{ds}\frac{du^b}{ds},$$

we may write these equations in the form

$$(19)\qquad B_a{}^\lambda\frac{d^2u^a}{ds^2} + 2\left(G^\lambda + \tfrac{1}{2}B^\lambda_{ab}\frac{du^a}{ds}\frac{du^b}{ds}\right) = 0,$$

or again, on writing

$$\bar{G}^c = B_\lambda{}^c\left(G^\lambda + \tfrac{1}{2}B^\lambda_{ab}\frac{du^a}{ds}\frac{du^b}{ds}\right) \tag{20}$$

and

$$H^p = C_\lambda{}^p\left(G^\lambda + \tfrac{1}{2}B^\lambda_{ab}\frac{du^a}{ds}\frac{du^b}{ds}\right), \tag{21}$$

we have

$$B_a{}^\lambda\left\{\frac{d^2u^2}{ds^2} + 2\bar{G}^a\left(u, \frac{du}{ds}\right)\right\} + 2C_p{}^\lambda H^p = 0. \tag{22}$$

From equation (11), on using the fact that $A^h_{jk}l^k = 0$, we have

$$'\Gamma^h_{00} = {}'\gamma^h_{00} = 2\,'G^h.$$

Now the 3-index symbols of Christoffel are deduced for the F_m immediately from the fact that the metric in F_m is induced from that in F_n, and we have

$$'\gamma^c_{ab} = B_\lambda{}^c(B^\lambda_{ab} + \gamma^\lambda_{\mu\nu}B^{\mu\nu}_{ab}),$$

so that

$$'G^c = B_\lambda{}^c\left(\tfrac{1}{2}B^\lambda_{ab}\frac{du^a}{ds}\frac{du^b}{ds} + G^\lambda\right) = \bar{G}^c.$$

Hence the equation (22) may be written

$$\frac{\varpi^\lambda}{ds} = B_a{}^\lambda\frac{\varpi^a}{ds} + 2C_p{}^\lambda H^p = 0, \tag{23}$$

where $\dfrac{\varpi^a}{ds} = \dfrac{{}'\varpi^a}{ds} = \dfrac{\bar{\varpi}^a}{ds}$ is the *geodesic curvature* of the curve in the F_m regarded as the locus of its elements of support, and where H^p is the *normal curvature* of the curve, the normal curvature vector being $H^\lambda \equiv C_p{}^\lambda H^p$. It is evident therefore that the geodesic curvature and the normal curvature of any curve of the F_m is the same whether we are dealing with the intrinsic or with the induced connection in the F_m.

4. *Relation between the induced and intrinsic connection parameters.* The difference between $\bar{\Gamma}^a_{bc}$ and $'\Gamma^a_{bc}$ may now be expressed fairly simply in terms of the normal curvature vector H^λ and its derivative with respect to the element of support.

We consider first

$$'\Gamma^e_{0d} = {}'G^e\|_d = \{B_\lambda{}^e(\tfrac{1}{2}B^\lambda_{bc}u'^b u'^c + G^\lambda)\}\|_d, \tag{24}$$

whence

$$'\Gamma^e_{0d} = B_\lambda{}^e(B^\lambda_{0d} + G^\lambda\|_\mu B_d{}^\mu) + B_\lambda{}^e\|_d(\tfrac{1}{2}B^\lambda_{bc}u'^b u'^c + G^\lambda). \tag{25}$$

Using $$B_\lambda{}^e\|_d = B_\mu{}^e(B_\lambda{}^\mu\|_d) = -B_\mu{}^e(C_\lambda{}^\mu\|_d)$$

and the fact that the first term on the right-hand side of (25) is $\bar\Gamma^e_{0d}$, we get

$$'\Gamma^e_{0d} = \bar\Gamma^e_{0d} - B_\mu{}^e\{C_\lambda{}^\mu(\tfrac{1}{2}B^\lambda_{bc}u'^b u'^c + G^\lambda)\}\|_d$$

or

$$'\Gamma^e_{0d} = \bar\Gamma^e_{0d} - B_\mu{}^e H^\mu\|_d. \tag{26}$$

Finally, using equation (12) for both $'S^a_{bc}$ and $\bar S^a_{bc}$, we have

$$'\Gamma^a_{bc} = {}'\gamma^a_{bc} + 2\,'g^{ad}\,'A_{[b|ce}\,'\Gamma^e_{0|d]},$$

with the corresponding equation for $\bar\Gamma^a_{bc}$. Using equation (26) we therefore have

$$'\Gamma^a_{bc} - \bar\Gamma^a_{bc} = 2\,'g^{ad}B_\lambda{}^e H^\lambda\|_{[d}\,'A_{b]ce}. \tag{27}$$

The relation (23) for ϖ^λ in terms of ϖ^a is valid only for derivation in the direction of the element of support itself. Let us now consider the absolute differential of the line-element of support in any direction $dx^\lambda = B_a{}^\lambda du^a$. We then have

$$\varpi^\lambda = dl^\lambda + \Gamma^\lambda_{0\mu}dx^\mu, \quad '\varpi^a = dl^a + {}'\Gamma^a_{0c}du^c, \quad \bar\varpi^a = dl^a + \bar\Gamma^a_{0c}du^c$$

and $$\varpi^\lambda - B_a{}^\lambda\bar\varpi^a = (B^\lambda_{0c} + \Gamma^\lambda_{0\mu}B_c{}^\mu - \bar\Gamma^a_{0c}B_a{}^\lambda)\,du^c,$$

where $$\bar\Gamma^a_{0c} = B_\nu{}^a(B^\nu_{0c} + \Gamma^\nu_{0\mu}B_c{}^\mu),$$

giving $$\varpi^\lambda - B_a{}^\lambda\bar\varpi^a = C_\nu{}^\lambda(B^\nu_{0c} + \Gamma^\nu_{0\mu}B_c{}^\mu)\,du^c.$$

On taking the expression for $H^\lambda = C_\mu{}^\lambda(\tfrac{1}{2}B^\mu_{bc}u'^b u'^c + G^\mu)$ and using (15) and (16) we immediately verify that

$$\varpi^\lambda - B_a{}^\lambda\bar\varpi^a = C_\mu{}^\lambda H^\mu\|_c\,du^c = H_c{}^\lambda\,du^c, \tag{28}$$

where $H_c{}^\lambda$ is defined by that equation. It can easily be proved that

$$\varpi^\lambda - B_a{}^\lambda\,'\varpi^a = H^\lambda\|_c\,du^c, \tag{29}$$

we gives $$B_\lambda{}^a\varpi^\lambda - {}'\varpi^a = \bar\varpi^a - {}'\varpi^a = B_\lambda{}^a H^\lambda\|_c\,du^c$$

for a differential du^c in a direction which is not that of the element of support. If du^c is in the direction of the element of support, then

$$H^\lambda\|_c\,du^c = \pounds H^\lambda\|_c\,l^c = 0,$$

since H^λ is homogeneous of degree zero in the elements l^c.

We are concerned in this paper with the subordination of F_m in F_n, so that the projected connection $\bar{\Gamma}^a_{bc}$ is the one that concerns us. Just as in the case of the F_n, there are connection parameters, denoted by $\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}$, which enter into the expression for the absolute differential of a vector when its element of support is transported parallely to itself, there are in F_m corresponding connection parameters ($\overset{*}{\bar{\Gamma}}{}^a_{bc}$ which we denote for simplicity by ${}_*\Gamma^a_{bc}$), given by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} - \bar{A}^a_{be}\,\bar{\Gamma}^e_{0c}$$

$$= B_\nu{}^a[B^\nu_{bc} + \Gamma^\nu_{\mu\omega}\,B^{\mu\omega}_{bc} + A^{\nu}_{\mu\omega}\,B_b{}^\mu\,B^\omega_{0c} - A^\nu_{\mu\tau}\,B_b{}^\mu\,B_\omega{}^\tau(B^\omega_{0c} + \Gamma^\omega_{0\rho}\,B_c{}^\rho)],$$

which we may easily modify to

$$\Gamma^a_{bc} = B_\nu{}^a[B^\nu_{bc} + \Gamma^\nu_{\mu c}\,B_b{}^\mu], \tag{30}$$

where, following Hombu (**4**, 50*a*),

$$\Gamma^{\nu}_{\mu c} = \overset{*}{\Gamma}{}^\nu_{\mu\omega}\,B_c{}^\omega + A^{\nu}_{\mu\omega}\,H_c{}^\omega, \tag{31}$$

so that, for a vector v^a of the submanifold,

$$\bar{D}v^a = v^a|_b\,du^b + (v^a\|_b + A^a_{cb}\,v^c)\,\bar{\omega}^b \tag{32}$$

with

$$v^a|_b = \partial_b\,v^a - v^a\|_d\,\Gamma^d_{0b} + \Gamma^a_{bc}\,v^c, \quad \text{and} \quad A^a_{cb} = \bar{A}^a_{cb} = {}'A^a_{cb}.$$

There exists, for vectors attached to points of F_m, but lying in the normal $(n-m)$-direction, a law of transport corresponding to the *induced* connection for vectors lying in the subspace. In that case there is no intrinsic connection, so that we have to determine the parameters by demanding that the absolute differential of a vector which is normal to F_m shall be the projection, on the normal $(n-m)$-direction, of the absolute differential of that vector regarded as a vector of F_n.

Let $w^\lambda = C_p{}^\lambda\,w^p$ be such a vector. We define $\bar{D}w^p$ by the equation

$$\bar{D}w^p = C_\lambda{}^p\,Dw^\lambda.$$

This gives

$$\bar{D}w^p = dw^p + \bar{\lambda}^p_{qc}\,w^q\,du^c + \bar{C}^p_{qc}\,w^q\,du'^c, \tag{33}$$

where

$$\bar{\lambda}^p_{qc} = C_\lambda{}^p(\partial_c\,C_q{}^\lambda + \Gamma^\lambda_{\mu\nu}\,C^\mu_q\,B_c{}^\nu + A^\lambda_{\mu\nu}\,C^\mu_q\,B^\nu_{0c}) \tag{34}$$

and $$\bar{C}^p_{qc} = C_\lambda{}^p\, C_q{}^\mu\, B_c{}^\nu\, C^\lambda_{\mu\nu}$$

when we use $C_\lambda{}^p\, C_q{}^\lambda\|_c = 0$ from equation (17).

The equation (33) may be written

$$\bar{D}w^p = dw^p + \lambda^p_{qc}\, w^q\, du^c + \bar{A}^p_{qc}\, w^q\, \bar{\varpi}^c \tag{35}$$

where

$$\lambda^p_{qc} = \bar{\lambda}^p_{qc} - \bar{A}^p_{qd}\, \bar{\Gamma}^d_{0c}, \tag{36}$$

which may easily be verified to be given by

$$\lambda^p_{qc} = C_\lambda{}^p(\partial_c\, C_q{}^\lambda + \Gamma^\lambda_{\mu e}\, C_q{}^\mu). \tag{37}$$

5. *D-symbolik, equations of Gauss, Codazzi, and Kühne.* For any contravariant vector field $X^\lambda(x, x')$ defined at points of the submanifold F_m, the absolute differential (along F_m) is

$$DX^\lambda = X^\lambda|_\mu\, dx^\mu + (X^\lambda\|_\mu + A^\lambda_{\rho\mu}\, X^\rho)\, \varpi^\mu \tag{38}$$

or, following Hombu, and defining $\overset{0}{\nabla}_\mu X^\lambda = X^\lambda|_\mu$ and $\overset{1}{\nabla}_\mu X^\lambda = X^\lambda\|_\mu + A^\lambda_{\rho\mu}\, X^\rho$ (where the $\overset{1}{\nabla}_\mu X^\lambda$ so defined differs by the factor $\mathfrak{L}$ from the corresponding expression in Hombu),

$$DX^\lambda = \overset{0}{\nabla}_\mu X^\lambda\, .\, dx^\mu + \overset{1}{\nabla}_\mu X^\lambda\, \varpi^\mu. \tag{39}$$

Using (28) we shall have

$$DX^\lambda = \overset{0}{D}_c X^\lambda\, .\, du^c + \overset{1}{D}_c X^\lambda\, \bar{\varpi}^c, \tag{40}$$

where

$$\overset{0}{D}_c X^\lambda = B_c{}^\mu\, \overset{0}{\nabla}_\mu X^\lambda + H_c{}^\mu\, \overset{1}{\nabla}_\mu X^\lambda, \quad \overset{1}{D}_c X^\lambda = B_c{}^\mu\, \overset{1}{\nabla}_\mu X^\lambda. \tag{41}$$

For a vector field $v^\lambda = B_a{}^\lambda\, v^a$ lying in the F_m itself, the corresponding $\overset{0}{D}_c v^a$ is defined by

$$\overset{0}{D}_c v^a = B_\lambda{}^a\left(B_c{}^\mu\, \overset{0}{\nabla}_\mu v^\lambda + H_c{}^\mu\, \overset{1}{\nabla}_\mu v^\lambda\right) \tag{42}$$

and, for a vector $w^\lambda = C_p{}^\lambda\, w^p$ normal to F_m,

$$\overset{0}{D}_c w^p = C_\lambda{}^p\left(B_c{}^\mu\, \overset{0}{\nabla}_\mu w^\lambda + H_c{}^\mu\, \overset{1}{\nabla}_\mu w^\lambda\right). \tag{43}$$

It is now obvious that these $\overset{0}{D}$-derivatives may be expressed in terms of the connection parameters $\Gamma^{\lambda}_{\mu c}$, Γ^{a}_{bc} and λ^{p}_{qc}, giving

$$\overset{0}{D}_c X^{\lambda} = \partial_c X^{\lambda} - X^{\lambda}\|_d \Gamma^{d}_{0c} + \Gamma^{\lambda}_{\mu c} X^{\mu}, \tag{44}$$

$$\overset{0}{D}_c v^{a} \equiv v^{a}|_c = \partial_c v^{a} - v^{a}\|_d \Gamma^{d}_{0c} + \Gamma^{a}_{bc} v^{b}, \tag{45}$$

$$\overset{0}{D}_c w^{p} = \partial_c w^{p} - w^{p}\|_d \Gamma^{d}_{0c} + \lambda^{p}_{qc} w^{q}, \tag{46}$$

and, for a mixed tensor $T^{\lambda}_{.a}{}^{p}$ involving the three kinds of indices, we can write immediately

$$\overset{0}{D}_c T^{\lambda}_{.a}{}^{p} = \partial_c T^{\lambda}_{.a}{}^{p} - T^{\lambda}_{.a}{}^{p}\|_d \Gamma^{d}_{0c} + \Gamma^{\lambda}_{\mu c} T^{\mu}_{.a}{}^{p} + \lambda^{p}_{qc} T^{\lambda}_{.a}{}^{q} - \Gamma^{d}_{ac} T^{\lambda}_{.d}{}^{p}. \tag{47}$$

The simplest mixed tensors to which the $\overset{0}{D}$-operator may be applied are the projection factors $B_a{}^{\lambda}$ and $C_p{}^{\lambda}$. We thus arrive at the natural generalizations of the tensors of Eulerian Curvature of the first and second kinds which occur in ordinary Riemannian Geometry. The first tensor is

$$\begin{aligned} \overset{0}{H}_{ba}{}^{\lambda} = \overset{0}{D}_b B_a{}^{\lambda} &= B_a{}^{\nu}\left(B_b{}^{\mu}\overset{0}{\nabla}_{\mu} B_{\nu}{}^{\lambda} + H_b{}^{\mu}\overset{1}{\nabla}_{\mu} B_{\nu}{}^{\lambda}\right) \\ &= \partial_b B_a{}^{\lambda} + \Gamma^{\lambda}_{\rho b} B_a{}^{\rho} - \Gamma^{d}_{ab} B_d{}^{\lambda} = C_{\rho}{}^{\lambda}(\partial_b B_a{}^{\rho} + \Gamma^{\rho}_{\sigma b} B_a{}^{\sigma}), \end{aligned} \tag{48}$$

so that the quantity $\overset{0}{H}_{ba}{}^{\lambda}$, which "lies with the index λ in the span", can also be written in the form $C_p{}^{\lambda}\overset{0}{H}_{ba}{}^{p}$, with

$$\overset{0}{H}_{ba}{}^{p} = C_{\lambda}{}^{p}(\partial_b B_a{}^{\lambda} + \Gamma^{\lambda}_{\rho b} B_a{}^{\rho}). \tag{49}$$

The second tensor is

$$\begin{aligned} \overset{0}{L}_{b.p}{}^{\lambda} = \overset{0}{D}_b C_p{}^{\lambda} &= C_p{}^{\nu}\left(B_b{}^{\mu}\overset{0}{\nabla}_{\mu} C_{\nu}{}^{\lambda} + H_b{}^{\mu}\overset{1}{\nabla}_{\mu} C_{\nu}{}^{\lambda}\right) \\ &= \partial_b C_p{}^{\lambda} - C_p{}^{\lambda}\|_d \Gamma^{d}_{0b} + \Gamma^{\lambda}_{\mu b} C_p{}^{\mu} - \lambda^{q}_{pb} C_q{}^{\lambda}, \end{aligned} \tag{50}$$

or

$$\overset{0}{L}_{b.p}{}^{\lambda} = B_{\rho}{}^{\lambda}(\partial_b C_p{}^{\rho} - C_p{}^{\rho}\|_d \Gamma^{d}_{0b} + \Gamma^{\rho}_{\mu b} C_p{}^{\mu}), \tag{51}$$

so that $\overset{0}{L}_{b.p}{}^{\lambda}$, which "lies with the index λ in the subspace", can be written in the form $B_c{}^{\lambda}\overset{0}{L}_{b.p}{}^{c}$.

The equations (30), (36), (49) and the one corresponding to (51) for $\overset{0}{L}_{b\cdot p}^{\cdot\lambda}$ may now be written

$$(52)\qquad \begin{cases} \Gamma^a_{bc} = B_\lambda{}^a \overset{0}{D}_b B^\lambda_{(c)}, & \lambda^p{}_c = C_\lambda{}^p \overset{0}{D}_c C^\lambda_{(q)}, \\ \overset{0}{H}{}^{\cdot p}_{bc} = C_\lambda{}^p \overset{0}{D}_b B^\lambda_{(c)}, & \overset{0}{L}_{b\cdot p}^{\cdot c} = B_\lambda{}^c \overset{0}{D}_b C^\lambda_{(p)}. \end{cases}$$

In view of the relations (13) between the projection factors, we can expect that the two tensors of Eulerian Curvature just introduced will not be independent. In fact, from (13*b*) we shall have, when we apply the $\overset{0}{D}$-operator,

$$B_\lambda{}^a \overset{0}{D}_b C_p{}^\lambda + C_p{}^\lambda \overset{0}{D}_b B_\lambda{}^a = 0$$

so that
$$\overset{0}{L}_{b\cdot p}^{\cdot a} = -C_p{}^\lambda {}'g^{ad} g_{\lambda\mu} \overset{0}{H}{}^{\cdot\mu}_{bd} = -{}'g^{ad} \overset{0}{H}{}^{\cdot p}_{bd}.$$

The corresponding tensors obtainable by applying the $\overset{1}{D}$-operator to the projection factors may be defined by

$$(53)\qquad \overset{1}{H}{}^{\cdot\lambda}_{ba} = B_b{}^\mu B_a{}^\nu \overset{1}{\nabla}_\mu B_\nu{}^\lambda, \qquad \overset{1}{L}_{b\cdot p}^{\cdot\lambda} = B_b{}^\mu C_p{}^\nu \overset{1}{\nabla}_\mu C_\nu{}^\lambda.$$

We can easily obtain generalizations of the equations of Gauss, Codazzi and Kühne for the relations between the various curvatures on giving alternation formulae for these tensors of Eulerian Curvature. Taking those deducible from the $\overset{0}{D}$-operator, we have

$$(54)\qquad 2\overset{0}{D}_{[c} \overset{0}{H}{}^{\cdot\lambda}_{b]a} = R^\lambda_{\cdot\rho,\,bc} - B_d{}^\lambda R^d_{\cdot a,\,bc},$$

$$(55)\qquad 2\overset{0}{D}_{[c} L_{b]\cdot p}^{\cdot\lambda} = R^\lambda_{\cdot\rho,\,bc} C_p{}^\rho - R^q_{\cdot p,\,bc} C_q{}^\lambda - C_p{}^\lambda \|_b R^d_{\cdot 0,\,bc},$$

where, when we write $*$ to indicate similar terms obtained on alternation with respect to c and b,

$$(56)\qquad R^\lambda_{\cdot\rho,\,bc} = \partial_c \Gamma^\lambda_{\rho b} - \Gamma^\lambda_{\rho b} \|_d \Gamma^d_{0c} + \Gamma^\lambda_{\sigma c} \Gamma^\sigma_{\rho b} - * ;$$

$$(57)\qquad R^d_{\cdot a,\,bc} = \partial_c \Gamma^d_{ab} - \Gamma^d_{ab} \|_e \Gamma^e_{0c} + \Gamma^d_{ec} \Gamma^e_{ab} - * ;$$

$$(58)\qquad R^q_{\cdot p,\,bc} = \partial_c \lambda^q_{pb} - \lambda^q_{pb} \|_d \Gamma^d_{0c} + \lambda^q_{rc} \lambda^r_{pb} - * .$$

Using now $\overset{0}{H}{}_{ba}^{\cdot\lambda} = C_p{}^{\lambda}\overset{0}{H}{}_{ba}^{\cdot p}$ and $\overset{0}{L}{}_{b\cdot p}^{\lambda} = B_a{}^{\lambda}\overset{0}{L}{}_{b\cdot p}^{a}$, we have the formulae

$$\overset{0}{D}_c\overset{0}{H}{}_{ba}^{\cdot\lambda} = \left(\overset{0}{D}_c C_p{}^{\lambda}\right)\overset{0}{H}{}_{ba}^{\cdot p} + C_p{}^{\lambda}\overset{0}{D}_c\overset{0}{H}{}_{ba}^{\cdot p} \tag{59}$$

$$= B_d{}^{\lambda}\overset{0}{L}{}_{c\cdot p}^{d}\overset{0}{H}{}_{ba}^{\cdot p} + C_p{}^{\lambda}\overset{0}{D}_c\overset{0}{H}{}_{ba}^{\cdot p},$$

$$\overset{0}{D}_c\overset{0}{L}{}_{b\cdot p}^{\lambda} = \left(\overset{0}{D}_c B_a{}^{\lambda}\right)\overset{0}{L}{}_{b\cdot p}^{a} + B_a{}^{\lambda}\overset{0}{D}_c\overset{0}{L}{}_{b\cdot p}^{a} \tag{60}$$

$$= C_q{}^{\lambda}\overset{0}{H}{}_{ca}^{\cdot q}\overset{0}{L}{}_{b\cdot p}^{a} + B_a{}^{\lambda}\overset{0}{D}_c\overset{0}{L}{}_{b\cdot p}^{a}.$$

Inserting these values from (59) and (60) in (54) and (55) and contracting successively with $B_\lambda{}^e$ and $C_\lambda{}^r$, and then putting for $C_p{}^{\lambda}\|_d$ its expression as given by (17), we have

$$2\overset{0}{L}{}_{[c\cdot|p|}^{e}\overset{0}{H}{}_{b]a}^{\cdot p} = R^{\lambda}_{\cdot\rho,bc}B^{\rho e}_{a\lambda} - R^{e}_{\cdot a,bc}, \tag{61}$$

$$2\overset{0}{D}_{[c}\overset{0}{H}{}_{b]a}^{\cdot r} = R^{\lambda}_{\cdot\rho,bc}B_a{}^{\rho}C_\lambda{}^{r}, \tag{62}$$

$$2\overset{0}{D}_{[c}\overset{0}{L}{}_{b]\cdot p}^{d} = R^{\lambda}_{\cdot\rho,bc}C_p{}^{\rho}B_\lambda{}^{d} + R^{e}_{\cdot 0,bc}B^{d\tau}_{\rho e}C_p{}^{\sigma}A^{\rho}_{\tau\sigma}, \tag{63}$$

$$2\overset{0}{H}{}_{[c|a|}^{\cdot r}\overset{0}{L}{}_{b]\cdot q}^{a} = R^{\lambda}_{\cdot\rho,bc}C^{\rho r}_{p\lambda} - R^{r}_{\cdot p,bc}. \tag{64}$$

(61) is the generalization of the Gauss equation, (62) and (63) of the Codazzi, and (64) of the Kühne equation.

6. *Deformation of a sub-space.* If we now consider the infinitesimal transformation

$$\bar{x}^{\lambda} = x^{\lambda} + v^{\lambda}(x)\,dt \tag{65}$$

applied to the points of F_m, the new points $\bar{x}^{\lambda}$ will form a new submanifold of F_n, which we can denote by $\bar{F}_m$, and the element of support will have been displaced so that it is tangential to $\bar{F}_m$. This, therefore, can be studied as a sub-space of F_n whose fundamental quantities differ by an infinitesimal amount from the corresponding quantities in F_m.

We shall now consider the variation of some of these fundamental quantities characterizing the subordination of F_m in F_n under the transformation (65). Taking first the projection factors $B_a{}^{\lambda}$, the value on $\bar{F}_m$ will be given by

$$\bar{B}_a{}^{\lambda} = B_a{}^{\lambda} + \partial_a v^{\lambda}\,dt,$$

so that, using the notation of Schouten and Struik (6, I, p. 150),

$$\overset{v}{d}B_a{}^\lambda = \partial_a v^\lambda \, dt. \tag{66}$$

But the "displaced" (*mitgeschleppt*) value, since the coordinate system in the F_m is supposed "taken on" in the transformation is also given by

$$\overset{m}{d}B_a{}^\lambda = \partial_a v^\lambda \, dt, \tag{67}$$

and consequently, defining $\overset{a}{\delta} = \overset{v}{d} - \overset{m}{d}$, we shall have

$$\overset{a}{\delta}B_a{}^\lambda = 0. \tag{68}$$

From the treatment of these quantities for Riemannian space as carried out by Schouten and Struik (6), it follows that the $\overset{a}{\delta}$ applied to any quantity of the F_n coincides with the Lie derivative, since the $\overset{v}{d}$ in that case is the $\overset{n}{d}$. We therefore have

$$\overset{a}{\delta} g_{\lambda\mu} = \underset{L}{\delta} g_{\lambda\mu}$$

and, for a Finsler space (7, p. 269),

$$\underset{L}{\delta} g_{\lambda\mu} = 2(v_{(\lambda|\mu)} + A_{\lambda\mu\nu} v^\nu |_0). \tag{69}$$

Hence, applying the $\overset{a}{\delta}$ operator to the equation (5), we have, using (68),

$$\overset{a}{\delta}{}' g_{ab} = \left(\underset{L}{\delta} g_{\lambda\mu}\right) B_{ab}^{\lambda\mu} \tag{70}$$

and

$$\overset{a}{\delta}{}' g^{ab} = -{}' g^{ac} \, {}' g^{bd} \overset{a}{\delta}{}' g_{cd}. \tag{71}$$

We are now in a position to give the value of $\overset{a}{\delta} B_\lambda{}^c$, on using (7), (69) and (71). We have, after a little calculation,

$$\overset{a}{\delta} B_\lambda{}^c = {}' g^{cd} B_d{}^\rho C_\lambda{}^\mu \left(\underset{L}{\delta} g_{\mu\rho}\right), \tag{72}$$

and consequently, on using (68) again,

$$\overset{a}{\delta} B_\lambda{}^\nu = B_c{}^\nu \overset{a}{\delta} B_\lambda{}^c = {}' g^{cd} B_c{}^\nu B_d{}^\rho C_\lambda{}^\mu \left(\underset{L}{\delta} g_{\mu\rho}\right) = -\overset{a}{\delta} C_\lambda{}^\nu. \tag{73}$$

To determine the $\overset{a}{\delta} C_p{}^\lambda$ and $\overset{a}{\delta} C_\mu{}^q$ separately we shall have to proceed in a manner similar to that followed in a previous paper (8, pp. 297–298), and put

$$\overset{a}{\delta} C_\lambda{}^p = 0 \tag{74}$$

and consequently

$$\overset{a}{\delta} C_p{}^\nu = -{}'g^{cd} B_{cd}^{\nu\rho} C_p{}^\mu \left(\underset{L}{\delta} g_{\mu\rho}\right). \tag{75}$$

Having now obtained the metric tensors and the projection tensors appropriate to the $\bar{F}_m$, we can reconstruct the geometry of the $\bar{F}_m$ completely, and in particular we can obtain the connection parameters in the new subspace and the new span respectively. Let us first consider the variation of the Γ^a_{bc}. If barred letters denote quantities referring to $\bar{F}_m$, we can write

$$\overset{v}{d}\,\Gamma^a_{bc} = \bar{B}_\nu{}^a(\bar{B}^\nu_{bc} + \bar{\Gamma}^\nu_{\mu c}\,\bar{B}_b{}^\mu) - \Gamma^a_{bc}. \tag{76}$$

Remembering the definition of $\Gamma^\nu_{\mu c}$ given in (31), and of $H_b{}^\nu$ given in (28), we must first of all find $\overset{v}{d}\,H_b{}^\nu$, where $H_b{}^\nu$ may be written

$$H_b{}^\nu = C_a{}^\nu (B^a_{ab} + \overset{*}{\Gamma}{}^a_{\rho\sigma}\, B^{\rho\sigma}_{ab})\, l^a. \tag{77}$$

On using

$$\overset{v}{d}\, C_a{}^\nu = \overset{a}{\delta}\, C_a{}^\nu + \overset{m}{d}\, C_a{}^\nu = \overset{a}{\delta}\, C_a{}^\nu + C_a{}^\rho\, \partial_\rho v^\nu\, dt - C_\rho{}^\nu\, \partial_\alpha v^\rho\, dt,$$

$$\overset{v}{d}\, B^a_{ab} = \frac{\partial^2 v^a}{\partial u^a\, \partial u^b}\, dt = \overset{m}{d}\, B^a_{ab},$$

$$\overset{v}{d}\, \overset{*}{\Gamma}{}^a_{\rho\sigma} = \overset{*}{\Gamma}{}^a_{\rho\sigma}(\bar{x}, \bar{x}') - \overset{*}{\Gamma}{}^a_{\rho\sigma}(x, x'),$$

we have, after a rather complicated calculation,

$$\overset{\bullet}{d}\, H_b{}^\nu = C_a{}^\nu\, B^{\rho\sigma}_{ab} \left(\underset{L}{\delta}\, \overset{*}{\Gamma}{}^a_{\rho\sigma}\right) l^a + \overset{a}{\delta}\, C_a{}^\nu\, H_b{}^a + H_b{}^\rho\, \partial_\rho v^\nu\, dt,$$

where

$$\underset{L}{\delta}\, \overset{*}{\Gamma}{}^a_{\rho\sigma} = v^a|_{\rho\sigma} + \overset{*}{R}{}^a_{\rho\tau\tau}\, v^\tau + \overset{*}{\Gamma}{}^a_{\rho\sigma}\|_\tau\, v^\tau|_0$$

is the Lie derivative of the connection parameters (7, p. 268), and finally, since $B_a{}^\rho\, l^a = l^\rho$,

$$\overset{a}{\delta}\, H_b{}^\nu = \left(\overset{v}{d} - \overset{m}{d}\right) H_b{}^\nu = C_a{}^\nu\, B_b{}^\sigma \left(\underset{L}{\delta}\, \overset{*}{\Gamma}{}^a_{0\sigma}\right) + \overset{a}{\delta}\, C_a{}^\nu\, H_b{}^a. \tag{78}$$

From this and the definition of $\Gamma^{\nu}_{\mu c}$, it follows that

$$\overset{a}{\delta}\Gamma^{\nu}_{\mu c} = \left(\underset{L}{\delta}\Gamma^{\nu}_{\mu\rho}\right) B_c^{\rho} + \left(\underset{L}{\delta} A^{\nu}_{\mu\rho}\right) H_c^{\rho} + A^{\nu}_{\mu\rho}\overset{a}{\delta} H_c^{\rho},$$

and since $\overset{m}{d}\Gamma^{a}_{bc} = 0$, we obtain

$$\overset{a}{\delta}\Gamma^{a}_{bc} = \left(\overset{a}{\delta}\Gamma^{\lambda}_{\mu c}\right) B^{a\mu}_{\lambda b} + \overset{a}{\delta} B_{\lambda}{}^{a}(B^{\lambda}_{bc} + \Gamma^{\lambda}_{\mu c} B_b{}^{\mu}).$$

Since the index λ in the expression for $\overset{a}{\delta} B_{\lambda}{}^{a}$ is a span index, it follows from the equation of definition (48) for the Eulerian Curvature tensor that

$$\overset{a}{\delta}\Gamma^{a}_{bc} = \left(\overset{a}{\delta}\Gamma^{\lambda}_{\mu c}\right) B^{a\mu}_{\lambda b} + \overset{a}{\delta} B_{\lambda}{}^{a}\overset{0}{H}{}^{\lambda}_{bc}. \tag{79}$$

A similar calculation would give us

$$\overset{a}{\delta}\lambda^{p}_{qb} = C^{p\mu}_{\lambda q}\left(\overset{a}{\delta}\Gamma^{\lambda}_{\mu b}\right) - C_q{}^{\sigma}\left(\overset{a}{\delta} B_{\sigma}{}^{e}\right)\overset{0}{H}{}^{p}_{bc}. \tag{80}$$

We may also give a formula for the interchange of the order of the $\overset{a}{\delta}$-operator and the $\overset{0}{D}$-operator. It follows in a manner completely analogous to that of Riemannian Geometry (**8**, p. 299) that for a mixed tensor of the form $T^{\lambda}_{\cdot b}{}^{p}$ we shall have

$$\overset{a}{\delta}\left(\overset{0}{D}_c T^{\lambda}_{\cdot b}{}^{p}\right) - \overset{0}{D}_c\left(\overset{a}{\delta} T^{\lambda}_{\cdot b}{}^{p}\right) = \left(\overset{a}{\delta}\Gamma^{\lambda}_{\mu c}\right) T^{\mu}_{\cdot b}{}^{p} + \overset{a}{\delta}\lambda^{p}_{qc} T^{\lambda}_{\cdot b}{}^{q} - \overset{a}{\delta}\Gamma^{d}_{bc} T^{\lambda}_{\cdot d}{}^{p}. \tag{81}$$

Applying this to the case of the simple tensor $B_b{}^{\lambda}$, we have

$$\overset{a}{\delta}\left(\overset{0}{D}_c B_b{}^{\lambda}\right) - \overset{0}{D}_c\left(\overset{a}{\delta} B_b{}^{\lambda}\right) = \left(\overset{a}{\delta}\Gamma^{\lambda}_{\mu c}\right) B_b{}^{\mu} - \left(\overset{a}{\delta}\Gamma^{d}_{bc}\right) B_d{}^{\lambda},$$

so that, on taking account of (68), (73) and (79), we get the following simple expression for the variation of the Eulerian Curvature tensor

$$\overset{a}{\delta} H_{bc}{}^{\lambda} = \left(\overset{a}{\delta}\Gamma^{\sigma}_{\mu c}\right) C_{\sigma}{}^{\lambda} B_b{}^{\mu} + \overset{0}{H}{}^{\sigma}_{bc}\overset{a}{\delta} C_{\sigma}{}^{\lambda}. \tag{82}$$

7. *The second fundamental form.* In the treatment of a subspace in a Riemannian space, the method of introducing the second fundamental form due to Bianchi (**11**, p. 450) consists in calculating the variation of the first fundamental form in passing from the given subspace to an infinitely near parallel one. A second method due to Bompiani (**10**, ch. I)

depends on transporting a vector of the subspace by parallelism firstly as a vector of the surrounding space and secondly as a vector of the subspace, and then finding the vectorial difference between the two results. In Riemannian Geometry these two methods give the same result (9, p. 291). In Finsler Geometry, however, the position is not so simple.

Let us consider, first of all, the variation of $ds^2 = {}'g_{bc}\,du^b\,du^c$. Since the du^b are just carried along in the deformation, the variation in the ds^2 will occur only through the coefficient ${}'g_{bc}$, and by the preceding section

$$\overset{a}{\delta}\,{}'g_{bc} = \left(\underset{L}{\delta}\, g_{\lambda\mu}\right) B_{bc}^{\lambda\mu}.$$

On quoting the expression (69) for $\underset{L}{\delta}\, g_{\lambda\mu}$ and using the fact that

$$\overset{0}{D}_b\, v^\rho = B_b{}^\mu\, \overset{0}{\nabla}_\mu v^\rho + H_b{}^\mu\, \overset{1}{\nabla}_\mu v^\rho = \partial_b v^\rho + \Gamma^\rho_{\sigma b}\, v^\sigma$$

(the term involving derivation with respect to the element of support being missing since v^ρ is supposed to be a function of position only), we have

$$\overset{a}{\delta}\,{}'g_{bc} = 2g_{\lambda\rho}\left\{ B^\lambda_{(b}\, \overset{0}{D}_{c)}\, v^\rho - B^\lambda_{(b}\, H^\mu_{c)}\, A^\rho_{\mu\nu}\, v^\nu + A^\rho_{\mu\nu}\, v^\nu \big|_0\, B^{\lambda\mu}_{bc} \right\}. \tag{83}$$

If v^ρ is taken to be orthogonal to F_m, then, by definition,

$$g_{\lambda\rho}\, B_b{}^\lambda\, v^\rho = 0$$

and consequently, on applying the $\overset{0}{D}$-operator to it, we have

$$g_{\lambda\rho}\, B_b{}^\lambda\, \overset{0}{D}_c\, v^\rho = -g_{\lambda\rho}\left(\overset{0}{D}_c\, B_b{}^\lambda\right) v^\rho = -v_\lambda\, \overset{0}{H}{}^\lambda_{cb},$$

which therefore gives

$$\overset{a}{\delta}\,{}'g_{bc} = -2v_\lambda\, \overset{0}{H}{}^\lambda_{(cb)} - 2g_{\lambda\rho}\, B^\lambda_{(b}\, H^\mu_{c)}\, A^\rho_{\mu\nu}\, v^\nu + 2A_{\lambda\mu\nu}\, v^\nu \big|_0\, B^{\lambda\mu}_{bc}, \tag{84}$$

an expression to which we shall return later when dealing with minimal subspaces.

Coming to Bompiani's method, let $X^a(u, u')$ be a vector field lying in F_m, which will therefore have components $Y^\lambda(x, x')$ in F_n connected with X^a by the relations

$$Y^\lambda = B_a{}^\lambda\, X^a. \tag{85}$$

Let P be the point of F_m of coordinates u, and Q the point of F_m of coordinates $u+du$. Then, if $Y^\lambda(Q\,|\,P)$ denotes the vector attached to Q

transported by parallelism to P, we have, for the transport of Y^λ as an ordinary vector of F_n,

$$Y^\lambda(Q|P) = Y^\lambda(Q) + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} Y^\mu dx^\nu + A^\lambda_{\mu\nu} Y^\mu \varpi^\nu. \tag{86}$$

Similarly, when transported as a vector of F_m, it becomes

$$X^a(Q|P) = X^a(Q) + \Gamma_{bc} X^b du^c + A^a_{bc} X^b \bar{\varpi}^c. \tag{87}$$

Its final component, as a vector of F_n, is therefore $B_a{}^\lambda(P) X^a(Q|P)$. On using (28) and (31) and the fact that

$$Y^\lambda(Q) = B_a{}^\lambda(Q) X^a(Q) = B_a{}^\lambda(P) X^a(Q) + (\partial_b B_a{}^\lambda) X^a(Q) du^b,$$

we have for the vector difference $V^\lambda = Y^\lambda(Q|P) - B_a{}^\lambda(P) X^a(Q|P)$, when we use (53) for the definition of $\overset{1}{H}{}^\lambda_{cb}$,

$$V^\lambda = \overset{0}{H}{}^\lambda_{cb} X^b du^c + \overset{1}{H}{}^\lambda_{cb} X^b \bar{\varpi}^c. \tag{88}$$

If Y^λ is an infinitesimal vector $\delta x^\lambda = B_a{}^\lambda \delta u^a$, then

$$V^\lambda = \overset{0}{H}{}^\lambda_{cb} \delta u^b du^c + \overset{1}{H}{}^\lambda_{cb} \delta u^b \bar{\varpi}^c. \tag{89}$$

The generalization of the form of the 4th order of Voss (**10**, *loc. cit.*) would therefore be $g_{\lambda\mu} V^\lambda V^\mu$, which would consist of a single term either (i) when $\bar{\varpi}^c = 0$, *i.e.* when the line in the direction du^c is a geodesic of F_m, or (ii) when the tensor of Eulerian Curvature associated with the $\overset{1}{D}$-operator $\overset{1}{H}{}^\lambda_{cb}$ vanishes, or, what is the same thing, if the tensor of torsion $A^\lambda_{\mu\nu}$ lies in the F_m with all its indices.

There appears a third quadratic form which plays the rôle of the second fundamental form in the works of Berwald (**2**) and Haimovici (**3**) on the theory of hypersurfaces. Its coefficients are the derivatives of the second order of the Normal Curvature H^p (which, since $m = n-1$ in this case, may be written as simply H) with respect to the u'^a. These coefficients, which we shall denote for the moment by Π_{ab}, given as

$$\Pi_{ab} = \tfrac{1}{2} \frac{\partial^2 H}{\partial u'^a \partial u'^b}, \tag{90}$$

are symmetric in the indices a and b, a property not possessed by the $\overset{0}{H}{}^p_{bc}$. By introducing a tangential Minkowski space Berwald has introduced a generalization of the Dupin Indicatrix (for a surface immersed in Finsler 3-space) as the curve of the second degree in that Minkowski

space which has the equation

$$\Pi_{ab}\,\eta^a\,\eta^b = \pm 1, \tag{91}$$

where η^a is a variable vector attached to the point u of the surface. The usual geometrical ideas, such as conjugate directions and asymptotic lines, can then be introduced in connection with the Dupin Indicatrix. The direction of the vector η^a in the line-element (u, u') is conjugate to the line element of support if

$$\Pi_{ab}\,l^a\,\eta^b = 0, \tag{92}$$

and in view of the symmetry of the coefficients Π_{ab} this relation between the two directions l^a and η^a is reciprocal. The asymptotic elements in a point of the surface have the directions of the asymptotes of the Dupin Indicatrix at the point. The direction of an asymptotic line is conjugate to itself.

A definition of conjugate directions can also be given in connection with (89). We could describe the directions d and δ as being conjugate provided that the corresponding $V^\lambda = 0$. In this case, however, it is evident that the lack of symmetry of the $\overset{0}{H}{}^{\lambda}_{cb}$ will prevent this relation of conjugacy from being reciprocal.

8. *Hypersurfaces with the element of support along the unique normal.* In this case the span is specified by the unique normal "element of support". In place of (13g) we have

$$B_\mu{}^\lambda + l^\lambda\,l_\mu = \delta_\mu{}^\lambda. \tag{93}$$

The theory of subordination in this case depends on the two equations

$$l_\lambda\,B_a{}^\lambda = 0, \quad l_\lambda\,\varpi^\lambda = 0, \tag{94}$$

so that we can write ϖ^λ in the form

$$\varpi^\lambda = B_a{}^\lambda(B_\mu{}^a\,\varpi^\mu) + l^\lambda(l_\mu\,\varpi^\mu) = B_a{}^\lambda\,\bar{\varpi}^a. \tag{95}$$

For a derivation in the direction $dx^\lambda = B_c{}^\lambda\,du^c$ of the hypersurface (95) may be written

$$B_\mu{}^\lambda\left(\frac{dl^\mu}{\partial u^c} + \overset{*}{\Gamma}{}^\mu_{\rho\sigma}\,l^\rho\,B_c{}^\sigma\right)du^c = B_a{}^\lambda\left[B_\mu{}^a\left(\frac{\partial l^\mu}{\partial u^c} + \overset{*}{\Gamma}{}^\mu_{\rho\sigma}\,l^\rho\,B_c{}^\sigma\right)\right]du^c. \tag{96}$$

We now put

$$B_\mu{}^a\left(\frac{\partial l^\mu}{\partial u^c} + \overset{*}{\Gamma}{}^\mu_{\rho\sigma}\,l^\rho\,B_\sigma{}^\sigma\right) = -\Omega^a_{.c}, \tag{97}$$

so that we can now write

$$\varpi^\lambda = -B_a^{\ \lambda}\,\Omega^a_{\cdot c}\,du^c, \tag{98}$$

and, when we use (98), the absolute derivative of any vector field X^λ defined along F_{n-1} may be written in the form

$$DX^\lambda = \left(\overset{0}{\nabla}_\nu X^\lambda\right) B_c^{\ \nu}\,du^c - \left(\overset{1}{\nabla}_\nu X^\lambda\right) B_a^{\ \nu}\,\Omega^a_{\cdot c}\,du^c = \overset{0}{D}_c X^\lambda\,du^c, \tag{99}$$

where

$$\overset{0}{D}_c X^\lambda = B_c^{\ \nu}\overset{0}{\nabla}_\nu X^\lambda - \overset{1}{\nabla}_\nu X^\lambda\, B_a^{\ \nu}\,\Omega^a_{\cdot c} \tag{100}$$

is the generalization of the D-symbolik for this case. If we apply this operator to the vector $B^\lambda_{(a)}$, where the index (a) is ordinal, then

$$\overset{0}{D}_c B^\lambda_{(a)} = \partial_c B_a^{\ \lambda} + \left(\overset{*}{\Gamma}{}^\lambda_{\mu\nu} B_c^{\ \nu} - A^\lambda_{\mu\nu} B_d^{\ \nu}\,\Omega^d_{\cdot c}\right) B_a^{\ \mu} = \partial_c B_a^{\ \lambda} + \Gamma^\lambda_{\mu c} B_a^{\ \mu}, \tag{101}$$

where, by analogy with the case of a tangential element of support, we have put

$$\Gamma^\lambda_{\mu c} = \overset{*}{\Gamma}{}^\lambda_{\mu\nu} B_c^{\ \nu} - A^\lambda_{\mu\nu} B_d^{\ \nu}\,\Omega^d_{\cdot c}. \tag{102}$$

We could then, as in the case of equation (52), define connection parameters for the F_{n-1} by

$$\gamma^a_{bc} = B_\lambda^{\ a}\overset{0}{D}_c B^\lambda_{(b)} \tag{103}$$

and the tensor of Eulerian Curvature by

$$K_{bc} = l_\lambda \overset{0}{D}_c B^\lambda_{(b)}. \tag{104}$$

It follows immediately from (94), however, that $K_{bc} = {}'g_{bd}\,\Omega^d_{\cdot c}$, so that we have (104) replaced by

$$\Omega_{bc} = l_\lambda \overset{0}{D}_c B^\lambda_{(b)} = l_\lambda\left(\partial_c B_b^{\ \lambda} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} B^{\mu\nu}_{bc}\right), \tag{105}$$

since $l_\lambda A^\lambda_{\mu\nu} = 0$.

The theory of the curve C on the hypersurface will not in this case be that of a curve regarded as the locus of its tangential elements of support, for the element of support is now normal to the hypersurface, and hence to the curve.

Let the equations of C be

$$x^\lambda = x^\lambda(s),$$

where s is taken for simplicity to be the arc-length. Then, since the element of support is normal to the curve,

$$l_\lambda \frac{dx^\lambda}{ds} = 0.$$

In view of our choice of s, we have

$$g_{\lambda\mu} \frac{dx^\lambda}{ds} \frac{dx^\mu}{ds} = 1$$

and, on differentiating covariantly,

$$g_{\lambda\mu} \frac{dx^\mu}{ds} \frac{D^2 x^\lambda}{ds^2} = 0, \tag{106}$$

where

$$\frac{D^2 x^\lambda}{ds^2} = \frac{d^2 x^\lambda}{ds^2} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + A^\lambda_{\mu\nu} \frac{Dl^\mu}{ds} \frac{dx^\nu}{ds} \tag{107}$$

is the first curvature vector of C.

On using (18) we may write this vector in the form

$$\frac{D^2 x^\lambda}{ds^2} = B_a{}^\lambda \frac{d^2 u^a}{ds^2} + \left(B^\lambda_{ab} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} B^{\mu\nu}_{ab} \right) \frac{du^a}{ds} \frac{du^b}{ds} + A^\lambda_{\mu\nu} \frac{Dl^\mu}{ds} B_a{}^\nu \frac{du^a}{ds}, \tag{108}$$

so that the *normal* curvature becomes

$$l_\lambda \frac{D^2 x^\lambda}{ds^2} = l_\lambda \left(B^\lambda_{ab} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} B^{\mu\nu}_{ab} \right) \frac{du^a}{ds} \frac{du^b}{ds} = \Omega_{ab} \frac{du^a}{ds} \frac{du^b}{ds}, \tag{109}$$

and, when we put

$$\frac{Dl^\mu}{ds} = \frac{\varpi^\mu}{ds} = -B_d{}^\mu \Omega^d_{.b} \frac{du^b}{ds},$$

the *geodesic* curvature becomes

$$B_\lambda{}^c \frac{D^2 x^\lambda}{ds^2} = \frac{d^2 u^c}{ds^2} + B_\lambda{}^c \left\{ B^\lambda_{ab} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} B^{\mu\nu}_{ab} - A^\lambda_{\mu\nu} B_d{}^\mu \Omega^d_{.b} B_a{}^\nu \right\} \frac{du^a}{ds} \frac{du^b}{ds} \tag{110}$$

$$= \frac{d^2 u^c}{ds^2} + \gamma^c_{ab} \frac{du^a}{ds} \frac{du^b}{ds}.$$

We notice that $\Omega_{ab} du^a du^b$ is the natural quadratic to take as the second fundamental form in this case. It remains to be seen whether the variation of ds^2 in passing from F_{n-1} to a parallel F_{n-1} can be expressed in terms of the coefficients of this form.

The variation of ds^2 is given by

$$\bar{ds}^2 - ds^2 = \left(\overset{a}{\delta}' g_{bc}\right) du^b du^c, \tag{111}$$

where $\overset{a}{\delta}' g_{bc}$ is given by (70). We can now proceed as we did previously in order to obtain equations corresponding to (83) and (84). The equation corresponding to (83) will be

$$\overset{a}{\delta}' g_{bc} = 2g_{\lambda\rho}\left[B^{\lambda}_{(b}\overset{0}{D}_{c)}v^{\rho} + B^{\lambda}_{(b}\Omega^{d}_{.c)}A^{\rho}_{\sigma\nu}B_{d}{}^{\nu}v^{\sigma} + A_{\lambda\mu\nu}v^{\nu}|_0 B^{\lambda\mu}_{bc}\right]. \tag{112}$$

On taking $v^{\rho} = \sigma l^{\rho}$ it is evident that the middle term inside the bracket will disappear. Also

$$g_{\lambda\rho}B_{b}{}^{\lambda}\overset{0}{D}_{c}v^{\rho} = -v_{\lambda}(l^{\lambda}\Omega_{bc}) = -\sigma\Omega_{bc}$$

and

$$v^{\nu}|_0 = \sigma\left(\frac{\partial l^{\nu}}{\partial x^{\rho}} + \overset{*}{\Gamma}{}^{\nu}_{\sigma\rho}l^{\sigma}\right)l^{\rho}$$

so that finally

$$\overset{a}{\delta}' g_{bc} = -2\sigma\Omega_{bc} + 2\sigma A_{\lambda\mu\nu}B^{\lambda\mu}_{bc}\left(\frac{\partial l^{\nu}}{\partial x^{\rho}} + \overset{*}{\Gamma}{}^{\nu}_{\sigma\rho}l^{\sigma}\right)l^{\rho}. \tag{113}$$

9. *Minimal subspaces.* Let us denote the volume element in F_m by

$$d\tau = \sqrt{'g}\, du^1 du^2 \ldots du^m, \quad 'g = \det('g_{ab}),$$

then, as in Riemannian Geometry, we can write, following Schouten and Struik (6, Vol. II, p. 165),

$$\frac{\overset{a}{\delta} d\tau}{d\tau} = \tfrac{1}{2}{}'g^{bc}\overset{a}{\delta}' g_{bc}. \tag{114}$$

On making use of (84) this gives

$$\frac{\overset{a}{\delta} d\tau}{d\tau} = -v_{\lambda}\,'g^{bc}\overset{0}{H}{}^{\lambda}_{cb} + A_{\nu}v^{\nu}|_0 \tag{115}$$

and for the normal element of support, making use of (113), we get

$$\frac{\overset{a}{\delta} d\tau}{d\tau} = -\sigma\,'g^{bc}\Omega_{bc} + \sigma A_{\nu}\left(\frac{\partial l^{\nu}}{\partial x^{\rho}} + \overset{*}{\Gamma}{}^{\nu}_{\sigma\rho}l^{\sigma}\right)l^{\rho}. \tag{116}$$

Hence in both cases the minimal subspaces will be characterized only by the vanishing of the "mean curvature" provided the vector of torsion A_{ν} vanishes. The spaces of Finsler for which this vector vanishes form

an important class of spaces between the Riemannian spaces and the general Finsler spaces. In particular they can be regarded as constituting a particular class of metric spaces based on the notion of area (see Cartan, **1**, pp. 30–32).

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SOME PROPERTIES OF CESÀRO-LEBESGUE INTEGRALS

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1. *Introduction.*

The Cesàro-Lebesgue integral may be defined inductively as follows*. *If* $\lambda \geqslant 0$ *and* $\mu = \max(\lambda - 1, 0)$, *we say that* $g(t)$ *is integrable* $C_\lambda L$ *in* $(0, a)$†, *and we denote its integral by*

$$\int_{\to 0\ (C,\lambda)}^{a} g(t)\,dt, \tag{1.1}$$

if (i) $g(t)$ *is integrable* L *in every interval* (ϵ, a), *where* $0 < \epsilon < a$ (a *fixed*), *and* (ii)

$$\left.\begin{aligned} &\lim_{\epsilon \to +0} \int_{\epsilon}^{a} g(t)\,dt \qquad (\lambda = 0) \\ \text{or} \quad &\lim_{\epsilon \to +0} \frac{\lambda}{\epsilon^{\lambda}} \int_{\to 0\ (C,\mu)}^{\epsilon} (\epsilon - t)^{\lambda-1}\,dt \int_{t}^{a} g(u)\,du \qquad (\lambda > 0) \end{aligned}\right\} \tag{1.2}$$

exists. The integral (1.1) *is then defined as the limit* (1.2).

If $g(t) \in C_\lambda L$ in $(0, a)$ for some unspecified λ we say that $g(t) \in (CL)$ in $(0, a)$, and we denote its integral, which by Theorem 10 is independent of λ, by

$$\int_{\to 0\ (C)}^{a} g(t)\,dt. \tag{1.3}$$

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* See 6. In the definition given previously it was *assumed* that $\int_t^a g(u)\,du$ is integrable $C_\mu L$. It will be shown in Theorem 1 that this is in fact *necessarily* true, so that the two definitions are equivalent. An alternative definition, shown to be equivalent by Theorem 7, was given for λ an integer in 4.

† Writing $g(t) \in C_\lambda L$ in $(0, a)$.

The importance of the (CL) integral has been demonstrated by its connection with the Cesàro summability problem for derived and successively derived Fourier series at a single point*. The properties collected in this paper are a development of those required in this and the conjugate problem.

The relation between the (CL) integral and the Cesàro-Perron integral of Burkill† is expressed by the following two theorems.

THEOREM A. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$, *then* $g(t) \in C_\lambda P$ *in* $(0, a)$, *and the integrals are equal.*

THEOREM B. *If* (i) $g(t) \in L$ *in every interval* (ϵ, a), *where* $0 < \epsilon < a$, (ii) $g(t) \in C_\lambda P$ *in* $(0, a)$, *then* $g(t) \in C_\lambda L$ *in* $(0, a)$.

If we use the fact‡ that the definition of integrability $C_\lambda L$ given here is equivalent to that given in 6, the proofs of Theorems A and B may be found elsewhere§.

In what follows only properties of Lebesgue integrals are assumed, and no use is made of Theorems A and B. The ideas involved belong to the theory of Cesàro limits rather than to the theory of integration, and most of the results have analogues for series.

2. We write, for $t > 0$,

$$\left.\begin{aligned} G_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \int_{\to 0\,(C)}^{t} (t-u)^{\alpha-1} g(u)\, du \quad (\alpha > 0), \\ G_0(t) &= g(t), \\ g_\alpha(t) &= \Gamma(\alpha+1)\, t^{-\alpha} G_\alpha(t) \qquad (\alpha \geqslant 0), \end{aligned}\right\} \tag{2.1}$$

and define $\Xi_\alpha(t)$, $\xi_\alpha(t)$, etc., similarly.

If $g(t) \in (CL)$, $p > -1$, $\alpha \geqslant 0$, and

$$G_\alpha(t) \sim l t^{\alpha+p}\, \Gamma(p+1)/\Gamma(\alpha+p+1) \tag{2.2}$$

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* **4, 6.** A similar definition of integration, under the assumption that $\int_t^a g(u)\, du \in L$, was used by Hardy and Littlewood, **14**, in their solution of the Cesàro summability problem for allied series.

† **7, 8, 9.** The $C_\lambda P$ integral is defined by means of upper and lower functions. An equivalent $C_\lambda D$ (Cesàro-Denjoy) integral, which directly includes the $C_\lambda L$ integral, has been defined by Sargent, **18**.

‡ Theorem 1.

§ 5 and 6.

as $t \to +0$, we write $g(t) \sim lt^p\ (C, a)$ as $t \to +0$, and the usual related notation, such as $g(t) = o(t^p)\ (C, a)$ is defined in the obvious way. The uniqueness of the definition of $G_\alpha(t)$ will follow from Theorem 10. Until this has been proved however, we must adopt the convention that, when it is given that $(t-u)^{\alpha-1} g(u) \in C_\lambda L$ in $(0, t)$, then $G_\alpha(t)$ is defined as a $C_\lambda L$ integral.

Theorems 1–10 are proved by a single inductive argument. We suppose throughout that $\lambda \geqslant 0$ and $\mu = \max(\lambda-1, 0)$.

THEOREM 1. *Necessary and sufficient conditions that* $g(t) \in C_\lambda L$ *in* $(0, a)$ *are that* (i) $g(t) \in L$ *in every interval* (ϵ, a), $0 < \epsilon < a$, (ii) $\int_0^a g(u)\,du \in C_\mu L$ *in* $(0, a)$ *and tends to a limit* (C, λ) *as* $t \to +0$.

THEOREM 2*. *If* $g(t) \in C_\mu L$ *in* $(0, a)$ *and is* $o(t^p)$ (C, λ) *as* $t \to +0$, *where* $p > -1$, (ii) $h(t)$ *is defined in* $(0, a)$ *and* $h^{(\rho)}(t) = O(t^{-r-\rho})$ *in* $(0, a)$ *for* $\rho = 0, 1, \ldots, m$†, *where* $r < p+1$ *and* m *is the least integer such that* $m \geqslant \lambda$, *then* $g(t)\,h(t) \in C_\mu L$ *in* $(0, a)$ *and is* $o(t^{p-r})$ (C, λ) *as* $t \to +0$.

THEOREM 3. *If* (i) $g(t) \in C_\lambda L$ *in* $(0, a)$, (ii) $G_1(t) = o(t^p)$ (C, λ) *as* $t \to +0$‡, *where* $p \geqslant 0$, (iii) $h(t)$ *is defined in* $(0, a)$ *and such that*

$$\int_{\to 0}^{a} t^{m+p} \left| dh^{(m)}(t) \right| < \infty\text{§}, \tag{2.3}$$

where m *is the least integer such that* $m \geqslant \lambda$, *then* $g(t)\,h(t) \in C_\lambda L$ *and*

$$\int_{\to 0\ (C,\,\lambda)}^{a} g(t)\,h(t)\,dt = G_1(a)\,h(a) - \int_{\to 0\ (C,\,\mu)}^{a} G_1(t)\,h'(t)\,dt \tag{2.4}$$

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* Proved for λ an integer in **4**. See also Theorem 23 below.

† To ensure (ii) it is sufficient to assume that $h(t) = O(t^{-r})$ and $h^{(m)}(t) = O(t^{-r-m})$. This is trivial unless for some ρ, say $\rho = \sigma$, we have $r+\rho = 1$, in which case the result is completed by using the identities

$$\int_0^a u h^{(\rho+1)}(u)\,du = C + h^{(\rho-1)}(t) - t h^{(\rho)}(t)$$

for $\rho = 1, 2, \ldots, \sigma-1$. Cf. Hardy and Littlewood, **12**.

‡ If $p = 0$, (ii) is included in (i).

§ In particular (iii) is satisfied if $h^{(\rho)}(t) = O(1)$ for $\rho = 0, 1, \ldots, m+1$. The case when $p = 0$ and $h^{(m)}(t)$ is of bounded variation in $(0, a)$ is included, in virtue of Theorems A and B, in a result of Burkill, **9**. See also Theorem 24 below.

if $\lambda > 0$, *while*

$$\int_{\to 0}^{a} g(t)\,h(t)\,dt = G_1(a)\,h(a) - \int_{\to 0}^{a} G_1(t)\,dh(t) \qquad (2.5)$$

if $\lambda = 0$.

THEOREM 4. *If* $g(u) \in C_\lambda L$ *in* $(0, a)$, *then* $(t-u)^{\alpha-1} g(u) \in C_\lambda L$ *in* $(0, t)$, *for all* t *in* $(0, a)$, *if* $\alpha \geqslant 1$, *and for almost all* t *in* $(0, a)$, *if* $0 < \alpha < 1$.

THEOREM 5. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$, *and* $\alpha > 0$, *then*

$$G_{\alpha+1}(t) = \frac{1}{\Gamma(\alpha)} \int_{\to 0\ (C,\,\mu)}^{t} (t-u)^{\alpha-1} G_1(u)\,du \qquad (2.6)$$

for all t *in* $(0, a)$.

THEOREM 6. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$, *and* $\alpha \geqslant 0$, *then* $G_{\alpha+1}(t)$ *is absolutely continuous in every interval* (ϵ, a), $0 < \epsilon < a$, *and*

$$\frac{d}{dt} G_{\alpha+1}(t) = G_\alpha(t) \qquad (2.7)$$

for all t *in* $(0, a)$ *if* $\alpha \geqslant 1$, *and for almost all* t *in* $(0, a)$ *if* $0 \leqslant \alpha < 1$.

THEOREM 7. *Necessary and sufficient conditions that* $g(t) \in C_\lambda L$ *in* $(0, a)$ *are that* (i) $g(t) \in L$ *in every interval* (ϵ, a), $0 < \epsilon < a$, (ii) *there should exist a function* $G(t)$ *which is an* $(n+1)$*-th integral of* $g(t)$ *for* $0 < t \leqslant a$*, where* $0 \leqslant n \leqslant \lambda$, *such that* $G(t) \in C_\gamma L$ *in* $(0, a)$, *where* $\gamma = \max(\lambda - n - 1, 0)$, *and* $G(t) = o(t^n)$ $(C, \lambda - n)$ *as* $t \to +0$.

THEOREM 8†. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$, *and* $\alpha \geqslant 0$, $\beta > 0$, *then*

$$G_\alpha(t) \in C_\rho L$$

in $(0, a)$, *where* $\rho = \max(\lambda - \alpha, 0)$, *and*

$$G_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_{\to 0\ (C,\,\rho)}^{t} (t-u)^{\beta-1} G_\alpha(u)\,du \qquad (2.8)$$

for all t *in* $(0, a)$, *if* $\alpha + \beta \geqslant 1$, *and whenever the left-hand side exists, if* $\alpha + \beta < 1$‡.

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* *I.e.* $G(t) = (-1)^{n+1} \int_t^a dt_1 \int_{t_1}^a dt_2 \ldots \int_{t_n}^a g(t_{n+1})\,dt_{n+1} + P(t)$, where $P(t)$ is a polynomial of degree not greater than n.

† Proved for $\alpha \geqslant \lambda + 1$, in virtue of Theorems A and B, by Burkill, 9.

‡ And so for almost all t in $(0, a)$.

THEOREM 9. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$ *and is* $o(t^p)$ (C, a) *as* $t \to +0$, *where* $p > -1$, $a \geqslant 0$, *then* $g(t) = o(t^p)$ (C, a'), *for* $a' > a$.

THEOREM 10*. *If* $g(t) \in C_\lambda L$ *in* $(0, a)$, *then* $g(t) \in C_{\lambda'} L$ *in* $(0, a)$, *for* $\lambda' > \lambda$, *and the integrals are equal.*

We shall require the following lemmas.

LEMMA 1†. *If* $g(t) \in C_0 L$ *in* $(0, x)$, *and* $0 < \alpha \leqslant 1$, $0 < x < y$, *then*

$$\left| \frac{1}{\Gamma(\alpha)} \int_{\to 0}^{x} (y-t)^{\alpha-1} g(t)\, dt \right| \leqslant \max_{0 \leqslant u \leqslant x} |G_\alpha(u)|, \tag{2.9}$$

where max *denotes the essential upper bound.*

Writing

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{\to 0}^{x} (y-t)^{\alpha-1} g(t)\, dt &= \frac{1}{\Gamma(\alpha)} \int_{\to 0}^{x} g(t)\, dt \int_{t}^{x} h(u)(u-t)^{\alpha-1}\, du \\ &= \frac{1}{\Gamma(\alpha)} \int_{\to 0}^{x} h(u)\, du \int_{\to 0}^{u} (u-t)^{\alpha-1} g(t)\, dt \\ &= \int_{\to 0}^{x} h(u)\, G_\alpha(u)\, du, \end{aligned}$$

where‡ $$h(u) = \frac{\sin \pi\alpha}{\pi} \frac{d}{du} \int_{x}^{y} (y-t)^{\alpha-1} (t-u)^{-\alpha}\, dt,$$

we obtain the result, since $h(u) > 0$ in $(0, x)$, and $\int_0^x h(u)\, du < 1$. The inversion of the repeated integral is justified since

$$\int_{\epsilon}^{x} h(u)\, du \int_{\to 0}^{\epsilon} (u-t)^{\alpha-1} g(t)\, dt = \int_{\epsilon}^{x} h(u)(u-\epsilon)^{\alpha-1} o(1)\, du = (y-\epsilon)^{\alpha-1} o(1) = o(1)$$

as $\epsilon \to +0$.

LEMMA 2. *If* $k(t)$ *is defined in* $0 < t \leqslant a$, *and*

$$\int_{\to 0}^{a} t^q |dk(t)| < \infty \tag{2.10}$$

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* Proved, in virtue of Theorems A and B, by Burkill, **9**. Cf. **6**.

† Proved essentially by M. Riesz, **17**, and by Verblunsky, **20**, for $g(t) \in L$, and attributed by Burkill, **9**, to Dr. Verblunsky in the case when $g(t)$ is integrable in the general Denjoy sense.

‡ Cf. **17**, **20**.

where $q \geqslant 0$, *then* (i) $k(t) = O(t^{-q})$ *in* $(0, a)$, *and*

(ii) *if* $q > 0$,

$$\int_0^a t^{q-1} |k(t)|\, dt < \infty. \tag{2.11}$$

We have, for $0 < t \leqslant a$,

$$|k(a)-k(t)| \leqslant \int_t^a |dk(u)| \leqslant \frac{1}{t^q} \int_t^a u^q |dk(u)| = O(t^{-q}),$$

from which (i) follows while, if $q > 0$, and $0 < \epsilon < a$,

$$\begin{aligned} \int_\epsilon^a t^{q-1} |k(a)-k(t)|\, dt &\leqslant \int_\epsilon^a t^{q-1}\, dt \int_t^a |dk(u)| \\ &= \int_\epsilon^a |dk(u)| \int_\epsilon^u t^{q-1}\, dt \\ &\leqslant \frac{1}{q} \int_\epsilon^a u^q |dk(u)|, \end{aligned}$$

and hence

$$\int_0^a t^{q-1} |k(a)-k(t)|\, dt \leqslant \frac{1}{q} \int_{\to 0}^a u^q |dk(u)|,$$

from which (ii) follows.

Proof of Theorems 1–10.

The case $\lambda = 0$. When $\lambda = 0$ the theorems are all trivial or well known*.

The case $\lambda > 0$. We shall now suppose that λ is given, and assume the truth of Theorems 1–10 when λ is replaced by $\mu = \max(\lambda-1, 0)$. We shall deduce the truth of Theorems 1–10, and thus they will have been proved by induction.

Theorem 1. Suppose that $0 < \eta < \epsilon \leqslant a$. Since the successive derivatives of $(\epsilon-t)^{\lambda-1}$ and $(\epsilon-t)^{1-\lambda}$ are all bounded in $(0, \eta)$, it follows

* Theorem 3 requires the first part of Lemma 1. For Theorems 4 and 8 see 1. Theorems 5, 6, 9 and 10 are easy consequences of Theorem 8, while Theorems 1, 2 and 7 are trivial. The case $\lambda = 0$ of Theorem 10 is proved by induction. Alternatively we may suppose throughout the proof of the theorem that $\lambda' \leqslant \lambda + 1$.

from Theorem 3, with λ replaced by μ, and $p=0$ that if either of the functions

$$\int_t^a g(u)\,du \quad \text{and} \quad (\epsilon-t)^{\lambda-1}\int^a g(u)\,du$$

is integrable $C_\mu L$ in $(0, \eta)$, then so is the other, while both functions are integrable L in (η, ϵ). The conclusion follows.

Theorem 2. *The case* $0<\lambda\leqslant 1$. Since $g(t)\,\epsilon\, C_0 L$ and is $o(t^p)$ (C, λ) as $t\to+0$, we have, by Theorem 9 with $\lambda=0$, $g(t)=o(t^p)$ $(C, 1)$, *i.e.* $G_1(t)=o(t^{p+1})$. Hence, if $0<\epsilon<t\leqslant a$,

$$\begin{aligned}\int_\epsilon^t g(u)\,h(u)\,du &= \Big[G_1(u)\,h(u)\Big]_\epsilon^t - \int_\epsilon^t G_1(u)\,h'(u)\,du\\ &= o(t^{p+1})\,O(t^{-r})+o(\epsilon^{p+1})\,O(\epsilon^{-r})+\int_\epsilon^t o(u^{p+1})\,O(u^{-r-1})\,du\\ &= o(t^{p+1-r})+o(\epsilon^{p+1-r}).\end{aligned}$$

Since $p+1-r>0$ it follows that $g(t)\,h(t)\,\epsilon\, C_0 L$ in $(0, a)$, and also that $g(t)\,h(t)=o(t^{p-r})$ $(C, 1)$ as $t\to+0$.

Suppose now that $p+\lambda>0$ *and* $r<p+\lambda$. Since $h(u)$ is continuous in (ϵ, t) and $(t-u)^{\lambda-1}g(u)\,\epsilon\, L$ in (ϵ, t) it follows that $(t-u)^{\lambda-1}g(u)\,h(u)\,\epsilon\, L$ in (ϵ, t). Also, on integration by parts, since $g(t)=o(t^p)$ (C, λ)

$$\begin{aligned}\int_\epsilon^t (t-u)^{\lambda-1}g(u)\,h(u)\,du &= G_\lambda(t)\,h(t)-h(\epsilon)\int_{\to 0}^{\epsilon}(t-v)^{\lambda-1}g(v)\,dv\\ &\quad -\int_\epsilon^t h'(u)\,du\int_{\to 0}^{u}(t-v)^{\lambda-1}g(v)\,dv\\ &= o(t^{\lambda+p})\,O(t^{-r})+O(\epsilon^{-r})\,o(\epsilon^{\lambda+p})+\int_\epsilon^t O(u^{-r-1})\,o(u^{\lambda+p})\,du\\ &= o(t^{\lambda+p-r})+o(\epsilon^{\lambda+p-r}),\end{aligned}$$

by Lemma 1, and hence it follows, since $\lambda+p-r>0$, that

$$(t-u)^{\lambda-1}g(u)\,h(u)\,\epsilon\, C_0 L$$

in $(0, t)$, and

$$\int_{\to 0}^{t}(t-u)^{\lambda-1}g(u)\,h(u)\,du = o(t^{\lambda+p-r}),$$

i.e. $g(t)\,h(t)=o(t^{p-r})$ (C, λ) as to $t\to+0$.

Next suppose only that $p+1>0$ *and* $r<p+1$. Since $g(t)\in C_0 L$ it follows, on integration by parts, that $tg(t)\in C_0 L$, and from the identity

$$t\int_{\to 0}^{t}(t-u)^{\lambda-1}g(u)\,du-\int_{\to 0}^{t}(t-u)^{\lambda}g(u)\,du=\int_{\to 0}^{t}(t-u)^{\lambda-1}ug(u)\,du,$$

by Theorem 9, with $\lambda=0$, since $g(t)=o(t^p)$ (C,λ), that $tg(t)=o(t^{p+1})$ (C,λ) as $t\to+0$. Since $p+1+\lambda>0$ and $r<p+1+\lambda$ it follows from the case already proved, with p replaced by $p+1$, that $tg(t)h(t)\in C_0 L$ and is $o(t^{p+1-r})$ (C,λ) as $t\to+0$.

Now we have already shown that $g(t)h(t)\in C_0 L$ and is $o(t^{p-r})$ $(C,1)$, and *a fortiori* $g(t)h(t)=o(t^{p-r})$ $(C,\lambda+1)$, by Theorem 9 with $\lambda=0$. It therefore follows from the identity

$$\int_{\to 0}^{t}(t-u)^{\lambda-1}ug(u)h(u)\,du+\int_{\to 0}^{t}(t-u)^{\lambda}g(u)h(u)\,du = t\int_{\to 0}^{t}(t-u)^{\lambda-1}g(u)h(u)\,du$$

that $$\int_{\to 0}^{t}(t-u)^{\lambda-1}g(u)h(u)\,du=o(t^{\lambda+p-r}),$$

i.e. $g(t)h(t)=o(t^{p-r})$ (C,λ) as $t\to+0$.

This completes the proof in the case $0<\lambda\leqslant 1$*.

The case $\lambda>1$†. If $0<\epsilon<t\leqslant a$, we have

$$\int_{\epsilon}^{t}g(u)h(u)\,du=\Big[G_1(u)h(u)\Big]_{\epsilon}^{t}-\int_{\epsilon}^{t}G_1(u)h'(u)\,du. \qquad (2.12)$$

Since $g(t)\in C_{\lambda-1}L$ and is $o(t^p)$ (C,λ), it follows from Theorems 1 and 5, with λ replaced by $\lambda-1$, that $G_1(t)\in C_\nu L$ and is $o(t^{p+1})$ $(C,\lambda-1)$, where $\nu=\max(\lambda-2,0)$, *i.e.* $G_1(t)$ satisfies the hypotheses of the present theorem, with λ replaced by $\lambda-1$ and p by $p+1$. Also $h(t)$ and $h'(t)$ satisfy the hypotheses of the present theorem, with λ replaced by $\lambda-1$, and r by r and $r+1$ respectively. Hence it follows that $G_1(t)h(t)\in C_\nu L$ and is $o(t^{p+1-r})$ $(C,\lambda-1)$, while $G_1(t)h'(t)\in C_\nu L$ and is $o(t^{p-r})$ $(C,\lambda-1)$. Thus, by Theorem 1, with λ replaced by ν, $\int_{\to 0\ (C,\nu)}^{t}G_1(u)h'(u)\,du$ is integrable

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* In the proof of Theorem 2 the induction does not come in until the case $\lambda>1$. An alternative arrangement would be to place Theorem 2 after Theorem 10, with μ replaced by λ, and λ by $\alpha\leqslant\lambda+1$. Then all that we have proved so far would be the case $\lambda=0$.

† In this case $\mu=\lambda-1$.

$C_\sigma L$ and is $o(t^{p+1-r})$ (C, ν), where $\sigma = \max(\lambda-3, 0)$, and hence, by Theorems 9 and 10, with λ replaced by σ, is integrable $C_\nu L$ and is $o(t^{p+1-r})$ $(C, \lambda-1)$.

It follows from (2.12), by the sufficiency part of Theorem 1, with λ replaced by $\lambda-1$, since $p+1-r > 0$, that $g(t)h(t) \in C_{\lambda-1} L$, and also

$$\int_{\to 0\ (C,\ \lambda-1)} g(u)h(u)\,du = G_1(t)h(t) - \int_{\to 0\ (C,\ \nu)} G_1(u)h'(u)\,du$$
$$= o(t^{p+1-r}) \quad (C, \lambda-1),$$

i.e. by Theorem 5, with λ replaced by $\lambda-1$, $g(t)h(t) = o(t^{p-r})$ (C, λ).

Theorem 3. If $0 < \epsilon < a$, we have

$$\int_\epsilon^a g(t)h(t)\,dt = \Big[G_1(t)h(t)\Big]_\epsilon^a - \int_\epsilon^a G_1(t)h'(t)\,dt. \qquad (2.13)$$

Now, by Theorem 1, $G_1(t) \in C_\mu L$ and by hypothesis $G_1(t) = o(t^p)$ (C, λ), while by Lemma 2 $h(t)$ satisfies the hypotheses of Theorem 2, with $r = p$. It follows from Theorem 2 that $G_1(t)h(t) \in C_\mu L$ and is $o(1)$ (C, λ).

On the other hand, by Theorems 5 and 9, with $g(t)$ replaced by $G_1(t)$, and λ by μ, $G_1(t)$ satisfies the hypotheses of the present theorem with λ replaced by μ and p by $p+1$, while, by Lemma 2, $h'(t)$ satisfies the corresponding hypotheses of $h(t)$, and hence $G_1(t)h'(t) \in C_\mu L$. It follows from Theorem 10, with λ replaced by μ, that $G_1(t)h'(t) \in C_\lambda L$, and hence, by the necessity part of Theorem 1, that $\int_l^a G_1(u)h'(u)\,du \in C_\mu L$ and tends to a limit (C, λ) as $t \to +0$. It then follows from (2.13), by the sufficiency part of Theorem 1, that $g(t)h(t) \in C_\lambda L$ and (2.4) holds.

Theorem 4. Suppose $0 < \eta < t \leqslant a$ (η fixed). Since $g(u) \in C_\lambda L$ and the successive derivatives of $(t-u)^{\alpha-1}$ with respect to u are all bounded in $(0, \eta)$, it follows from Theorem 3 that $(t-u)^{\alpha-1}g(u) \in C_\lambda L$ in $(0, \eta)$. Also, since $g(u) \in L$ in (η, a), $(t-u)^{\alpha-1}g(u) \in L$ in (η, t), for all t in (η, a) if $\alpha \geqslant 1$, while if $0 < \alpha < 1$, $(t-u)^\alpha g(u) \in L$ in (η, t) and

$$\int_\eta^t (t-u)^\alpha g(u)\,du = \alpha \int_\eta^t g(u)\,du \int_u^t (v-u)^{\alpha-1}\,dv$$
$$= \alpha \int_\eta^t dv \int_\eta^v (v-u)^{\alpha-1} g(u)\,du,$$

by Fubini's Theorem, and hence $(v-u)^{\alpha-1}g(u) \in L$ in (η, v), for almost all v in (η, a)*. The result follows, since η may be as small as we please.

* Cf. Tonelli, **19**, and also **1**.

Theorem 5. As in the proof of Theorem 4, if $0<\eta<t\leqslant a$,

$$(t-u)^{\alpha}g(u)\,\epsilon\, C_{\lambda}\,L$$

in $(0,\eta)$, by Theorem 3, and

$$\frac{1}{\Gamma(\alpha+1)}\int_{\to 0\,(C,\lambda)}^{\eta}(t-u)^{\alpha}g(u)\,du$$

$$=\frac{1}{\Gamma(\alpha+1)}(t-\eta)^{\alpha}G_1(\eta)+\frac{1}{\Gamma(\alpha)}\int_{\to 0\,(C,\mu)}^{\eta}(t-u)^{\alpha-1}G_1(u)\,du.$$

Also, since $g(u)\,\epsilon\, L$ in (η,t), and $\alpha>0$,

$$\frac{1}{\Gamma(\alpha+1)}\int_{\eta}^{t}(t-u)^{\alpha}g(u)\,du$$

$$=-\frac{1}{\Gamma(\alpha+1)}(t-\eta)^{\alpha}G_1(\eta)+\frac{1}{\Gamma(\alpha)}\int_{\eta}^{t}(t-u)^{\alpha-1}G_1(u)\,du.$$

The result follows on addition.

Theorem 6. If $\alpha=0$ the result is immediate, since $g(t)\,\epsilon\, L$ in (ϵ,a), $0<\epsilon<a$. If $\alpha>0$, suppose that $0<\eta<\epsilon<t<a$ (η, ϵ fixed), and write, by Theorem 4,

$$G_{\alpha+1}(t)=\frac{1}{\Gamma(\alpha+1)}\left\{\int_{\to 0\,(C,\lambda)}^{\eta}+\int_{\eta}^{t}\right\}(t-u)^{\alpha}g(u)\,du$$

$$=I_1+I_2.$$

Then, since $g(u)\,\epsilon\, L$ in (η,a), as in the proof of Theorem 4,

$$I_2=\frac{1}{\Gamma(\alpha)}\int_{\eta}^{t}dv\int_{\eta}^{v}(v-u)^{\alpha-1}g(u)\,du,$$

which is absolutely continuous in (η,a), and hence

$$\frac{d}{dt}I_2=\frac{1}{\Gamma(\alpha)}\int_{\eta}^{t}(t-u)^{\alpha-1}g(u)\,du,$$

for all t in (η,a) if $\alpha\geqslant 1$, since the last integral is continuous, and for almost all t in (η,a) if $0<\alpha<1$.

Next applying Theorems 3 and 5 m times, where m is the least integer such that $m\geqslant\lambda$, with λ replaced successively by λ, $\lambda-1$, ..., $\lambda-m+1$ and $g(t)$ by $g(t)$, $G_1(t)$, ..., $G_{m-1}(t)$ respectively, and then integrating the

resulting $C_0 L$ integral once by parts, we find that*

$$I_1 = \sum_{\rho=0}^{m} \frac{1}{\Gamma(\alpha+1-\rho)} (t-\eta)^{\alpha-\rho} G_{\rho+1}(\eta) + \frac{1}{\Gamma(\alpha-m)} \int_0^\eta (t-u)^{\alpha-m-1} G_{m+1}(u),$$

and hence

$$\frac{d}{dt} I_1 = \sum_{\rho=0}^{m} \frac{1}{\Gamma(\alpha-\rho)} (t-\eta)^{\alpha-1-\rho} G_{\rho+1}(\eta) + \frac{1}{\Gamma(\alpha-m-1)} \int_0^\eta (t-u)^{\alpha-m-2} G_{m+1}(u)\, du,$$

which is continuous in (ϵ, a). Hence I_1 is absolutely continuous in (ϵ, a) and, by an integration by parts followed by m more applications of Theorems 3 and 5, we obtain

$$\frac{d}{dt} I_1 = \frac{1}{\Gamma(\alpha)} \int_{\to 0\ (C,\lambda)}^{\eta} (t-u)^{\alpha-1} g(u)\, du$$

for all t in (ϵ, a). The result follows on addition, since η and ϵ may be as small as we please.

Theorem 7. *Necessity.* If $g(t) \epsilon C_\lambda L$ in $(0, a)$, we find, after $n+1$ applications of Theorems 1 and 5, that $G_{n+1}(t) \epsilon C_\gamma L$ in $(0, a)$, where $\gamma = \max(\lambda-n-1, 0)$, and

$$G_{\lambda+1}(t) = \frac{1}{\Gamma(\lambda-n)} \int_{\to 0\ (C,\gamma)}^{t} (t-u)^{\lambda-n-1} G_{n+1}(u)\, du = o(t^\lambda),$$

since $g(t) \epsilon C_\lambda L$, *i.e.* $G_{n+1}(t) = o(t^n)$ $(C, \lambda-n)$. Since $G_{n+1}(t)$ is an $(n+1)$-th Lebesgue integral of $g(t)$ in $0 < t \leqslant a$ we may thus take $G(t) = G_{n+1}(t)$.

Sufficiency. Suppose that (i) holds and that $G(t)$ satisfies (ii).

If $n = 0$, the result follows from Theorem 1.

If $n \geqslant 1$, $G'(t)$ satisfies the conditions of $G(t)$, with $n-1$ in place of n. For $G'(t)$ is an n-th integral of $g(t)$ for $0 < t \leqslant a$, and $\int_t^a G'(u)\, du \epsilon C_\gamma L$ in $(0, a)$ and tends to $G(a)$ $(C, \lambda-n)$ as $t \to +0$. Hence $G'(t) \epsilon C_{\lambda-n} L$ in $(0, a)$, by Theorem 1, and

$$\int_{\to 0\ (C,\lambda-n)}^{t} G'(u)\, du = G(t) = o(t^n) \quad (C, \lambda-n),$$

i.e. by Theorem 5 $G'(t) = o(t^{n-1})$ $(C, \lambda-n+1)$.

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* $1/\Gamma(\sigma)$ is to be interpreted as zero if $\sigma = 0, -1, \ldots$.

By repetition of the argument we find that $G^{(n)}(t) \in C_\mu L$ and is $o(1)$ (C, λ) as $t \to +0$. Since $G^{(n)}(t)$ is an integral of $g(t)$ for $0 < t \leqslant a$, it follows from the case $n = 0$ that $g(t) \in C_\lambda L$ in $(0, a)$.

Theorem 8. *The case* $\alpha \geqslant \lambda + 1$*. Since $g(t) \in C_\lambda L$ in $(0, a)$ it follows from Theorem 7 that $G_m(t) \in C_0 L$ in $(0, a)$, where m is the least integer such that $m \geqslant \lambda$. Then, by repeated applications of Theorem 5, if $0 < t \leqslant a$,*

$$\begin{aligned} G_{\alpha+\beta}(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_{\to 0\ (C,\lambda)}^{t} (t-u)^{\alpha+\beta-1} g(u)\, du \\ &= \frac{1}{\Gamma(\alpha+\beta-m)} \int_{\to 0}^{t} (t-u)^{\alpha+\beta-m-1} G_m(u)\, du \\ &= \frac{1}{\Gamma(\beta)\,\Gamma(\alpha-m)} \int_{\to 0}^{t} (t-u)^{\beta-1} du \int_{\to 0}^{u} (u-v)^{\alpha-m-1} G_m(v)\, dv \\ &= \frac{1}{\Gamma(\beta)} \int_{\to 0}^{t} (t-u)^{\beta-1} G_\alpha(u)\, du, \end{aligned}$$

by the case $\lambda = 0$ of the present theorem and further applications of Theorem 5. The integrability $C_0 L$ of $G_\alpha(u)$ appears incidentally, *e.g.* from the case $\beta = 1$.

The case $\lambda \leqslant \alpha < \lambda + 1$. If $\alpha > \lambda$, it follows from the case $\alpha = \lambda + 1$ that

$$\begin{aligned} G_{\alpha+1}(t) &= \int_{\to 0}^{t} (t-u)^{\alpha-\lambda-1} G_{\lambda+1}(u)\, du \\ &= \int_{0}^{t} (t-u)^{\alpha-\lambda-1} o(u^\lambda)\, du \\ &= o(t^\alpha) \end{aligned}$$

as $t \to +0$, since $g(t) \in C_\lambda L$, and the conclusion is still true if $\alpha = \lambda$. Thus in particular $G_{\alpha+1}(t) = o(1)$ and, since $G_{\alpha+1}(t)$ is a Lebesgue integral of $G_\alpha(t)$ in every interval (ϵ, a), $0 < \epsilon < a$, by Theorem 6, it follows that $G_\alpha(t) \in C_0 L$ in $(0, a)$, and

$$G_{\alpha+1}(t) = \int_{\to 0}^{t} G_\alpha(u)\, du.$$

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* This case is included in the corresponding result of Burkill, **9**, for Cesàro-Perron integrals. The argument is formally the same, but we give the details for completeness.

We now have, using Theorems 5 and 6, and the present theorem in the case $\alpha \geqslant \lambda+1$,

$$G_{\alpha+\beta}(t) \equiv \frac{d}{dt} G_{\alpha+\beta+1}(t)$$

$$= \frac{d}{dt} \frac{1}{\Gamma(\beta)} \int_{\to 0}^{t} (t-u)^{\beta-1} G_{\alpha+1}(u)\, du$$

$$= \frac{d}{dt} \frac{1}{\Gamma(\beta+1)} \int_{\to 0}^{t} (t-u)^{\beta} G_{\alpha}(u)\, du$$

$$\equiv \frac{1}{\Gamma(\beta)} \int_{\to 0}^{t} (t-u)^{\beta-1} G_{\alpha}(u)\, du.$$

Further, there is equality between the extreme members for all t in $(0, a)$ such that $G_{\alpha+\beta}(t)$ exists. For, if $0 < \eta < a$, we may write $g(u) = j(u) + k(u)$, where $j(u) = g(u)$ in $(0, \eta)$ and $k(u) = g(u)$ in (η, a). Then the result with $g(u)$ replaced by $j(u)$ holds for all t in (η, a), since $j(u) = 0$ in (η, a)*, while it holds with $g(u)$ replaced by $k(u)$ for all relevant t in (η, a), since $k(u) \in L$ in $(0, t)$†. The result follows, since η may be as small as we please.

The case $0 < \alpha < \lambda$. Let n be the least integer such that $\alpha+n+1 \geqslant \lambda$. Then, by the case $\alpha \geqslant \lambda$, $G_{\alpha+n+1}(t) \in C_0 L$ in $(0, a)$, and

$$G_{\lambda+1}(t) = \frac{1}{\Gamma(\lambda-\alpha-n)} \int_{\to 0}^{t} (t-u)^{\lambda-\alpha-n-1} G_{\alpha+n+1}(u)\, du = o(t^{\lambda})$$

as $t \to +0$, since $g(t) \in C_{\lambda} L$, *i.e.* $G_{\alpha+n+1}(t) = o(t^{\alpha+n})$ $(C, \lambda-\alpha-n)$. Also $G_{\alpha+n+1}(t)$ is an $(n+1)$-th Lebesgue integral of $G_{\alpha}(t)$ for $0 < t \leqslant a$, by Theorem 6, and hence $G_{\alpha}(t) \in C_{\lambda-\alpha} L$ in $(0, a)$, by Theorem 7, with λ replaced by $\lambda-\alpha$, and $g(t)$ by $G_{\alpha}(t)$. Further, as in the proof of Theorem 7, by Theorem 6,

$$G_{\alpha+\mu+1}(t) = \left(\frac{d}{dt}\right)^{n-\mu} G_{\alpha+n+1} = o(t^{\alpha+\mu}) \quad (C, \lambda-\alpha-\mu)$$

as $t \to +0$, for $\mu = n-1, n-2, \ldots, 0$, and

$$G_{\alpha+\mu+1}(t) = \int_{\to 0\ (C,\ \lambda-\alpha-\mu)}^{t} G_{\alpha+\mu}(u)\, du$$

for $\mu = n, n-1, \ldots, 0$.

Finally, using Theorems 5 and 6, and the present theorem in the case

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* So that the extreme members are both continuous in $\eta < t \leqslant a$.

† See **1**.

$\alpha \geqslant \lambda$, we obtain

$$\begin{aligned} G_{\alpha+\beta}(t) &\equiv \left(\frac{d}{dt}\right)^{n+1} G_{\alpha+\beta+n+1}(t) \\ &= \left(\frac{d}{dt}\right)^{n+1} \frac{1}{\Gamma(\beta)} \int_{\to 0}^{t} (t-u)^{\beta-1} G_{\alpha+n+1}(u)\, du \\ &= \left(\frac{d}{dt}\right)^{n+1} \frac{1}{\Gamma(\beta+n+1)} \int_{\to 0\ (C,\ \lambda-\alpha)}^{t} (t-u)^{\beta+n} G_{\alpha}(u)\, du \\ &\equiv \frac{1}{\Gamma(\beta)} \int_{\to 0\ (C,\ \lambda-\alpha)}^{t} (t-u)^{\beta-1} G_{\alpha}(u)\, du. \end{aligned}$$

Further, there is equality between the extreme members for all t in $(0, a)$ such that $G_{\alpha+\beta}(t)$ exists, by reasoning similar to that in the case $\lambda \leqslant \alpha < \lambda+1$.

Theorem 9. Since $(t-u)^{\alpha'-\alpha}$ is of bounded variation in (η, t), and its derivatives are all bounded in $(0, \eta)$, where $0 < \eta < t$, $G_{\alpha'}(t)$ exists whenever $G_{\alpha}(t)$ does, by Theorem 3, and hence it follows from Theorem 8 that

$$\begin{aligned} G_{\alpha'}(t) &= \frac{1}{\Gamma(\alpha'-\alpha)} \int_{\to 0\ (C,\ \lambda-\alpha)}^{t} (t-u)^{\alpha'-\alpha-1} G_{\alpha}(u)\, du \\ &= \int_{0}^{t} (t-u)^{\alpha'-\alpha-1} o(u^{\alpha+p})\, du \\ &= o(t^{\alpha'+p}) \end{aligned}$$

as $t \to +0$.

Theorem 10. Since $g(t) \in C_{\lambda} L$ in $(0, a)$, it follows from the necessity part of Theorem 1 that $G_1(t) \in C_{\mu} L$ in $(0, a)$ and is $o(1)$ (C, λ) as $t \to +0$, while $(t-u)^{\lambda-1} G_1(u) \in C_{\mu} L$ in $(0, t)$. Hence by Theorem 9 and the present theorem, with λ replaced by μ, $G_1(t) \in C_{\mu'} L$ in $(0, a)$ and is $o(1)$ (C, λ') as $t \to +0$, where $\mu' = \max(\lambda'-1, 0)$, while $(t-u)^{\lambda-1} G_1(u) \in C_{\mu'} L$ in $(0, t)$. It follows that $g(t) \in C_{\lambda'} L$ in $(0, a)$ and $\int_{\to 0\ (C,\ \lambda')}^{a} g(t)\, dt = \int_{\to 0\ (C,\ \lambda)}^{a} g(t)\, dt$.

This completes the proof of Theorems 1–10.

3. In this section we give some further results which suggest themselves naturally.

THEOREM 11. *If $g(t) \in (CL)$ in $(0, a)$ and is $o(t^p)$ (C, λ), where $p > -1$, $\lambda \geqslant 0$, then $g(t) \in C_{\mu} L$ in $(0, a)$, where $\mu = \max(\lambda-1, 0)$.*

Proof. If $\lambda = 0$, the result is trivial. Suppose now that λ is given and that the theorem is true when λ is replaced by μ, and μ by $\nu = \max(\mu-1, 0)$. It follows from the hypotheses, by Theorems 1, 8 and 9, that $G_1(t) \in (CL)$ in $(0, a)$ and is $o(t^{p+1})$ (C, μ) as $t \to +0$. Hence by the case assumed, with p replaced by $p+1$, $G_1(t) \in C_\nu L$ in $(0, a)$ and, since $p+1 > 0$, $G_1(t) = o(1)$ (C, μ). Thus, by Theorem 1, $g(t) \in C_\mu L$, and the result follows by induction.

THEOREM 12. *If $g(t) \in (CL)$ in $(0, a)$, a necessary and sufficient condition that $g(t) \in C_\lambda L$ in $(0, a)$ is that $G_{\alpha+1}(t) = o(t^\alpha)$ $(C, \lambda-\alpha)$ as $t \to +0$, where $-1 < \alpha \leqslant \lambda$.*

Proof. By Theorem 8, the condition $G_{\alpha+1}(t) = o(t^\alpha)$ $(C, \lambda-\alpha)$ is equivalent to $G_1(t) = o(1)$ (C, λ), and the result follows from Theorems 11 and 1.

THEOREM 13*. *If $g(t) \in (CL)$ in $(0, a)$, a necessary and sufficient condition that $g(t) = o(t^p)$ (C, λ) as $t \to +0$, where $p > -1$, $\lambda \geqslant 0$, is that $g_\alpha(t) = o(t^p)$ $(C, \lambda-\alpha)$ as $t \to +0$, where $0 \leqslant \alpha \leqslant \lambda$.*

Proof. Necessity. If $g(t) = o(t^p)$ (C, λ), it follows from Theorem 8 that $G_\alpha(t) = o(t^{\alpha+p})$ $(C, \lambda-\alpha)$, and hence from Theorem 11 and Theorem 2, with $h(t) = \Gamma(\alpha+1)t^{-\alpha}$, that $g_\alpha(t) = o(t^p)$ $(C, \lambda-\alpha)$.

Sufficiency. If $g_\alpha(t) = o(t^p)$ $(C, \lambda-\alpha)$, it follows from Theorem 11 and Theorem 2, with $h(t) = t^\alpha/\Gamma(\alpha+1)$, that $G_\alpha(t) = o(t^{\alpha+p})$ $(C, \lambda-\alpha)$, *i.e.* by Theorem 8, $g(t) = o(t^p)$ (C, λ).

COROLLARY†. *If $g(t) \in (CL)$ in $(0, a)$, a necessary and sufficient condition that $g(t) = o(t^p)$ (C, κ) as $t \to +0$, where κ is a positive integer, is that*

$$\frac{1}{t}\int_{\to 0\,(C)}^{t} \frac{dt_1}{t_1}\int_{\to 0\,(C)}^{t_1} \frac{dt_2}{t_2} \dots \int_{\to 0\,(C)}^{t_{\kappa-1}} g(t_\kappa)\,dt_\kappa = o(t^p), \qquad (3.1)$$

i.e. $g(t) = o(t^p)$ (H, κ) *as* $t \to +0$.

THEOREM 14‡. *If $g(t) \in (CL)$ in $(0, a)$, a necessary and sufficient condition that $g(t) \in C_\lambda L$ in $(0, a)$ is that $tg(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$.*

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* This completes a result given in **2**, for $g(t) \in L$ and, in the necessity part, for λ an integer.

† Stated for $g(t) \in L$ by Hardy and Littlewood, **13**, and proved for $g(t) \in C_0 L$ in **2**.

‡ See footnote to Theorem 18.

Proof. Necessity. Suppose that $g(t) \in C_\lambda L$ in $(0, a)$. Then

$$\chi(t) = tg(t) \in C_\lambda L,$$

by Theorem 3, with $h(t) = t$, and hence, for $\sigma \geqslant 1$, $0 < t \leqslant a$,

$$\begin{aligned} \mathrm{X}_\sigma(t) &= \frac{1}{\Gamma(\sigma)} \int_{\to 0\ (C)}^{t} (t-u)^{\sigma-1} u g(u)\, du \\ &= \frac{t}{\Gamma(\sigma)} \int_{\to 0\ (C)}^{t} (t-u)^{\sigma-1} g(u)\, du - \frac{\sigma}{\Gamma(\sigma+1)} \int_{\to 0\ (C)}^{t} (t-u)^{\sigma} g(u)\, du, \end{aligned}$$

i.e.

$$\mathrm{X}_\sigma(t) = tG_\sigma(t) - \sigma G_{\sigma+1}(t). \tag{3.2}$$

Since $g(t) \in C_\lambda L$ we have $G_{\lambda+1}(t) = o(t^\lambda)$. It follows that

$$tG_{\lambda+1}(t) = o(t^{\lambda+1}),$$

and also, by Theorem 8, that $G_{\lambda+2}(t) = o(t^{\lambda+1})$. Hence by (3.2), with $\sigma = \lambda+1$, $\mathrm{X}_{\lambda+1}(t) = o(t^{\lambda+1})$, *i.e.* $tg(t) = o(1)$ $(C, \lambda+1)$.

Sufficiency. Suppose that $tg(t) = o(1)$ $(C, \lambda+1)$, and $g(t) \in C_{\lambda+n} L$ in $(0, a)$, where by Theorem 10 we may suppose that n is a positive integer. Then, by Theorem 9, $tg(t) = o(1)$ $(C, \lambda+n)$, *i.e.* $\mathrm{X}_{\lambda+n}(t) = o(t^{\lambda+n})$, and also $G_{\lambda+n+1}(t) = o(t^{\lambda+n})$. Hence by (3.2), with $\sigma = \lambda+n$, $tG_{\lambda+n}(t) = o(t^{\lambda+n})$, *i.e.* $G_{\lambda+n}(t) = o(t^{\lambda+n-1})$, and thus, by Theorem 12, $g(t) \in C_{\lambda+n-1} L$ in $(0, a)$. By repetition of the argument, with n replaced by $n-1, n-2, \ldots, 1$, we find that $g(t) \in C_\lambda L$ in $(0, a)$.

The first hypothesis in Theorem 14 is not superfluous, for, if

$$g(t) = t^{-1}(\log t)^{-1},$$

then $tg(t) = o(1)$ as $t \to +0$, but $g(t)$ is not integrable (CL) in $(0, \frac{1}{2})$.

THEOREM 15. *If $g(t) \in C_\lambda L$ in $(0, a)$, then $tg_\alpha(t) = o(1)$ (C, τ) as $t \to +0$, where $\alpha > 0$ and $\tau = \max(\lambda+1-\alpha, 0)$.*

Proof. Since $g(t) \in C_\lambda L$, it follows from Theorem 12, if $0 < \alpha \leqslant \lambda+1$ and Theorem 8, if $\alpha > \lambda+1$, that $G_\alpha(t) = o(t^{\alpha-1})$ (C, τ). Hence, by Theorems 11 and 2, with $h(t) = \Gamma(\alpha+1)/t^{\alpha-1}$, $tg_\alpha(t) = \Gamma(\alpha+1)\, G_\alpha(t)/t^{\alpha-1} = o(1)$ (C, τ) as $t = \to +0$.

THEOREM 16. *If $g(t) \in C_\lambda L$ and $g_\alpha(t) \in (CL)$ in $(0, a)$, where $\alpha > 0$, then $g_\alpha(t) \in C_\rho L$ in $(0, a)$, where $\rho = \max(\lambda-\alpha, 0)$.*

Proof. By Theorem 15, since $g(t) \in C_\lambda L$ in $(0, a)$, $tg_\alpha(t) = o(1)$ (C, τ), and hence, by Theorem 9, $tg_\alpha(t) = o(1)$ $(C, \rho+1)$. The result follows from Theorem 14, since $g_\alpha(t) \in (CL)$.

The second hypothesis in Theorem 16 is not superfluous, for if

$$g(t) = t^{-1}(\log t)^{-2},$$

then $g(t) \in L$ in $(0, \frac{1}{2})$, but $g_1(t) = -t^{-1}(\log t)^{-1}$, which is not integrable (CL).

THEOREM 17. *If $g(t) \in L$ in every interval (ϵ, a), $0 < \epsilon < a$, a necessary and sufficient condition that $\int_t^a g(u)\,du \in C_\lambda L$ in $(0, a)$ is that $tg(t) \in C_{\lambda+1} L$ in $(0, a)$.*

This is the case $\sigma = 1$ of Theorem 20, with $g(t)$ replaced by $tg(t)$ and $g(t) = 0$ for $t > a$.

4. In this section we consider functions defined in the range $(0, \infty)$. We begin by restating Theorems 14 and 17 in a form relevant to the theory of conjugate functions.

Suppose that $g(t) \in L$ in every interval (ϵ, Δ), for $0 < \epsilon < \Delta < \infty$, and write, for $t > 0$,

$$\vartheta(t) = \int_t^\infty \frac{g(u)}{u}\,du, \tag{4.1}$$

supposing this to be convergent. Then we have the following two theorems.

THEOREM 18*. *If $\vartheta(t) \in (CL)$ in $(0, a)$ and tends to l (C) as $t \to +0$, a necessary and sufficient condition that $\vartheta(t) \to l$ (C, λ), where $\lambda \geqq 0$, is that $g(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$.*

THEOREM 19†. *A necessary and sufficient condition that $\vartheta(t) \in C_\lambda L$ in $(0, a)$ is that $g(t) \in C_{\lambda+1} L$ in $(0, a)$.*

Theorem 19 is the case $\sigma = 1$ of Theorem 20.

Suppose that $g(t) \in L$ in every interval (ϵ, Δ), for $0 < \epsilon < \Delta < \infty$, and write, for $t > 0$,

$$\xi(t) = t^{\sigma-1} \int^\infty \frac{g(u)}{u^\sigma}\,du, \tag{4.2}$$

where σ is real, the integral being assumed convergent. Then we have the following three theorems.

THEOREM 20. (i) *If $\xi(t) \in C_\lambda L$ in $(0, a)$, then $g(t) \in C_{\lambda+1} L$ in $(0, a)$.*

(ii) *If $\sigma \neq 0$, and $g(t) \in C_{\lambda+1} L$ in $(0, a)$, there is a number s such that $\xi(t) - st^{\sigma-1} \in C_\lambda L$ in $(0, a)$. If $\sigma > 0$, s is arbitrary.*

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* Given by Paley, **16**, for $g(t) \in L$ in $(0, a)$. See also **3**.

† It was proved by Hardy, **11**, that if $g(t) \in L$ then $\vartheta(t) \in L$.

THEOREM 21. *If* $\sigma > 0$, *and* $g(t) \in (CL)$ *in* $(0, \Delta)$, *for* $\Delta > 0$, *then* $\xi(t) \in (CL)$ *in* $(0, \Delta)$ *and, for* $\alpha \geqslant 0$, $t > 0$,

$$\xi_\alpha(t) = t^{\sigma-1} \int_0^\infty \frac{g_\alpha(u)}{u^\sigma}\, du. \tag{4.3}$$

THEOREM 22. *If* $\sigma > 0$ *and* $g(t) \in (CL)$ *in* $(0, \Delta)$, *for* $\Delta > 0$, *a necessary and sufficient condition that* $g(t)/t^\sigma \in C_\lambda L$ *and**

$$\int_{\to 0\ (C,\,\lambda)}^{\infty} \frac{g(u)}{u^\sigma}\, du = l \tag{4.4}$$

is that $g_\alpha(t)/t^\sigma \in C_{\lambda-\alpha} L$ *and*

$$\int_{\to 0\ (C,\,\lambda-\alpha)}^{\infty} \frac{g_\alpha(u)}{u^\sigma}\, du = \frac{\Gamma(\sigma)\,\Gamma(\alpha+1)}{\Gamma(\sigma+\alpha)}\, l, \tag{4.5}$$

where $0 \leqslant \alpha \leqslant \lambda$.

We require the following lemmas.

LEMMA 3. *If* $\sigma > 0$, $g(t) \in (CL)$ *in* $(0, \Delta)$, *for* $\Delta > 0$, *and* $\xi(t)$, *defined by* (4.2), *exists for* $t > 0$, *then* $g(t) = o(t^{\sigma-1})$ $(C, 1)$ *as* $t \to +\infty$.

We have, for $0 < \zeta < t$,

$$G_1(t) = \left\{\int_0^\zeta + \int_\zeta^t\right\} g(u)\, du = I_1 + I_2,$$

where, by the Second Mean Value Theorem,

$$|I_2| \leqslant t^\sigma \max_{\zeta \leqslant t' \leqslant t} \left| \int_{t'}^t \frac{g(u)}{u^\sigma}\, du \right| < \epsilon t^\sigma$$

provided $\zeta > \zeta(\epsilon)$, while I_1 is independent of t. Hence $G_1(t) = o(t^\sigma)$, *i.e.* $g(t) = o(t^{\sigma-1})$ $(C, 1)$ as $t \to +\infty$.

LEMMA 4. *If* $g(t) \in (CL)$ *in* $(0, \Delta)$, *for* $\Delta > 0$, *and is* $o(t^p)$ (C, α) *as* $t \to +\infty$, *where* $p > -1$, $\alpha \geqslant 0$, *then* $g(t) = o(t^p)$ (C, α') *as* $t \to +\infty$, *for* $\alpha' > \alpha$.

The proof of Theorem 9 holds if t is large and tends to $+\infty$.

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* $\int_{\to 0\ (C,\,\lambda)}^{\infty}$ means $\lim_{\Delta \to +\infty} \int_{\to 0\ (C,\,\lambda)}^{\Delta}$. If $\sigma = r+1$, (4.4) is a form of the condition for the Cesàro summability of the r-th derived allied series of a Fourier series. Cf. **6**, where the condition is stated, and a paper to be published in these *Proceedings* entitled "The Cesàro summability of the successively derived allied series of a Fourier series", where the details are given.

Proof of Theorem 20. (i) Suppose $\xi(t) \in C_\lambda L$ in $(0, a)$. Then, integrating by parts, we have, if $0 < \epsilon < a$,

$$\int_\epsilon^a g(t)\,dt = -\int_\epsilon^a t^\sigma\, d\left\{\frac{\xi(t)}{t^{\sigma-1}}\right\}$$

$$= \left[-t\xi(t)\right]_\epsilon^a + \sigma\int_\epsilon^a \xi(t)\,dt.$$

Since $\xi(t) \in C_\lambda L$ in $(0, a)$ it follows from Theorem 14 that $\epsilon\xi(\epsilon) = o(1)$ $(C, \lambda+1)$ as $\epsilon \to +0$, and hence, by Theorems 11 and 1, that $g(t) \in C_{\lambda+1} L$ in $(0, a)$.

(ii) Suppose that $g(t) \in C_{\lambda+1} L$ in $(0, a)$, and that $\xi(t)$ exists for $t > 0$.

The case $\sigma > 0$. We first show that $t\xi(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$. Integrating by parts ν times, where ν is the least integer such that $\nu \geqslant \lambda+2$, we have, since by Lemmas 3 and 4 $G_\rho(u) = o(u^{\rho+\sigma-1})$ as $u \to +\infty$, for $\rho = 1, 2, \ldots, \nu$,

$$t\xi(t) = \lim_{\Delta\to+\infty} t^\sigma \int_t^\Delta \frac{g(u)}{u^\sigma}\,du$$

$$= \lim_{\Delta\to+\infty} \left\{t^\sigma \left[\sum_{\rho=1}^{\nu} \frac{\Gamma(\sigma+\rho-1)}{\Gamma(\sigma)} \frac{G_\rho(u)}{u^{\sigma+\rho-1}}\right]_t^\Delta + \frac{\Gamma(\sigma+\nu)}{\Gamma(\sigma)} t^\sigma \int_t^\Delta \frac{G_\nu(u)}{u^{\sigma+\nu}}\,du\right\}$$

$$= -\sum_{\rho=1}^{\nu} \frac{\Gamma(\sigma+\rho-1)}{\Gamma(\sigma)\,\Gamma(\rho+1)}\, tg_\rho(t) + \frac{\Gamma(\sigma+\nu)}{\Gamma(\sigma)\,\Gamma(\nu+1)}\, t^\sigma \int_t^\infty \frac{g_\nu(u)}{u^\sigma}\,du.$$

Now, by Theorem 15, $tg_\rho(t) = o(1)$ (C, τ) as $t \to +0$, where $\tau = \max(\lambda+2-\rho, 0)$ for $\rho = 1, 2, \ldots, \nu$. Hence, by Theorem 9, $tg_\rho(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$, while

$$t^\sigma \int_t^\infty \frac{g_\nu(u)}{u^\sigma}\,du = t^\sigma \int_t^a o(u^{-\sigma-1})\,du + O(t^\sigma) = o(1)$$

as $t \to +0$, and so it follows from Theorem 9 that $t\xi(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$.

Next integrating by parts, if $0 < \epsilon < a$, we have

$$\sigma\int_\epsilon^a \xi(t)\,dt = \sigma\int_\epsilon^a t^{\sigma-1}\,\frac{\xi(t)}{t^{\sigma-1}}\,dt$$

$$= \left[t^\sigma\,\frac{\xi(t)}{t^{\sigma-1}}\right]_\epsilon^a - \int_\epsilon^a t^\sigma\, d\left\{\frac{\xi(t)}{t^{\sigma-1}}\right\}$$

$$= a\xi(a) - \epsilon\xi(\epsilon) + \int_\epsilon^a g(t)\,dt.$$

Since $g(t) \in C_{\lambda+1} L$ in $(0, a)$ and $\epsilon\xi(\epsilon) = o(1)$ $(C, \lambda+1)$ as $t \to +0$, it follows by Theorems 11 and 1, that $\xi(t) \in C_{\lambda+1} L$ in $(0, a)$.

Finally, since $\xi(t) \in (CL)$ and $t\xi(t) = o(1)$ $(C, \lambda+1)$ as $t \to +0$, we deduce from Theorem 14 that $\xi(t) \in C_\lambda L$ in $(0, a)$, while the arbitrariness of s is trivial.

The case $\sigma < 0$. Since $g(t) \in C_{\lambda+1} L$ in $(0, a)$ and $\sigma < 0$, $g(t)/t^\sigma \in C_{\lambda+1} L$ in $(0, a)$, by Theorem 3, and

$$t\xi(t) = t^\sigma \int_t^\infty \frac{g(u)}{u^\sigma}\, du = st^\sigma - t^\sigma \int_{\to 0\ (C,\ \lambda+1)}^t \frac{g(u)}{u^\sigma}\, du.$$

Hence, applying Theorem 3 ν times, where ν is the least integer such that $\nu \geqslant \lambda+2$, we have, for $t > 0$,

$$\begin{aligned} t\xi(t) - st^\sigma &= -t^\sigma \int_{\to 0\ (C,\ \lambda+1)}^t \frac{g(u)}{u^\sigma}\, du \\ &= -t^\sigma \sum_{\rho=1}^{\nu} \frac{\Gamma(\sigma+\rho-1)}{\Gamma(\sigma)} \frac{G_\rho(t)}{t^{\sigma+\rho-1}} + \frac{\Gamma(\sigma+\nu)}{\Gamma(\sigma)} t^\sigma \int_{\to 0}^t \frac{G_\nu(u)}{u^{\sigma+\nu}}\, du \\ &= o(1) \quad (C, \lambda+1) \end{aligned}$$

as $t \to +0$, as in the first part, since

$$t^\sigma \int_{\to 0}^t \frac{G_\nu(u)}{u^{\sigma+\nu}}\, du = t^\sigma \int_0^t o(u^{-\sigma-1})\, du = o(1).$$

Finally

$$\begin{aligned} \sigma \int_\epsilon^a \{\xi(t) - st^{\sigma-1}\}\, dt &= \sigma \int_\epsilon^a t^{\sigma-1} \left\{\frac{\xi(t)}{t^{\sigma-1}} - s\right\} dt \\ &= \left[t^\sigma \left\{\frac{\xi(t)}{t^{\sigma-1}} - s\right\}\right]_\epsilon^a - \int_\epsilon^a t^\sigma\, d\left\{\frac{\xi(t)}{t^{\sigma-1}}\right\} \\ &= A - \epsilon\{\xi(\epsilon) - s\epsilon^{\sigma-1}\} + \int_\epsilon^a g(t)\, dt, \end{aligned}$$

which tends to a limit $(C, \lambda+1)$ as $\epsilon \to +0$, so that, by Theorems 11 and 1, $\xi(t) - st^{\sigma-1} \in C_{\lambda+1} L$ in $(0, a)$. Since $t\{\xi(t) - st^{\sigma-1}\} = o(1)$ $(C, \lambda+1)$ as $t \to +0$, it follows from Theorem 14 that $\xi(t) - st^{\sigma-1} \in C_\lambda L$ in $(0, a)$.

Theorem 20 (ii) is shown to be false for $\sigma = 0$ by the example

$$g(t) = t^{-1} (\log t)^{-2}$$

in $(0, \frac{1}{2})$.

Proof of Theorem 21. Since $g(t) \epsilon (CL)$ in $(0, \Delta)$, and $\xi(t)$ exists by hypothesis, it follows from Theorem 20 that $\xi(t) \epsilon (CL)$ in $(0, \Delta)$, for $\Delta > 0$, and, as in the proof of Theorem 20, for $u > 0$,

$$G_1(u) = \sigma \Xi_1(u) + (t-u)\,\xi(u) - t\xi(u). \tag{4.6}$$

Multiplying by $(t-u)^{\alpha-2}/\Gamma(\alpha-1)$ and integrating, we obtain, for $\alpha > 1$, $t > 0$,

$$G_\alpha(t) = (\sigma+\alpha-1)\,\Xi_\alpha(t) - t\,\Xi_{\alpha-1}(t), \tag{4.7}$$

from which it follows that, for $\alpha \geqslant 1$, $t > 0$,

$$\frac{g_\alpha(t)}{t^\sigma} = -\frac{d}{dt}\left\{\frac{\xi_\alpha(t)}{t^{\sigma-1}}\right\}. \tag{4.8}$$

If $0 < \alpha < 1$, (4.8) is obtained, for almost all $t > 0$, by differentiating (4.7), with α replaced by $\alpha+1$. Also, for $\alpha \geqslant 0$, $\xi_\alpha(t)/t^{\sigma-1}$ is absolutely continuous in every interval (ϵ, Δ), for $0 < \epsilon < \Delta < \infty$, since $\xi(t)$ is a Lebesgue integral for $t > 0$.

It follows that, for $0 < t < \zeta < \infty$,

$$\int_t^\zeta \frac{g_\alpha(u)}{u^\sigma}\,du = \frac{\xi_\alpha(t)}{t^{\sigma-1}} - \frac{\xi_\alpha(\zeta)}{\zeta^{\sigma-1}},$$

and the result follows on letting $\zeta \to +\infty$, by Lemma 4, since $\xi(t) = o(t^{\sigma-1})$ as $t \to +\infty$, by the convergence of the integral (4.2).

Proof of Theorem 22. *Necessity.* Suppose that $g(t)/t^\sigma \epsilon C_\lambda L$ and (4.4) holds, *i.e.* $\xi(t)/t^{\sigma-1} \to l$ (C, λ) as $t \to +0$. It follows from Theorem 11 and Theorem 2, with $g(t)$ replaced by $\xi(t)/t^{\sigma-1} - l$ and $h(t) = t^{\sigma-1}$, that $\xi(t) \sim lt^{\sigma-1}$ (C, λ) as $t \to +0$. Hence, by Theorem 13, with $g(t)$ replaced by $\xi(t) - lt^{\sigma-1}$ and $p = \sigma-1$, we have $\xi_\alpha(t) \sim lt^{\sigma-1}\Gamma(\sigma)\Gamma(\alpha+1)/\Gamma(\sigma+\alpha)$ $(C, \lambda-\alpha)$, and so, by Theorem 2, with $g(t)$ replaced by

$$\xi_\alpha(t) - lt^{\sigma-1}\Gamma(\sigma)\Gamma(\alpha+1)/\Gamma(\sigma+\alpha)$$

and $h(t) = t^{1-\sigma}$, we obtain $\xi_\alpha(t)/t^{\sigma-1} \sim l\Gamma(\sigma)\Gamma(\alpha+1)/\Gamma(\sigma+\alpha)$ $(C, \lambda-\alpha)$ as $t \to +0$, which by Theorem 21 is (4.5).

Sufficiency. Since $\xi(t) \epsilon (CL)$ in $(0, \Delta)$, by Theorem 20, the sufficiency is established by reversing the argument.

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* Cf. 3.

5. It would be natural to replace the number m, occurring in Theorems 2 and 3, by λ, $h^{(\lambda)}(t)$ being interpreted appropriately*. We end by stating two results in this direction.

We write†, for $0<\lambda<1$,

$$\left.\begin{aligned} &h^{(\lambda)}(t) = -\frac{1}{\Gamma(1-\lambda)}\frac{d}{dt}\int_t^a (u-t)^{-\lambda}\{h(u)-h(a)\}\,du \quad (0<t<a),\\ &h^{(\lambda)}(a) = 0. \end{aligned}\right\} \tag{5.1}$$

Then we have the following two theorems.

THEOREM 23. *If* (i) $g(t)\,\epsilon\, C_0 L$ *in* $(0, a)$ *and is* $o(t^p)$ (C, λ) *as* $t\to+0$, *where* $0<\lambda<1$, $p>-1$, (ii) $h(t)$ *is continuous in* $0<t\leqslant a$, $h(t)=O(t^{-r})$ *and* $h^{(\lambda)}(t)=O(t^{-\lambda-r})$ *in* $(0, a)$, *where* $0\leqslant r<p+1$, *then* $g(t)\,h(t)\,\epsilon\, C_0 L$ *in* $(0, a)$ *and is* $o(t^{p-r})$ (C, λ) *as* $t\to+0$.

THEOREM 24. *If* (i) $g(t)\,\epsilon\, C_\lambda L$ *in* $(0, a)$, *where* $0<\lambda<1$, (ii) $h(t)$ *is continuous in* $0<t\leqslant a$ *and*

$$\int_{\to 0}^{a} t^{\lambda}\,|dh^{(\lambda)}(t)| < \infty\ddagger, \tag{5.2}$$

then $g(t)\,h(t)\,\epsilon\, C_\lambda L$ *in* $(0, a)$, *and* (2.4) *holds.*

Further

$$\int_{\to 0}^{a} g(t)\{h(t)-h(a)\}\,dt = \int_{\to 0}^{a} G_\lambda(t)\,h^{(\lambda)}(t)\,dt\S. \tag{5.3}$$

That Theorems 23 and 24 are proper extensions of cases of Theorems 2 and 3 is shown by the following two lemmas.

LEMMA 5. *If* $h(t)=O(t^{-r})$ *and* $h'(t)=O(t^{-1-r})$ *in* $(0, a)$, *then* $h^{(\lambda)}(t)=O(t^{-\lambda-r})$ *in* $(0, a)$, *where* $0<\lambda<1$, $r\geqslant 0$.

LEMMA 6. *If*

$$\int_{\to 0}^{a} t\,|dh'(t)| < \infty, \tag{5.4}$$

then (5.2) *holds.*

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* Cf. Cossar, **10**.

† The notation is chosen to make the enunciations simple. Strictly, $h^{(\lambda)}(t)$ is a fractional derivative of $h(t)-h(a)$.

‡ (5.2) is satisfied in particular if $h^{(\lambda)}(t)$ is of bounded variation in $(0, a)$, when the first part of the theorem is included in a result of Burkill, **9**.

§ Cf. Love and Young, **15**.

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THE CESÀRO SUMMABILITY OF THE SUCCESSIVELY DERIVED ALLIED SERIES OF A FOURIER SERIES

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1. *Introduction.*

I have proved in earlier papers* that *a necessary and sufficient condition that the series obtained by formally differentiating r times a Lebesgue-Fourier series should be summable* $(C, \alpha+r)$ *to sum s at a given point, where* $\alpha \geqslant 0$, *is that a certain even function should be integrable in the Cesàro-Lebesgue sense in* $(0, \pi)$ *and its Fourier series be summable* (C, α) *to s at* $t = 0$. I have also stated an analogous result for the successively derived allied series†. In this paper I prove the latter result. The method is essentially the same as in the case of Fourier series, but the details are different, and involve the establishment of some subsidiary results of independent interest. Some of these have been given in another paper‡.

Notation. The notation and definitions employed are consistent with those in D_rFS and CL. Some of the definitions in CL, however, contain slightly "lighter" hypotheses than those in D_rFS (though the definitions are in fact equivalent§). Either set may be employed in this paper.

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* *Quart. J. of Math.* (Oxford), 10 (1939), 67–74, where the case $r = 1$ is given, and *Proc. London Math. Soc.* (2), 46 (1940), 270–289, where the general case is discussed. These papers will be referred to as DFS and D_rFS respectively. The case $r = 1$ of an analogous result for absolute Cesàro summability is given in *Quart. J. of Math.* (Oxford), 12 (1941), 15–25.

† D_rFS, Theorem 4. For references to other papers on successively derived allied series see D_rFS.

‡ *Proc. London Math. Soc.* (2), 49 (1945), 40–62. This will be referred to as CL.

§ Notably the definition of integrability $C_\lambda L$ of $g(t)$, where it need not be assumed that $\int_t^a g(u)\,du$ is integrable $C_\mu L$, where $\mu = \max(\lambda-1, 0)$, and the definition of $g(t) \sim st^p$ (C, λ), where it need not be assumed that $g(t)$ is integrable $C_\mu L$, but only (CL), though in both cases the omitted condition is implied by the others.

We write, for $t>0$,

$$G_\sigma(t)=\frac{1}{\Gamma(\sigma)}\int_{\to 0\,(C)}^{t}(t-u)^{\sigma-1}g(u)\,du \quad (\sigma>0), \tag{1.1}$$

and similarly with $g(t)$ replaced by $\theta(t)$, $\psi(t)$, $\chi(t)$, etc.*

We write†

$$C_\sigma(t)+i\bar{C}_\sigma(t)=\frac{1}{\Gamma(\sigma)}\int_0^t(t-u)^{\sigma-1}e^{iu}\,du \quad (\sigma>0) \tag{1.2}$$

and

$$\gamma_\sigma(t)+i\bar{\gamma}_\sigma(t)=\Gamma(\sigma)t^{-\sigma}\{C_\sigma(t)+i\bar{C}_\sigma(t)\}. \tag{1.3}$$

We require the identities

$$C_\sigma(t)=\sum_{\nu=0}^{\rho-1}(-1)^\nu\frac{t^{\sigma+2\nu}}{\Gamma(\sigma+2\nu+1)}+(-1)^\rho\bar{C}_{\sigma+2\rho-1}(t) \tag{1.4}$$

and

$$\bar{C}_\sigma(t)=\sum_{\nu=1}^{\rho}(-1)^{\nu-1}\frac{t^{\sigma+2\nu-1}}{\Gamma(\sigma+2\nu)}+(-1)^\rho\bar{C}_{\sigma+2\rho}(t), \tag{1.5}$$

and the inequalities

$$|\Delta^\kappa\gamma_\sigma^{(\lambda)}(nt)|\leqslant\begin{cases}At^\kappa\\ At^{\kappa-\sigma}n^{-\sigma}+At^{-\lambda-2}n^{-\kappa-\lambda-2}\end{cases} \tag{1.6}$$

and

$$|\Delta^\kappa\bar{\gamma}_\sigma^{(\lambda)}(nt)|\leqslant\begin{cases}At^\kappa\\ At^{\kappa-\sigma}n^{-\sigma}+At^{-\lambda-1}n^{-\kappa-\lambda-1},\end{cases} \tag{1.7}$$

where $\Delta u_n=u_n-u_{n+1}$, $\Delta^\kappa u_n=\Delta(\Delta^{\kappa-1}u_n)$, $\Delta^0u_n=u_n$, and A denotes a number independent of the variables.

We write

$$\kappa^\sigma(n,t)+i\bar{\kappa}^\sigma(n,t)=\tfrac{1}{2}+\sum_{\nu=1}^{n}\binom{\sigma+n-\nu}{n-\nu}e^{i\nu t}\Big/\binom{n+\sigma}{n} \tag{1.8}$$

and require the inequalities

$$\left|\left(\frac{\partial}{\partial t}\right)^\kappa\kappa^\sigma(n,t)\right|\leqslant\begin{cases}An^{\kappa+1}\\ An^{\kappa-\sigma}t^{-\sigma-1}+An^{-1}t^{-\kappa-2}\end{cases} \tag{1.9}$$

* This is consistent with the definition of $C_\sigma(t)$ and $\bar{C}_\sigma(t)$ if we write $c(t)+i\bar{c}(t)=e^{it}$.

† For references see D, FS. Plainly $\bar{C}_\sigma(t)=C_{\sigma+1}(t)$.

and

$$\left|\left(\frac{\partial}{\partial t}\right)^{\kappa}\bar{\kappa}^{\sigma}(n,t)\right| \leqslant \begin{cases} An^{\kappa+1} \\ An^{\kappa-\sigma}t^{-\sigma-1}+At^{-\kappa-1}. \end{cases} \tag{1.10}$$

Finally we write Σ for $\sum_{1}^{\infty}$ throughout the paper.

2. *Preliminary lemmas.*

LEMMA 1. *If $g(t) \in C_\lambda L$ in $(0, \pi)$, where $\lambda \geqslant 0$, and if $h(t)$ and its successive derivatives are all bounded in $(0, \pi)$, then $g(t)h(t) \in C_\lambda L$ in $(0, \pi)$, and*

$$\int_{\to 0\ (C,\lambda)}^{\pi} g(t)h(t)\,dt = G_1(\pi)h(\pi) - \int_{\to 0\ (C,\mu)}^{\pi} G_1(t)h'(t)\,dt, \tag{2.1}$$

where $\mu = \max(\lambda-1, 0)$.

This is included in a result proved in CL*

LEMMA 2†. *If $g(t) \in C_\lambda L$ in $(0, \pi)$, where $\lambda \geqslant 0$, then so do $g(t)\cos\mu t$ and $g(t)\sin\mu t$.*

LEMMA 3. *If $g(t)/t \in C_\lambda L$ in $(0, \pi)$, where $\lambda \geqslant 0$, then so do $g(t)$ and $g(t)\cot\frac{1}{2}t$.*

Lemmas 2 and 3 are corollaries of Lemma 1, since the successive derivatives of $\cos\mu t$, $\sin\mu t$, t and $t\cot\frac{1}{2}t$ are all bounded in $(0, \pi)$.

LEMMA 4. *If $g(t)/t \in (CL)$ in $(0, \pi)$, a necessary and sufficient condition that $g(t)/t \in C_\lambda L$, where $\lambda \geqslant 0$, is that $g(t) = o(1)\ (C, \lambda+1)$ as $t \to +0$.*

This has been proved in CL‡.

LEMMA 5. *If $g(t) \in (CL)$ in $(0, \Delta)$, for every $\Delta > 0$, and if*

$$\theta(t) = \int_t^{\infty} \frac{g(u)}{u^{\tau}}\,du \tag{2.2}$$

converges for every $t > 0$, where $\tau > 0$, a necessary and sufficient condition that $g(t)/t^{\tau} \in C_\sigma L$ and

$$\int_{\to 0\ (C)}^{\infty} \frac{g(u)}{u^{\tau}}\,du = l, \tag{2.3}$$

* CL, Theorem 3.

† Proved for λ an integer in DFS, Lemma 2.

‡ CL, Theorems 14 and 18.

where $\sigma > 0$, *is that* $G_\sigma(t)/t^{\sigma+\tau} \in C_0 L$ *and*

$$\int_{\to 0}^{\infty} \frac{G_\sigma(u)}{u^{\sigma+\tau}}\, du = \frac{\Gamma(\tau)}{\Gamma(\sigma+\tau)}\, l. \tag{2.4}$$

This is included in a result proved in CL*.

LEMMA 6. *If* $\sigma > 0$, *then*

$$\int_0^{\infty} \frac{\bar{C}_\sigma(u)}{u^{\sigma+1}}\, du = \frac{\frac{1}{2}\pi}{\Gamma(\sigma+1)}. \tag{2.5}$$

We have $$\theta(t) = \int_t^{\infty} \frac{\sin u}{u}\, du \to \tfrac{1}{2}\pi$$

as $t \to +0$, and hence†

$$\frac{\Gamma(\sigma+1)\,\Theta_\sigma(t)}{t^\sigma} = \Gamma(\sigma+1) \int_t^{\infty} \frac{\bar{C}_\sigma(u)}{u^{\sigma+1}}\, du \to \tfrac{1}{2}\pi$$

as $t \to +0$, by the consistency theorem for Cesàro limits.

LEMMA 7. *If* Σu_n *is summable* (C, κ), *where* $\kappa \geqslant 0$, *then*

$$\lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^\sigma}\, \bar{C}_\sigma(nt)\, \rho^n = o(t^\sigma) \tag{2.6}$$

as $t \to +0$, *for* $\sigma > \kappa + 1$.

Since $\bar{C}_\sigma(t) = o(t^\sigma)$ as $t \to +0$ we may, by altering u_1, suppose without loss of generality that Σu_n is summable (C, κ) to zero. *If* κ *is an integer*, writing $S_n{}^\kappa$ for the n-th Cesàro sum of order σ of Σu_n‡, and arguing as in D, FS, Lemma 4, we have

$$\begin{aligned} S(t) &= \lim_{\rho \to 1-0} \Sigma u_n \bar{\gamma}_\sigma(nt)\, \rho^n = \Sigma S_n{}^\kappa \Delta^{\kappa+1} \bar{\gamma}_\sigma(nt) \\ &= \sum_{n \leqslant t^{-1}} o(n^\kappa)\, O(t^{\kappa+1}) + \sum_{n > t^{-1}} o(n^\kappa)\{O(t^{-1} n^{-\kappa-2}) + O(t^{\kappa+1-\sigma} n^{-\sigma})\} \\ &= o(1) + o(1) \end{aligned}$$

as $t \to +0$, for $\sigma > \kappa + 1$.

The case when κ is not an integer may be treated by an elaboration of the same argument.

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* CL, Theorem 22.

† Cf. CL, Theorem 21, or L. S. Bosanquet, *Proc. London Math. Soc.* (2), 37 (1934), 17–32, Lemma 2. This paper will be referred to as FS. Lemma 6 is also a corollary of Lemma 5, since $(\sin u)/u \in C_\sigma L$.

‡ *I.e.*, $$S_n = \sum_{\nu=1}^{n} \binom{\kappa+n-\nu}{n-\nu} u_\nu.$$

LEMMA 8. *If $\kappa \geqslant 0$ and Σu_n is summable (C, κ) to s, then*

$$\int_t^\infty \frac{1}{u^{\sigma+1}} \left\{ \lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^\sigma} \bar{C}_\sigma(nu)\, \rho^n \right\} du \to \frac{\frac{1}{2}\pi s}{\Gamma(\sigma+1)} \qquad (2.7)$$

as $t \to +0$, for $\sigma > \kappa+1$.*

By Lemma 6 we may suppose without loss of generality that $s = 0$, for if u_1 is replaced by $u_1 - s$, then s is replaced by zero. *If κ is an integer*, in the notation of the proof of Lemma 7,

$$\int_t^\infty \frac{S(u)}{u}\, du = \int_t^\infty \frac{1}{u} \{\Sigma S_n{}^\kappa \Delta^{\kappa+1} \bar{\gamma}_\sigma(nu)\}\, du$$

$$= \Sigma S_n{}^\kappa \int_t^\infty \frac{1}{u} \Delta^{\kappa+1} \bar{\gamma}_\sigma(nu)\, du,$$

the inversion being justified by absolute convergence, for $\sigma > \kappa+1$, since

$$S_n{}^\kappa u^{-1} \Delta^{\kappa+1} \bar{\gamma}_\sigma(nu) = O(n^{\kappa-\sigma} u^{\kappa-\sigma}) + O(n^{-2} u^{-2})$$

for $n \geqslant 1$, $u \geqslant t > 0$ (t fixed).

Now

$$J(n, t) = \int_t^\infty \frac{1}{u} \Delta^{\kappa+1} \bar{\gamma}_\sigma(nu)\, du = \Delta^{\kappa+1} \int_t^\infty \frac{\bar{\gamma}_\sigma(nu)}{u}\, du$$

$$= \Delta^{\kappa+1} \int_n^\infty \frac{\bar{\gamma}_\sigma(tx)}{x}\, dx = t\sigma^{-1} \Delta^{\kappa+1} \int_n^\infty \gamma_{\sigma+1}(tx)\, dx$$

$$= (-1)^\kappa \sigma^{-1} t^{\kappa+1} \int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \ldots \int_{x_\kappa}^{x_\kappa+1} \gamma_{\sigma+1}^{(\kappa)}(tx)\, dx,$$

and hence it follows that

$$|J(n, t) \leqslant \begin{cases} At^{\kappa+1} \\ At^{\kappa-\sigma} n^{-\sigma-1} + At^{-1} n^{-\kappa-2}. \end{cases}$$

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* The integral in (2.7) may be written

$$\int_t^\infty \frac{1}{u^{\sigma+1}}\, d\left\{ \lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^{\sigma+1}} \bar{C}_{\sigma+1}(nu)\, \rho^n \right\},$$

since $$\Sigma S_n{}^\kappa \Delta^{\kappa+1} \{\bar{C}_\sigma(nu)/n^\sigma\} = (d/du) \Sigma S_n{}^\kappa \Delta^{\kappa+1} \{\bar{C}_{\sigma+1}(nu)/n^{\sigma+1}\}.$$

In this form the lemma holds for all $\sigma > \kappa$, since

$$\Gamma(\sigma+1) \int_t^\infty \frac{1}{u^{\sigma+1}}\, d\left\{ \lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^{\sigma+1}} \bar{C}_{\sigma+1}(nu)\, \rho^n \right\}$$

$$= -\frac{\Gamma(\sigma+1)}{t^{\sigma+1}} \lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^{\sigma+1}} \bar{C}_{\sigma+1}(nt)\, \rho^n + \Gamma(\sigma+2) \int_t^\infty \frac{1}{u^{\sigma+2}} \left\{ \lim_{\rho \to 1-0} \Sigma \frac{u_n}{n^{\sigma+1}} \bar{C}_{\sigma+1}(nu)\, \rho^n \right\} du,$$

and the result follows from the lemma with $\sigma+1$ in place of σ, by Lemma 7.

Thus

$$\int_t^\infty \frac{S(u)}{u}\,du = \sum_{n\leqslant t^{-1}} o(n^\kappa)\,O(t^{\kappa+1}) + \sum_{n>t^{-1}} o(n^\kappa)\{O(t^{\kappa-\sigma}n^{-\sigma-1}) + O(t^{-1}n^{-\kappa-2})\}$$

$$= o(1)+o(1)$$

as $t\to+0$, for $\sigma>\kappa+1$.

The result for non-integral κ may be deduced from the formula

$$\Gamma(\sigma+1)\int_t^\infty \frac{1}{u^{\sigma+1}}\left\{\lim_{\rho\to1-0}\Sigma\frac{u_n}{n^\sigma}\,\bar{C}_\sigma(nu)\,\rho^n\right\}du$$

$$= -\frac{\Gamma(\sigma+1)}{t^{\sigma+1}}\lim_{\rho\to1-0}\Sigma\frac{u_n}{n^{\sigma+1}}\,\bar{C}_{\sigma+1}(nt)\,\rho^n$$

$$+\Gamma(\sigma+2)\int_t^\infty \frac{1}{u^{\sigma+2}}\left\{\lim_{\rho\to1-0}\Sigma\frac{u_n}{n^{\sigma+1}}\,\bar{C}_{\sigma+1}(nu)\,\rho^n\right\}du$$

by means of Lemma 7 and the consistency theorem for Cesàro summability.

LEMMA 9. *If* $g(t)\in C_\kappa L$ *in* $(0,\pi)$, *where* $\kappa\geqslant0$, *and*

$$a_n+i\beta_n=\int_{\to0\ (C)}^{\pi} g(t)\,e^{int}\,dt,$$

then a_n *and* β_n *are* $o(1)$ (C,σ) *as* $n\to\infty$, *for* $\sigma>\kappa+1$.

If κ *is an integer*, and $a_n{}^\sigma$ denotes the n-th Cesàro mean of order σ of the sequence* $\frac{1}{2}a_0, a_1, \ldots, a_n, \ldots$, we have, by repeated applications of Lemma 1,

$$a_n{}^\sigma=\frac{\sigma}{n+\sigma}\int_{\to0\ (C)}^{\pi} g(t)\,\kappa^{\sigma-1}(n,t)\,dt$$

$$=\frac{\sigma}{n+\sigma}\left\{\sum_{\rho=0}^{\kappa}(-1)^\rho G_{\rho+1}(t)\left(\frac{\partial}{\partial t}\right)^\rho \kappa^{\sigma-1}(n,t)\right\}_{t=\pi}$$

$$+(-1)^{\kappa+1}\frac{\sigma}{n+\sigma}\int_{\to0}^{\pi} G_{\kappa+1}(t)\left(\frac{\partial}{\partial t}\right)^{\kappa+1}\kappa^{\sigma-1}(n,t)\,dt$$

$$=o(1)+O\left(\frac{1}{n}\right)\int_0^{n^{-1}} o(t^\kappa)\,O(n^{\kappa+2})$$

$$+O\left(\frac{1}{n}\right)\int_{n^{-1}}^{\pi} o(t^\kappa)\{O(t^{-\sigma}n^{\kappa-\sigma+2})+O(t^{-\kappa-3}n^{-1})\}\,dt$$

$$=o(1)$$

as $n\to\infty$, for $\sigma>\kappa+1$.

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* *I.e.* $\left\{\binom{\sigma-1+n}{n}\frac{1}{2}a_0+\sum_{\nu=1}^{n}\binom{\sigma-1+n-\nu}{n-\nu}a_\nu\right\}\Big/\binom{\sigma+n}{n}$.

The result for non-integral κ is obtained by an elaboration of the same process, and the result for β_n is obtained similarly*.

3. *The main theorem.*

In this section we suppose that $f(t)$ is integrable L in $(-\pi, \pi)$, and periodic with period 2π. We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\}, \quad \psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}, \tag{3.1}$$

and suppose that the Fourier series of $f(t)$ is

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nt + b_n \sin nt). \tag{3.2}$$

Then the allied series is

$$\Sigma(b_n \cos nt - a_n \sin nt). \tag{3.3}$$

THEOREM I. *If $f(t)$ is integrable L in $(-\pi, \pi)$, and periodic with period 2π, a necessary and sufficient condition that the series*

$$\Sigma(d/dx)^r(b_n \cos nx - a_n \sin nx)$$

should be summable $(C, \alpha+r)$ to sum s, where $\alpha \geqslant 0$ and r is a positive integer, is that constants α_ν should exist such that (i) *the odd function $g(t)$ is integrable (CL) in $(0, \pi)$ and its allied series† summable (C, α) at $t = 0$,* (ii) *$g(t)/t$ is integrable (CL) in $(0, \pi)$ and*

$$\frac{2}{\pi}\int_{\to 0\,(C)}^{\infty} \frac{g(t)}{t}\,dt = s, \tag{3.4}$$

where‡

$$g(t) = \begin{cases} \dfrac{r!}{t^r}\left\{\psi(t) - \displaystyle\sum_{\nu=1}^{\frac{1}{2}r} \frac{\alpha_{2\nu-1}}{(2\nu-1)!}\, t^{2\nu-1}\right\} & (r \text{ even}) \\ \dfrac{r!}{t^r}\left\{\phi(t) - \displaystyle\sum_{\nu=0}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu}}{(2\nu)!}\, t^{2\nu}\right\} & (r \text{ odd}). \end{cases} \tag{3.5}$$

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* More generally $\Sigma n^{-1}\beta_n$ is summable $(C, r-1)$. For $\Sigma n^{-1}\beta_n \cos nt$ is the Fourier series of a $C_\kappa L$ integral, which is thus C_κ-continuous. The result follows, for κ an integer, from a theorem of Burkill, *Journal London Math. Soc.*, 10 (1935), 254–259, and may be completed in the ordinary way.

† *I.e.*, $\Sigma\beta(n)\cos nt$, where

$$\beta(n) = \frac{2}{\pi}\int_{\to 0\,(C)}^{\pi} g(t)\sin nt\,dt.$$

‡ In a single formula

$$g(t) = \frac{r!}{t^r}\left(\tfrac{1}{2}\{f(x+t)-(-1)^r f(x-t)\} - \sum_{\nu=0}^{[\frac{1}{2}(r-1)]} \frac{\alpha_{r-1-2\nu}}{(r-1-2\nu)!}\, t^{r-1-2\nu}\right).$$

Cf. A. F. Moursund, *Annals of Math.*, 36 (1935), 182–193.

Moreover, for $\nu = r-1, r-3, \ldots,$

$$a_\nu = \lim_{\rho\to 1-0} \Sigma(d/dx)^\nu (a_n \cos nx + b_n \sin nx)\rho^n. \qquad (3.6)$$

LEMMA 10. *If* $f(t) \in L$ *and* $\Sigma(d/dx)^r(b_n \cos nx - a_n \sin nx)$ *is summable* (C) *to* s, *there exist constants* a_ν *such that* (i) $g(t)/t \in (CL)$ *in* $(0, \pi)$, *where* $g(t)$ *is given by* (3.5), (ii) (3.4) *is satisfied. Moreover the constants* a_ν *satisfy* (3.6).

Suppose that r *is even* and that

$$\Sigma(d/dx)^r(b_n \cos nx - a_n \sin nx) = \Sigma(-1)^{\frac{1}{2}r} n^r B_n$$

is summable (C, κ) to s, where $\kappa \geqslant 0$. Then*, if $\sigma \geqslant 1$,

$$\Psi_\sigma(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-u)^{\sigma-1}\psi(u)\,du$$

$$= \Sigma B_n n^{-\sigma} \bar{C}_\sigma(nt)$$

$$= \sum_{n=1}^{\infty} B_n \left\{ \sum_{\nu=1}^{\frac{1}{2}r} (-1)^{\nu-1} \frac{n^{2\nu-1} t^{\sigma+2\nu-1}}{\Gamma(\sigma+2\nu)} + (-1)^{\frac{1}{2}r} n^{-\sigma} \bar{C}_{\sigma+r}(nt) \right\}$$

$$= \sum_{\nu=1}^{\frac{1}{2}r} \frac{t^{\sigma+2\nu-1}}{\Gamma(\sigma+2\nu)} \lim_{\rho\to 1-0} \sum_{n=1}^{\infty} (-1)^{\nu-1} n^{2\nu-1} B_n \rho^n$$

$$+ \lim_{\rho\to 1-0} \Sigma(-1)^{\frac{1}{2}r} B_n n^{-\sigma} \bar{C}_{\sigma+r}(nt)\rho^n$$

$$= \sum_{\nu=1}^{\frac{1}{2}r} \frac{a_{2\nu-1}}{\Gamma(\sigma+2\nu)} t^{\sigma+2\nu-1} + \lim_{\rho\to 1-0} \Sigma(-1)^{\frac{1}{2}r} B_n n^{-\sigma} \bar{C}_{\sigma+r}(nt)\rho^n.$$

Now let $\chi(t) = \psi(t) - \sum_{\nu=1}^{\frac{1}{2}r} \frac{a_{2\nu-1}}{(2\nu-1)!} t^{2\nu-1}$. Then

$$X_\sigma(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-u)^{\sigma-1} \left\{ \psi(u) - \sum_{\nu=1}^{\frac{1}{2}r} \frac{a_{2\nu-1}}{(2\nu-1)!} u^{2\nu-1} \right\} du$$

$$= \Psi_\sigma(t) - \sum_{\nu=1}^{\frac{1}{2}r} \frac{a_{2\nu-1}}{\Gamma(\sigma+2\nu)} t^{\sigma+2\nu-1}$$

$$= \lim_{\rho\to 1-0} \Sigma(-1)^{\frac{1}{2}r} B_n n^{-\sigma} \bar{C}_{\sigma+r}(nt)\rho^n.$$

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* Since $(t-u)^{\sigma-1}$ is of bounded variation in $(0, t)$. Cf. E. W. Hobson, *The theory of functions of a real variable*, 2 (1926), 583.

It follows that

$$\int_t^\infty \frac{X_\sigma(u)}{u^{\sigma+r+1}}\,du = \int_t^\infty \frac{1}{u^{\sigma+r+1}} \left\{ \lim_{\rho\to 1-0} \Sigma(-1)^{\frac{1}{2}r} n^r B_n n^{-\sigma-r} \bar{C}_{\sigma+r}(nu)\,\rho^n \right\} du$$

$$\to \tfrac{1}{2}\pi s/\Gamma(\sigma+r+1)$$

as $t\to+0$, for* $\sigma+r>\kappa+1$, by Lemma 8, and hence, by Lemma 5, that $r!\,\chi(t)/t^{r+1}\in C_\sigma L$ and

$$r!\int_{\to 0\ (C)}^\infty \frac{\chi(t)}{t^{r+1}}\,dt = \tfrac{1}{2}\pi s.$$

On taking $g(t)=r!\,\chi(t)/t^r$ the lemma is established for r even.

The case when r is odd may be treated similarly†.

LEMMA 11. *If $g(t)$ is odd, a necessary and sufficient condition that $g(t)$ should be integrable (CL) in $(0,\pi)$ and its allied series be summable (C) at $t=0$ is that $g(t)/t\in(CL)$ in $(0,\pi)$.*

Sufficiency. Suppose that $g(t)/t\in C_\kappa L$, where $\kappa\geqslant 0$. Then, by Lemma 3, $g(t)$ and $g(t)\cot\frac{1}{2}t$ are both integrable $C_\kappa L$. If $\bar{s}_n$ denotes the n-th partial sum of the allied series of $g(t)$ at $t=0$, we have, by Lemma 2,

$$\bar{s}_n = \frac{1}{\pi}\int_{\to 0\ (C)}^\pi g(t)\,\frac{\cos\frac{1}{2}t-\cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{\pi}\int_{\to 0\ (C)}^\pi g(t)\cot\tfrac{1}{2}t(1-\cos nt)\,dt + \frac{1}{\pi}\int_{\to 0\ (C)}^\pi g(t)\sin nt\,dt$$

$$\to \frac{1}{\pi}\int_{\to 0\ (C)}^\pi g(t)\cot\tfrac{1}{2}t\,dt \quad (C,\sigma)$$

as $n\to\infty$, where $\sigma>\kappa+1$, by Lemma 9.

Necessity‡. Suppose that $g(t)\in C_r L$ in $(0,\pi)$, where r is a positive integer, and that its allied series is summable (C,κ) at $t=0$, where $\kappa\geqslant r$. Then the allied series of $g(t)$ is the r-th derived allied series of a function which is integrable $C_0 L$ and of the form $G_r(t)+P(t)$, where $P(t)$ is a certain polynomial§. A repetition of the argument of Lemma 10 shows

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* In fact the result holds for $\sigma+r>\kappa$, since

$$\Gamma(\sigma+r+1)\int_t^\infty \frac{X_\sigma(u)}{u^{\sigma+r+1}}\,du = -\Gamma(\sigma+r+1)\,\frac{X_{\sigma+1}(t)}{t^{\sigma+r+1}} + \Gamma(\sigma+r+2)\int_t^\infty \frac{X_{\sigma+1}(u)}{u^{\sigma+r+2}}\,du,$$

while $X_{\sigma+1}(t)=o(t^{\sigma+r+1})$ as $t\to+0$, by Lemma 7.

† Cf. D$_r$FS, Lemma 8.

‡ We do not need this part in the proof of Theorem 1.

§ Cf. D$_r$FS, Lemma 10, footnote.

that $G_{r+\sigma}(t)/t^{r+\sigma+1} \in (C_0 L)$, and hence by Lemma 5 that $g(t)/t \in (CL)$. Further by Lemma 7 $G_{r+\sigma}(t) = o(t^{r+\sigma})$, *i.e.* $g(t) = o(1)$ $(C, r+\sigma)$ as $t \to +0$, and hence by Lemma 4* $g(t)/t \in C_\tau T$, where $\tau = r+\sigma-1 > \kappa$.

LEMMA 12. *If constants a_ν exist such that $g(t)$, defined by (3.5), is integrable (CL) in $(0, \pi)$, then constants β_ν exist such that $G(t)$ is integrable (CL) in $(0, \pi)$, and the allied series of $g(t)$ and $G(t)$ are equi-summable (C, α), for $\alpha \geqslant 0$, where*

$$G(t) = \begin{cases} \dfrac{r!}{(2\sin\frac{1}{2}t)^r}\left\{\psi(t) - \sum_{\nu=1}^{\frac{1}{2}r} \dfrac{\beta_{2\nu-1}}{(2\nu-1)!}\sin^{2\nu-1}t\right\} & (r \text{ even}) \\ \dfrac{r!}{(2\sin\frac{1}{2}t)^r}\left\{\phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-1)} \dfrac{\beta_{2\nu}}{(2\nu)!}\sin^{2\nu}t\right. & (r \text{ odd}). \end{cases} \tag{3.7}$$

This is proved by arguments similar to ones in $D_r FS$†.

LEMMA 13. *If* (i) $G(t) \in (CL)$ *in* $(0, \pi)$, (ii) $t^r G(t) \in L$ *in* $(0, \pi)$, *where r is a positive integer, and if*

$$\beta(\mu) = \frac{2}{\pi}\int_{\to 0\,(C)}^{\pi} G(t)\sin\mu t\,dt, \tag{3.8}$$

then the series $\Sigma\beta(n)$ and $\Sigma\beta(n-\frac{1}{2}r)$ are equi-summable (C, α), for $\alpha \geqslant 0$.

This is proved by an argument similar to one in $D_r FS$‡.

LEMMA 14. *If $G(t)$ is defined by (3.7) and $\beta(\mu)$ by (3.8), then*

$$n^r\Delta^r\beta(n-\tfrac{1}{2}r) = r!(d/dx)^r(b_n\cos nx - a_n\sin nx) \tag{3.9}$$

for $n \geqslant r$.

This is similar to a result in $D_r FS$§.

LEMMA 15. *If Σu_n is summable (C), then a necessary and sufficient condition for $\Sigma n^r\Delta^r u_n$ to be summable $(C, \alpha+r)$, where $\alpha \geqslant 0$, is that Σu_n be summable (C, α).*

This is proved in $D_r FS$‖.

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* Or the remark in the footnote to Lemma 10.
† Cf. $D_r FS$, Lemmas 11 and 15.
‡ Cf. $D_r FS$, Lemma 12.
§ Cf. $D_r FS$, Lemma 13.
‖ $D_r FS$, Lemma 14.

Proof of Theorem 1. *Necessity.* Suppose that $\Sigma(d/dx)^r(b_n \cos nx - a_n \sin nx)$ is summable $(C, \alpha+r)$ to s. Then by Lemma 10 constants α_ν exist such that $g(t)/t$ is integrable (CL) in $(0, \pi)$, where $g(t)$ is defined by (3.5), while (3.4) is satisfied, the constants α_ν being unique and satisfying (3.6). It follows by the sufficiency part of Lemma 11 that $g(t)$ is integrable (CL) and its allied series is summable (C) at $t=0$.

Now, by Lemma 12, $g(t)$ defines a function $G(t)$, given by (3.7), which is integrable (CL) in $(0, \pi)$ and its allied series is also summable (C) at $t=0$, *i.e.* $\Sigma\beta(n)$ is summable (C), where $\beta(n)$ is defined by (3.8). It follows by Lemma 13 that $\Sigma\beta(n-\frac{1}{2}r)$ is summable (C). Now, by Lemma 14, $n^r\Delta^r\beta(n-\frac{1}{2}r) = r!(d/dx)^r(b_n \cos nx - a_n \sin nx)$ for $n \geqslant r$, and so by hypothesis $\Sigma n^r\Delta^r\beta(n-\frac{1}{2}r)$ is summable $(C, \alpha+r)$. Hence, by the necessity part of Lemma 15, $\Sigma\beta(n-\frac{1}{2}r)$ is summable (C, α), and, by Lemma 13, so also is $\Sigma\beta(n)$, *i.e.* the allied series of $G(t)$ is summable (C, α) at $t=0$. It follows, by Lemma 12, that the allied series of $g(t)$ is summable (C, α) at $t=0$, and we have already seen that (3.4) is satisfied.

Sufficiency. Suppose that constants α_ν exist such that the odd function $g(t)$, given by (3.5), is integrable (CL) in $(0, \pi)$, that its allied series is summable (C, α) at $t=0$, and that (3.4) is satisfied. Then $g(t)$ defines $G(t)$, given by (3.7), which is also integrable (CL) and its allied series summable (C, α) at $t=0$, *i.e.* $\Sigma\beta(n)$ is summable (C, α). Then, by Lemma 13, $\Sigma\beta(n-\frac{1}{2}r)$ is summable (C, α) and, by Lemma 14 and the sufficiency part of Lemma 15, $\Sigma(d/dx)^r(b_n \cos nx - a_n \sin nx)$ is summable $(C, \alpha+r)$. Finally by (3.4) and the necessity part of the theorem the sum is s, while the constants α_ν being unique must satisfy (3.6).

This completes the proof.

The following theorem is a corollary of Theorem 1, in the light of Lemma 11.

THEOREM 2. *If $f(t) \in L$ in $(-\pi, \pi)$, a necessary and sufficient condition that the r-th successively derived allied series of $f(t)$ should be summable (C) at $t=x$ is that there should exist constants α_ν such that $g(t)/t$, given by* (3.5) *is integrable (CL) in $(0, \pi)$.*

4. *Supplementary remarks.*

If we write

$$k_r(t) = \begin{cases} \dfrac{r!}{t^r}\left\{\phi(t) - \displaystyle\sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{\alpha_{2\nu}}{(2\nu)!}\, t^{2\nu}\right\} & (r \text{ even}), \\ \dfrac{r!}{t^r}\left\{\psi(t) - \displaystyle\sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu-1}}{(2\nu-1)!}\, t^{2\nu-1}\right\} & (r \text{ odd}), \end{cases} \tag{4.1}$$

i.e.

$$k_r(t) = \frac{r!}{t^r}\,\tfrac{1}{2}\{f(x+t)+(-1)^r f(x-t)\} - \sum_{\nu=1}^{[\frac{1}{2}r]} \frac{a_{r-2\nu}}{(r-2\nu)!}\, t^{r-2\nu}, \qquad (4.2)$$

the constants a_0, a_2, ..., or a_1, a_3, ..., occurring in Theorem 1 or in D_rFS, Theorems 2 and 3, may be defined inductively by the relations

$$a_\nu = (C)\text{-}\lim_{t\to 0} k_\nu(t) \qquad (4.3)$$

for $\nu = r-1, r-3, \ldots$, or $\nu = r-2, r-4, \ldots$, respectively. These relations follow, by D_rFS, Theorem 3 and Lemma 10, footnote, from the summability (C) of $\Sigma(d/dx)^r(a_n \cos nx + b_n \sin nx)$ for $\nu = r-1, r-3, \ldots$, under the hypotheses of Theorem 1, or for $\nu = r-2, r-4, \ldots$, under the hypotheses of D_rFS, Theorems 2 and 3.

If we now write $f_{(\nu)}(x) = a_\nu$ we see that $f_{(\nu)}(x)$ is a generalisation of the ν-th de la Vallée Poussin symmetric derivative of $f(x)$. A crude form of the main result in D_rFS may then be expressed as follows.

THEOREM 3. *If $f(t) \in L$ a necessary and sufficient condition that its r-th derived Fourier series should be summable (C) to s at $t = x$ is that $f_{(r)}(x)$ should exist and have the value s.*

We may also write*

$$\tilde{f}_{(r)}(x) = \frac{2}{\pi} \int_{\to 0\,(C)}^{\infty} \frac{\bar{k}_r(t)}{t}\, dt, \qquad (4.4)$$

where $\bar{k}_r(t) = g(t)$, as defined by (3.5), with $a_\nu = f_{(\nu)}(x)$. Then a crude form of Theorem 1 may be expressed as follows.

THEOREM 4. *If $f(t) \in L$ a necessary and sufficient condition that its r-th derived allied series should be summable (C) to s at $t = x$ is that $\tilde{f}_{(r)}(x)$ should exist and have the value s.*

If $f^{(r-1)}(t)$ is absolutely continuous the r-th derived Fourier series and allied series of $f(t)$ are respectively the Fourier series and allied series of $f^{(r)}(t)$. Writing

$$k(t) = \tfrac{1}{2}\{f^{(r)}(x+t)+f^{(r)}(x-t)\}, \quad \bar{k}(t) = \tfrac{1}{2}\{f^{(r)}(x+t)-f^{(r)}(x-t)\}, \qquad (4.5)$$

we have†

$$k_r(t) = r!\,t^{-r} K_r(t), \quad \bar{k}_r(t) = r!\,t^{-r} \bar{K}_r(t), \qquad (4.6)$$

i.e. $k_r(t)$ and $\bar{k}_r(t)$ are the mean values of order r of $k(t)$ and $\bar{k}(t)$ respectively.

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* Cf. Moursund, *loc. cit.*

† $k_r(t)$ and $\bar{k}_r(t)$ being defined as in (4.1) and (4.4) and $K_r(t)$, $\bar{K}_r(t)$ as in (1.1).

We also have*

$$\frac{2}{\pi}\int_{\to 0\,(C)}^{\infty}\frac{\bar{k}_r(t)}{t}\,dt=\frac{2r!}{\pi}\int_{\to 0\,(C)}^{\infty}\frac{\bar{K}_r(t)}{t^{r+1}}\,dt$$

$$=\frac{2}{\pi}\int_{\to 0\,(C)}^{\infty}\frac{\bar{k}(t)}{t}\,dt.$$

The main theorems of D_rFS and the present paper then reduce, in this special case, to the following known results†.

THEOREM 5. *If $f(t)\in L$ a necessary and sufficient condition that its Fourier series should be summable $(C, \alpha+r)$ at $t=x$, where $\alpha\geqslant 0$ and r is a positive integer, is that $r!\,t^{-r}\Phi_r(t)$ should be integrable C_0L‡ and its Fourier series summable (C, α) to s at $t=0$.*

THEOREM 6. *If $f(t)\in L$ a necessary and sufficient condition that its allied series should be summable $(C, \alpha+r)$ to s at $t=x$ is that $r!\,t^{-r}\Psi_r(t)$ should be integrable C_0L, its allied series summable (C, α) and*

$$\frac{2}{\pi}\int_{\to 0\,(C)}^{\infty}\frac{\psi(t)}{t}\,dt=s. \qquad (4.7)$$

We end by obtaining an alternative to (3.4), with the integration taken over the range $(0, \pi)$ instead of $(0, \infty)$. We write $g(t)=r!\,t^{-r}\chi(t)$, and let $\chi_1(t)=\chi(t)$ in $(-\pi, \pi)$, $\chi_1(t+2\pi)=\chi_1(t)$. Then writing

$$\frac{2}{\pi}\int_{\to 0\,(C)}^{\infty}\frac{g(t)}{t}\,dt=\frac{2r!}{\pi}\left\{\int_{\to 0\,(C)}^{\pi}\frac{\chi(t)}{t^{r+1}}\,dt+\int_{\pi}^{\infty}\frac{\chi_1(t)}{t^{r+1}}\,dt+\int_{\pi}^{\infty}\frac{\chi(t)-\chi_1(t)}{t^{r+1}}\,dt\right\}$$

$$=J_1+J_2+J_3,$$

we have§

$$J_2=\frac{r!}{\pi}\sum_{-\infty}^{\infty}{}'\int_{-\pi}^{\pi}\frac{\chi(t)}{(t+2k\pi)^{r+1}}\,dt$$

$$=\frac{2}{\pi}\int_0^{\pi}\chi(t)\left\{(-1)^r\left(\frac{d}{dt}\right)^r\tfrac{1}{2}\cot\tfrac{1}{2}t-r!\,t^{-r-1}\right\}dt,$$

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* Cf. Lemma 5 or CL, Theorem 22.

† FS, Theorems 2 and 3a. Here $f(t)$, $\phi(t)$, $\psi(t)$ play the rôle of $f^{(r)}(t)$, $k(t)$, $\bar{k}(t)$.

‡ This is a consequence of the integrability (CL) of $r!\,t^{-r}\Phi_r(t)$ and C_0L of $\phi(t)$. Cf. CL, Theorem 16.

§ Cf. G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1924), 211–246. We write

$$\sum_{-\infty}^{\infty}{}' \quad \text{for} \quad \lim_{K\to\infty}\left(\sum_{-K}^{-1}+\sum_{1}^{K}\right).$$

by uniform convergence, while*

$$J_3 = \frac{2r!}{\pi}\left(\tfrac{1}{2}\sum_{-\infty}^{\infty}{}' \int_{-\pi}^{\pi} \frac{1}{(t+2k\pi)^{r+1}} - \int_{\pi}^{\infty} \frac{1}{t^{r+1}}\right) \sum_{\nu=0}^{[\frac{1}{2}(r-1)]} \frac{a_{r-1-2\nu}}{(r-1-2\nu)!}\, t^{r-1-2\nu}\, dt$$

$$= \frac{2}{\pi} \sum_{\nu=0}^{[\frac{1}{2}(r-2)]} \frac{a_{r-1-2\nu}}{(r-1-2\nu)!}$$

$$\times\left(\int_0^{\pi} t^{r-1-2\nu}\left\{(-1)^r\left(\frac{d}{dt}\right)^r \tfrac{1}{2}\cot\tfrac{1}{2}t - r!\, t^{-r-1}\right\} dt - r! \int_{\pi}^{\infty} t^{-2-2\nu}\, dt\right)$$

$$= \sum_{\nu=0}^{[\frac{1}{2}(r-2)]} a_{r-1-2\nu} \sum_{\mu=\nu}^{[\frac{1}{2}(r-2)]} \frac{\pi^{2\mu-2\nu}}{(2\mu+1-2\nu)!} \left\{\left(\frac{d}{dt}\right)^{2\mu+1} \cot\tfrac{1}{2}t\right\}_{t=\pi}$$

$$= C_r.$$

It follows that (3.4) may be replaced by

$$\frac{(-1)^r}{\pi} \int_{\to 0\ (C)}^{\pi} \chi(t)\left(\frac{d}{dt}\right)^r \cot\tfrac{1}{2}t\, dt + C_r = s \qquad (4.8)$$

This includes a formula obtained by Moursund† in the case when the de la Vallée Poussin derivatives and the integral exist as ordinary limits.

If $f^{(r-1)}(t)$ is absolutely continuous (4.8) may be replaced by

$$\frac{1}{2\pi} \int_{\to 0\ (C)}^{\pi} \{f^{(r)}(x+t) - f^{(r)}(x-t)\} \cot\tfrac{1}{2}t\, dt = s, \qquad (4.9)$$

which is connected to (4.8) by r formal integrations by parts‡.

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* When $r = 1$, $J_3 = 0$.

† Moursund, *loc. cit.*

‡ Cf. Moursund, *loc. cit.*

THE INDUCTION OF CURRENTS IN INFINITE PLANE CURRENT-SHEETS (II)

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1. *Introduction.*

In this paper I complete the discussion (begun in an earlier paper*) of the induction of electric currents in an infinite uniform plane current-sheet; in (I) the problem was considered with special reference to the currents induced in the sheet by a magnetic system moving uniformly towards the sheet. In the first part of this paper we consider the more general problem of the induction of currents by an aperiodic electromagnetic field, supposing this to have been varying for an infinite time before the instant considered, and supposing the sheet to conduct according to Ohm's Law. Formal expressions will be given for the electromagnetic field variables due to the currents induced in the sheet, but the results are in general too complicated to be of physical interest. The electric polarisation of the sheet in the case of the system considered in (I), and the case of an electrostatic system in uniform relative motion to the sheet, are discussed in some detail.

The second part deals with the modifications of the theory necessary when the currents in the sheet flow without resistance. Assuming that the inducing electromagnetic field was initially zero at an infinitely remote epoch, it will be shown that the constitutive equation for the current, as was mentioned in (I), is identical with the equation of conduction for super-conductors. The relevant equations of the problem are derived and a formal solution given. The interpretation of this is simpler than in the case when the currents flow according to Ohm's Law.

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* *Proc. London Math. Soc.* (2), 46 (1940), 99–112; this paper will be referred to as (I).

PART I.

Theory of an infinite uniform plane current-sheet.

(*Flow according to Ohm's Law.*)

2. For the sake of completeness, we shall briefly state some of the relevant equations in (I) (§§ 2–3 . 2), and in addition we shall derive the expression for the electric polarisation of the sheet, which was omitted in (I): as in (I), the plane of the sheet is taken to be the plane $z = 0$ of a system of Cartesian coordinates (x, y, z); σ_x and σ_y denote the x- and y-components of the current density σ at any point of the sheet, for a system of currents flowing in the sheet. We shall denote further by ν the surface charge-density of the polarisation in the sheet. The vector potential $\mathbf{A}^{(s)}$ of the electromagnetic field due to the currents flowing in the sheet, at the field point $(\mathbf{r}, t)$ is taken to be

$$\mathbf{A}^{(s)} = \int \frac{[\sigma]\, dS}{|\mathbf{r}-\mathbf{r}_1|}, \tag{1}$$

where $\mathbf{r}_1$ denotes the position vector of a point of the sheet and $[\sigma]$ the "retarded" value of σ for that point. The scalar potential $\phi^{(s)}$ due to the polarisation of the sheet is likewise taken to be

$$\phi^{(s)} = \int \frac{[\nu]\, dS}{|\mathbf{r}-\mathbf{r}_1|}, \tag{2}$$

where $[\nu]$ is the retarded value of ν at the point $\mathbf{r}_1$ of the sheet; as noted in (I), σ and hence $\mathbf{A}^{(s)}$ possess no z-component.

The magnetic field $\mathbf{H}^{(s)}$ due to the currents in the sheets has components given by

$$H_x^{(s)} = -\partial A_y^{(s)}/\partial z, \tag{3}$$

$$H_y^{(s)} = \partial A_x^{(s)}/\partial z, \tag{4}$$

$$H_z^{(s)} = (\partial A_y^{(s)}/\partial x) - (\partial A_x^{(s)}/\partial y), \tag{5}$$

whilst the electric field $\mathbf{E}^{(s)}$ due to the sheet has components

$$E_x^{(s)} = -\partial A_x^{(s)}/\partial t - \partial \phi^{(s)}/\partial x, \tag{6}$$

$$E_y^{(s)} = -\partial A_y^{(s)}/\partial t - \partial \phi^{(s)}/\partial y, \tag{7}$$

$$E_z^{(s)} = \qquad -\partial \phi^{(s)}/\partial z. \tag{8}$$

As shown in (I), $\mathbf{A}^{(s)}$ and $\phi^{(s)}$ are even functions of z, so that $H_x^{(s)}$, $H_y^{(s)}$, $E_z^{(s)}$ are odd functions of z. The magnetic field due to the currents in the

sheet on one side of it is therefore the negative* mirror image of that on the other side, whilst the electric field due to the polarisation is the positive mirror image of that on the other.

We denote by $f^{(i)}$ and $f^{(s)}$ the contributions to any field variable f arising from the inducing system and from the currents in the sheet, so that

$$f = f^{(i)} + f^{(s)}.$$

Then the electromagnetic field at any point may be derived from a scalar potential ϕ and a vector potential A, the latter being chosen so as to have no z-component in conformity with (1). It will be sometimes convenient to denote the four vector (A, ϕ) by the symbol M and we remark that M is arbitrary to the extent of a four vector $(-\text{grad}\,\psi, \partial\psi/c\,\partial t)$, where ψ satisfies the wave equation

$$\nabla^2\psi = \partial^2\psi/c^2\,\partial t^2.$$

The boundary conditions† to be satisfied are‡

$$\left.\begin{aligned} (H_x)_+ - (H_x)_- &= 4\pi\sigma_y, \quad (9)\\ (H_y)_+ - (H_y)_- &= -4\pi\sigma_x, \quad (10)\\ (E_x)_+ - (E_z)_- &= 4\pi\nu. \quad (11) \end{aligned}\right\} \quad (z = 0)$$

Since the field variables for the inducing system are continuous across the sheet, the discontinuities in the components of E and H in (9)–(11) may be replaced by the discontinuities in the components of $\mathrm{E}^{(s)}$ and $\mathrm{H}^{(s)}$. Also, from what has been said above,

$$(E_z^{(s)})_+ = -\partial\phi^{(s)}/\partial|z| = -(E_z^{(s)})_-$$

so that (11) may be rewritten

$$\partial\phi^{(s)}/\partial|z| = -2\pi\nu \quad (z = 0), \quad (12)$$

and, as in (I),

$$\partial\mathbf{A}^{(s)}/\partial|z| = -2\pi\sigma \quad (z = 0). \quad (13)$$

These are the required boundary conditions.

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* This was not made clear in (I), § 2.

† (I), equations (8), (9) and reference given therein.

‡ $(f)_+$ and $(f)_-$ denote the value of a variable f on the positive and negative sides of the sheet respectively.

3. *The differential equations of the problem.*

The system producing the inducing electromagnetic field $M^{(i)}(x, y, z, t)$ will be supposed to lie on the positive side of the sheet (*i.e.* the region defined by $z \geqslant 0$) and to be such that $M^{(i)}$ and its derivatives vanish for $t = -\infty$, and that if t is finite, these become vanishingly small as x, y, z become infinitely large; then $M^{(i)}$ possesses no singularities in this region $z \leqslant 0$. Let $\bar{M}^{(i)}(x, y, z, t)$ be defined by

$$\bar{M}^{(i)} = M^{(i)}(x, y, -|z|, t). \tag{14}$$

This may then be termed the conjugate function of $M^{(i)}$ since it is the value of this function at the image-point in the sheet. There $M^{(i)}$ and $\bar{M}^{(i)}$ are equal, together with their derivatives with respect to x, y, t, for $z = 0$; moreover $\bar{M}^{(i)}$ satisfies the wave equation and possesses no singularities in any region of space.

If R (e.m.u.) denotes the resistance per unit area of the sheet, (13) becomes, by (6), (7),

$$\frac{\partial \mathbf{A}^{(s)}}{\partial |z|} = \frac{2\pi}{R}\left(\frac{\partial \mathbf{A}}{\partial t} + c\frac{\partial \phi}{\partial \boldsymbol{\alpha}}\right) \quad (z = 0), \tag{15}$$

where $\partial/\partial\boldsymbol{\alpha}$ denotes the vector operator $(\partial/\partial x, \partial/\partial y, 0)$, and where

$$\phi = \phi^{(i)} + \phi^{(s)}, \qquad \mathbf{A} = \mathbf{A}^{(i)} + \mathbf{A}^{(s)}.$$

We shall now show that the equations (34), (37) of (I) are valid for all aperiodic fields defined as in the beginning of this paragraph; the proof is essentially as given in § 5 of (I), but as this was given for a particular case only we repeat it here for the sake of completeness.

Consider the function u defined by

$$u \equiv \frac{\partial \mathbf{A}^{(s)}}{\partial |z|} - \frac{2\pi}{R}\left(\frac{\partial \mathbf{A}^{(s)}}{\partial t} + \frac{\partial \bar{\mathbf{A}}^{(i)}}{\partial t} + c\frac{\partial \phi^{(s)}}{\partial \boldsymbol{\alpha}} + c\frac{\partial \bar{\phi}^{(i)}}{\partial \boldsymbol{\alpha}}\right);$$

u satisfies the wave equation, possesses no singularities in the region $z > 0$ and, by (15), vanishes for $z = 0$ for all values of the time t; it also vanishes for $t = -\infty$. Since we require a solution $M^{(s)}$ which vanishes at infinity, let us suppose that $u \to 0$ as $r \to \infty$ (where $r^2 = x^2 + y^2 + z^2$) by an order higher than $1/r$. It then follows by the uniqueness theorem* for such a wave function that u vanishes everywhere in the region $z \geqslant 0$. Hence

$$\frac{\partial \mathbf{A}^{(s)}}{\partial |z|} = \frac{2\pi}{R}\left(\frac{\partial \mathbf{A}^{(s)}}{\partial t} + \frac{\partial \bar{\mathbf{A}}^{(i)}}{\partial t} + c\frac{\partial \phi^{(s)}}{\partial \boldsymbol{\alpha}} + c\frac{\partial \bar{\phi}^{(i)}}{\partial \boldsymbol{\alpha}}\right); \tag{16}$$

* A. E. H. Love, *Proc. London Math. Soc.* (2), 1 (1903), 37, 42.

operating with div and remembering that, for any four vector $(\mathbf{A}, \phi)$, we have

$$\operatorname{div}\mathbf{A}+\partial\phi/c\,\partial t=0,$$

we find that

$$\frac{\partial^2\phi^{(s)}}{\partial|z|\partial t}=\frac{2\pi c^2}{R}\frac{\partial^2}{\partial|z|^2}(\phi^{(s)}+\bar{\phi}^{(i)}).$$

Integrating this equation with respect to $|z|$ from infinity to $|z|$, we find

$$\frac{\partial\phi^{(s)}}{\partial t}=\frac{2\pi c^2}{R}\left(\frac{\partial\phi^{(s)}}{\partial|z|}+\frac{\partial\bar{\phi}^{(i)}}{\partial|z|}\right), \tag{17}$$

which is the other equation of the problem.

4. Let us now consider the decay of a system of currents and charges in the sheet under the action of resistance and self-induction; to examine this effect we may suppose that at any given instant the inducing system $M^{(i)}$ ceases to act. We must then solve (16) and (17) for the case in which $\bar{\phi}^{(i)}$ and $\bar{\mathbf{A}}^{(i)}$ are both zero, viz.:

$$\frac{\partial\phi^{(s)}}{\partial t}=\frac{2\pi c^2}{R}\frac{\partial\phi^{(s)}}{\partial|z|}, \tag{18}$$

$$\frac{\partial\mathbf{A}^{(s)}}{\partial t}=\frac{2\pi}{R}\left(\frac{\partial\mathbf{A}^{(s)}}{\partial t}+c\,\frac{\partial\phi^{(s)}}{\partial\boldsymbol{\alpha}}\right). \tag{19}$$

The general solution of (18) is

$$\phi^{(s)}=E(x, y, |z|+2\pi c^2 t/R); \tag{20}$$

following Maxwell, this may be interpreted as follows: suppose that at the instant at which $M^{(i)}$ ceases to act $\phi^{(s)}\equiv E(x, y, z)$ over the sheet; then, by (12), at a later time the decay of the electric polarisation can be obtained by moving this field in the negative direction perpendicularly to itself with speed $2\pi c^2/R$. In the case of a perfectly conducting sheet, $R=0$, and the speed is infinite: the polarisation would therefore vanish immediately (a result agreeing with the general theory of perfect conductors). The fact that this speed may exceed the speed of light is not in contradiction with relativity, since we are not here concerned with a material sheet.

Eliminating $\phi^{(s)}$ between (18) and (19), we obtain a differential equation of the second order for $\mathbf{A}^{(s)}$, the general solution of which is

$$\mathbf{A}^{(s)}=\mathbf{F}(x, y, |z|+Rt/2\pi)+\mathbf{G}(x, y, |z|+2\pi c^2 t/R), \tag{21}$$

where F and G are arbitrary functions (possessing no z-components), provided that $R/2\pi \neq c$.

Since the solution of (18) and (19) contain, at most, two arbitrary functions, there is a relation connecting E, F, G. By substituting (21) in (19) this is found to be

$$\mathrm{G} = -\frac{R/2\pi c}{1-(R/2\pi c)^2}\int \frac{\partial E}{\partial a}\, d|z|. \tag{22}$$

In (21) the second term represents the contribution due to the current induced by the electric field set up by the polarisation of the sheet; the first term is that given by Maxwell. Thus Maxwell's approximate solution is exact in the case when $E = 0$; in this case, supposing $\mathrm{F}(x, y, z)$ to be the value of $\mathrm{A}^{(s)}$ at the instant when $\mathrm{A}^{(i)}$ ceases to act, the field at later times can be obtained by moving this field in the negative direction perpendicularly to itself with speed $R/2\pi$.

We shall write hereafter

$$R/2\pi = w, \tag{23}$$

$$2\pi c^2/R(= c^2/w) = w'. \tag{24}$$

In the case when $w = c$, (21) must be replaced by

$$\mathrm{A}^{(s)} = \mathrm{F}(x, y, |z|+wt)+|z|\,\mathrm{G}(x, y, |z|+w' t) \tag{25}$$

and (22) by

$$\mathrm{G} = \partial E/\partial a. \tag{26}$$

5. *Solution of the equations.*

In the interesting discussion in Ch. XII of his *Treatise on electricity and magnetism*, Maxwell has shown how the result of the last paragraph may be utilized to obtain the solution of the equations of the problem. We shall not follow this procedure here, but shall give a formal solution following the lines of the solution in (I).

We rewrite (17) as

$$\frac{\partial \phi^{(s)}}{\partial |z|} - \frac{w}{c^2}\frac{\partial \phi^{(s)}}{\partial t} = -\frac{\partial \phi^{(i)}}{\partial |z|}$$

and denote by $\bar{\phi}_\tau^{(i)}$ the function derived from $\bar{\phi}^{(i)}$ by replacing $|z|$ and t respectively by $z+\tau$ and $t-w\tau/c^2$. Then following the argument given

in (I), § 7.1, the solution of this equation is found to be

$$\phi^{(s)} = -\int_0^\infty \frac{\partial \bar{\phi}_\tau^{(i)}}{\partial |z|}\, d\tau. \tag{27}$$

The equation for the vector potential $\mathbf{A}^{(s)}$ may be written

$$\frac{\partial \mathbf{A}^{(s)}}{\partial |z|} - \frac{2\pi}{R}\frac{\partial \mathbf{A}^{(s)}}{\partial t} = \frac{2\pi c}{R}\left(\frac{\partial \phi^{(s)}}{\partial \boldsymbol{a}} + \frac{\partial \bar{\phi}^{(i)}}{\partial \boldsymbol{a}}\right) + \frac{2\pi}{R}\frac{\partial \mathbf{A}^{(i)}}{\partial t}.$$

Let $\bar{\mathbf{A}}_\sigma^{(i)}$ denote the function derived from $\bar{\mathbf{A}}^{(i)}$ by replacing $|z|$ and t by $|z|+w\sigma$ and $t-\sigma$ respectively, and let $\bar{\phi}_\sigma^{(i)}$ and $\bar{\phi}_{\tau\sigma}^{(i)}$ be similarly constructed from $\bar{\phi}^{(i)}$ and $\bar{\phi}_\tau^{(i)}$. Then as in (I), § 7.2, the solution of this equation is

$$\mathbf{A}^{(s)} = -\int_0^\infty \frac{\partial \bar{\mathbf{A}}_\sigma^{(i)}}{\partial t}\, d\sigma - c\int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}}{\partial \boldsymbol{a}}\, d\sigma + c\int_0^\infty d\sigma \int_0^\infty \frac{\partial^2 \bar{\phi}_{\tau\sigma}^{(i)}}{\partial \boldsymbol{a}\, \partial |z|}\, d\tau. \tag{28}$$

When $w \neq c$, the double integral on the right-hand side of (28) may be replaced by the sum of two ordinary integrals. We then find

$$\mathbf{A}^{(s)} = -\int_0^\infty \frac{\partial \bar{\mathbf{A}}_\sigma^{(i)}}{\partial t}\, d\sigma + \frac{w^2/c}{1-w^2/c^2}\int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}}{\partial \boldsymbol{a}}\, d\sigma - \frac{w/c}{1-w^2/c^2}\int_0^\infty \frac{\partial \bar{\phi}_\tau^{(i)}}{\partial \boldsymbol{a}}\, d\tau. \tag{29}$$

In this equation the first term is the same as the one given by Maxwell; the remaining two terms are due to the polarisation of the sheet.

These results may be interpreted in terms of Maxwell's moving trail of images; in this case there are two such moving trails of images. But whereas in Maxwell's approximate solution the interpretation is comparatively simple, the polarisation terms in the exact solution render this interpretation too complicated to be of interest. For this reason we shall not consider it further here.

6. We return to consider the case of a perfectly conducting sheet: as has been pointed out in (I), the solution in this case is extremely simple: we have

$$\phi^{(s)} = -\bar{\phi}^{(i)}, \quad \mathbf{A}^{(s)} = -\bar{\mathbf{A}}^{(i)}, \tag{3}$$

showing that the electromagnetic field is completely reflected by the sheet. We shall consider the electric polarisation produced in the sheet in a simple case, principally because it is in this that the exact solution differs from Maxwell's. We consider the case of a magnetic pole moving uniformly towards the sheet. The solution has been given in (I), to which

reference must be made for details*. The electric field due to the sheet was not, however, considered explicitly; it is easily seen that on the positive side of the sheet this is given by

$$\mathbf{E}^{(s)} = -\frac{1}{c}\,\bar{\mathbf{U}} \wedge \mathbf{H}^{(s)}, \tag{31}$$

where $\bar{\mathbf{U}}$ is the mirror image of the velocity vector $\mathbf{U}$ of the moving pole in the sheet. This result may also be readily obtained by applying the Lorentz transformation to this field due to the moving image of the pole.

If the direction of U is perpendicular to the sheet, then $E^{(s)}$ has no component perpendicular to the sheet and so there is no polarisation in this case; this has already been shown in (I). The polarisation is greatest when the pole moves parallel to the sheet; we shall restrict ourselves to this case to obtain some idea of the distribution of the charges.

The surface charge density on the positive face of the sheet is given by (12), and using (32), this gives

$$\nu = UH_x/(2\pi c), \tag{32}$$

so that ν is of the first order in U/c. Putting $l = 0$, $m = 0$, $R = 0$, in (61) and (62), or by considering the image of the pole in the sheet, we find

$$\nu = Ux/(2\pi c)\,.\{x^2+\kappa^2(y-z_0-Ut)^2+z^2\}^{-\frac{3}{2}}, \tag{33}$$

where $\kappa = (1-U^2/c^2)^{-\frac{1}{2}}$.

Further, ignoring for simplicity the relativity correction, and taking axes moving with the pole (writing $Y = y-z_0-Ut$), we have

$$\nu \sim \frac{U}{2\pi c}\cdot\frac{x}{(x^2+Y^2+z^2)^{\frac{3}{2}}}. \tag{34}$$

The lines of equidensity of charge are given by $\nu = \text{const.}$, or, say

$$x = \lambda(x^2+Y^2+z^2)^{\frac{3}{2}},$$

where λ is a constant. Using polar coordinates defined by $x = \rho\cos\theta$, $Y = \rho\sin\theta$, this becomes

$$\rho\cos\theta = \lambda(\rho^2+z^2)^{\frac{3}{2}},$$

* The second term in (61) of (I) was incorrectly given; it should read

$$\kappa m U(l^2+m^2\kappa^{-2})\int_0^\infty d\sigma\int_0^\infty \frac{\partial^2}{\partial\alpha\,\partial y}(\alpha^2+\beta_{\tau\sigma}^2+\gamma_{\tau\sigma}^2)^{-\frac{1}{2}}d\tau;$$

also the last term in (31) should read $-\partial B/\partial x$.

and since $|\cos\theta| \leqslant 1$, we find that

$$\lambda < \frac{2}{3z^2\sqrt{3}}.$$

The lines of equidensity shown in Fig. 1 have been drawn taking $z = 1$; the Y-axis divides the positively charged from the negatively charged regions of the sheet. The curves have two foci corresponding to the points of maximum density of charge; they are the points $(\pm 0.707, 0)$ and correspond to $\lambda = \pm\frac{2}{3}\sqrt{3}$.

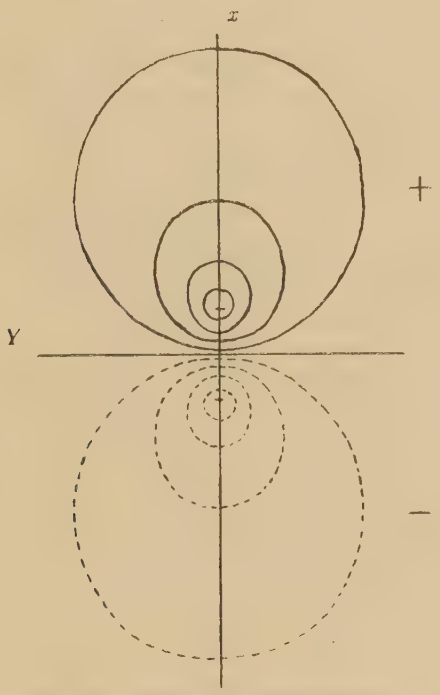

Fig. 1.

When the pole is moving obliquely to the sheet, the curves are no longer symmetrical about the x-axis, but otherwise will be similar to these. It is to be noted that in the case when a magnetic system moves parallel to the sheet with constant velocity the field is carried with the system.

The case where the pole is replaced by a magnetic doublet whose axis is parallel to the sheet and at right angles to the direction of motion (this being also parallel to the sheet) is of some interest in connection with geomagnetic storms. The sheet being perfectly conducting as before, the

lines of equidensity of charge are similarly found to be given by

$$Y^2+z^2-2x^2=\lambda(x^2+Y^2+z^2)^{\frac{5}{2}}, \tag{35}$$

where λ is a constant. The curve dividing the positive from the negative regions of the sheet is the hyperbola

$$Y^2-2x^2+z^2=0,$$

z being here the (constant) distance of the doublet from the sheet.

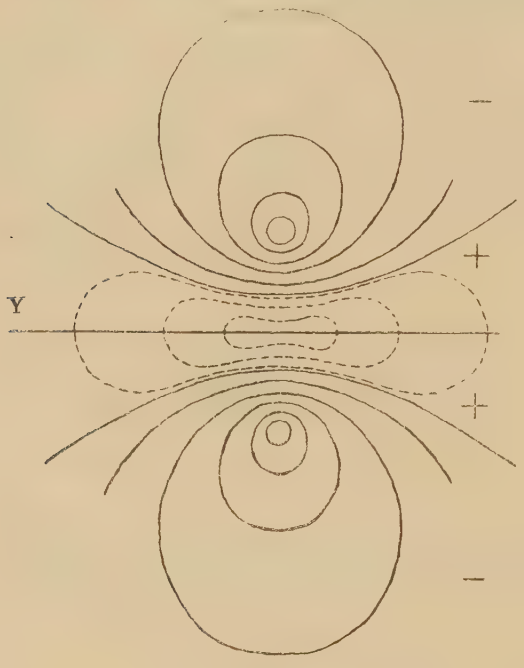

Fig. 2.

Introducing polar coordinates as before, (35) may be written

$$3\cos^2\theta=\frac{\rho^2+z^2}{\rho^2}\,[1-\lambda(\rho^2+z^2)^{\frac{3}{2}}].$$

Since the left-hand side of this equation is positive, if $\lambda>0$, we must have

$$0\leqslant\rho^2\leqslant\lambda^{-\frac{2}{3}}-z^2. \tag{36}$$

Thus the curves of equidensity in the positively charged regions are closed and, for any given value of z, there is an upper limit to the value of λ.

If $\lambda<0$, say $\lambda=-\lambda_1$, then, since $\cos^2\theta\leqslant 1$, we must have

$$2\rho^2-z^2\geqslant\lambda_1(\rho^2+z^2)^{\frac{5}{2}}; \tag{37}$$

hence $\rho^2 > z^2/2$ and, writing also the left-hand side of (37) as $2(\rho^2+z^2)-3z^2$, we have

$$\lambda_1(\rho^2+z^2)^{\frac{5}{2}} \leqslant 2.$$

Thus $\rho^2 \leqslant (2/\lambda_1)^{\frac{2}{3}}-z^2$ also, and it follows that if $\lambda < 0$, ρ is bounded above and below. The curves are again closed, and are roughly as indicated in the Fig. 2.

7. *An electrostatic system in uniform motion.*

The solution of the problem of an electrostatic system moving with uniform velocity U from infinity towards the sheet will now be given. In what follows the notation used is as in (I), and reference to this paper must be made for details, especially § 3.3. We have in this case

$$\mathbf{A}^{(i)\prime} = 0, \quad \phi^{(i)\prime} = \phi^{(i)\prime}(x', y', z'); \tag{38}$$

hence by (I) (19) and (2) we have

$$A_{1x}^{(i)} = 0 = A_x^{(i)\prime}, \quad A_{1y}^{(i)} = 0 = A_y^{(i)\prime}, \quad A_{1z}^{(i)} = -\kappa U\phi^{(i)\prime}/c, \quad \phi_1^{(i)} = \kappa\phi^{(i)\prime}. \tag{39}$$

Hence, by (I) (15),

$$\begin{cases} A_x^{(i)} = A_{1x}^{(i)} = 0, \quad A_y^{(i)} = lA_{1y}^{(i)} - mA_{1z}^{(i)} = -mA_{1z}^{(i)}, \\ A_z^{(i)} = mA_{1z}^{(i)} + lA_{1z}^{(i)} = lA_{1z}^{(i)}, \quad \phi^{(i)} = \phi_1^{(i)}. \end{cases} \tag{40}$$

Since we require that the vector potentials shall possess no z-component, we choose a new four-vector $(\mathcal{A}, \Phi)$ such that (cf. I, § 4)

$$\mathcal{A} = \mathbf{A} - \operatorname{grad} f, \quad \Phi = \phi + \frac{1}{c}\frac{\partial f}{\partial t}, \tag{41}$$

where

$$f = l\int A_{1z}\,dz = -(lU/c)\int \phi\,dz. \tag{42}$$

Thus we have

$$\mathcal{A}_x^{(i)} = -\frac{\partial f^{(i)}}{\partial x}, \quad \mathcal{A}_y^{(i)} = -mA_{1z} - \frac{\partial f^{(i)}}{\partial y}, \quad \mathcal{A}_z^{(i)} = 0, \quad \Phi^{(i)} = \phi^{(i)} + \frac{1}{c}\frac{\partial f^{(i)}}{\partial t}; \tag{43}$$

it will be convenient to write

$$'P = \int \phi^{(i)} dz \tag{44}$$

and we then find

$$\mathcal{A}_x^{(i)} = \frac{lU}{c}\frac{\partial P}{\partial x}, \quad \mathcal{A}_y^{(i)} = \frac{mU\phi}{c} + \frac{lU}{c}\frac{\partial P}{\partial y}, \quad \mathcal{A}_z^{(i)} = 0, \quad \Phi^{(i)} = \phi^{(i)} - \frac{lU}{c^2}\frac{\partial P}{\partial t}. \tag{45}$$

Defining $\bar{\mathcal{A}}^{(i)}$, $\bar{\Phi}^{(i)}$ as in § 3, and defining $\bar{P}$ by

$$\bar{P} = \int \bar{\phi}^{(i)} dz, \tag{46}$$

we have, from (17), (45), (46), the following equation for the scalar potential $\phi^{(s)}$ due to the sheet:

$$\frac{\partial \phi^{(s)}}{\partial t} = w' \left(\frac{\partial \phi^{(s)}}{\partial |z|} + \frac{\partial \bar{\phi}^{(i)}}{\partial |z|} + \frac{lU}{c^2} \frac{\partial \bar{\phi}^{(i)}}{\partial t} \right). \tag{47}$$

To solve this equation, write

$$\phi^{(s)} = -\bar{\phi}^{(i)} + \psi. \tag{48}$$

In the right-hand side the first term represents the negative image of the system so that ψ represents the correction required in the case when the sheet is not perfectly conducting. Substituting this in (47) we find

$$\frac{\partial \psi}{\partial |z|} - \frac{1}{w'} \frac{\partial \psi}{\partial t} = -\frac{1}{c^2} (w + lU) \frac{\partial \bar{\phi}^{(i)}}{\partial t}.$$

Proceeding as before (I, § 7.1) we find the solution of this equation to be

$$\psi = \frac{1}{c^2} (w + lU) \int_0^\infty \frac{\partial \bar{\phi}_\tau^{(i)}}{\partial \tau} d\tau, \tag{49}$$

where all variables are defined as before.

The vector potential $\mathbf{A}^{(s)}$ due to the sheet satisfies the equation

$$w \frac{\partial \mathbf{A}^{(s)}}{\partial |z|} - \frac{\partial \mathbf{A}^{(s)}}{\partial t} = \frac{\partial \bar{\mathcal{A}}^{(i)}}{\partial t} + c \frac{\partial \phi^{(s)}}{\partial \boldsymbol{\alpha}} + c \frac{\partial \bar{\Phi}^{(i)}}{\partial \boldsymbol{\alpha}}; \tag{50}$$

using (45) and (48) the components $A_x^{(s)}$, $A_y^{(s)}$ satisfy respectively the equations

$$w \frac{\partial A_x^{(s)}}{\partial |z|} - \frac{\partial A_x^{(s)}}{\partial t} = c \frac{\partial \psi}{\partial x}, \tag{51}$$

$$w \frac{\partial A_y^{(s)}}{\partial |z|} - \frac{\partial A_y^{(s)}}{\partial t} = c \frac{\partial \psi}{\partial y} + \frac{mU}{c} \frac{\partial \bar{\phi}^{(i)}}{\partial t}. \tag{52}$$

The solution of these equations are

$$A_x^{(s)} = -\frac{1}{c} (w + lU) \int_0^\infty d\sigma \int_0^\infty \frac{\partial^2 \bar{\phi}_{\tau\sigma}^{(i)}}{\partial t \, \partial x} d\tau, \tag{53}$$

$$A_y^{(s)} = -\frac{1}{c} (w + lU) \int_0^\infty d\sigma \int_0^\infty \frac{\partial^2 \bar{\phi}_{\tau\sigma}^{(i)}}{\partial t \, \partial y} d\tau - mU \int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}}{\partial t} d\sigma, \tag{54}$$

the notation being as in § 5; in the case when $w \neq c$, the double integrals may be replaced by simple ones; we find

$$A_x^{(s)} = \frac{c(w+lU)}{c^2-w^2}\left[\int_0^\infty \frac{\partial \bar{\phi}_\tau^{(i)}}{\partial x}\, d\tau - w\int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}}{\partial x}\, d\sigma\right], \tag{55}$$

$$A_y^{(s)} = \frac{c(w+lU)}{c^2-w^2}\left[\int_0^\infty \frac{\partial \bar{\phi}_\tau^{(i)}}{\partial y}\, d\tau - w\int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}\, d\sigma}{\partial y}\right] - mU\int_0^\infty \frac{\partial \bar{\phi}_\sigma^{(i)}}{\partial t}\, d\sigma. \tag{56}$$

It will suffice here also to consider the case of a single electric charge, since an electrostatic system can always be derived from point charges as in Maxwell's theory of poles.

In this case

$$\phi^{(i)\prime} = \frac{1}{r'}, \tag{57}$$

where

$$r'^2 = x'^2+y'^2+z'^2 = x'^2+\eta^2+\zeta^2. \tag{58}$$

in the notation of (I), § 7. Since $\phi^{(i)} = \phi_1^{(i)} = \kappa\phi^{(i)\prime}$, we have

$$\phi^{(i)} = \kappa(x'^2+\eta^2+\zeta^2)^{-\frac{1}{2}};$$

so that

$$\bar{\phi}^{(i)} = \kappa(l^2+m^2\kappa^{-2})(\alpha^2+\beta^2+\gamma^2)^{-\frac{1}{2}}, \tag{59}$$

where [see I, (51)–(53)]

$$\alpha = (l^2+m^2\kappa^{-2})\,x,$$

$$\beta = y - mz_0 - mUt$$

$$\gamma = \kappa[(lmU^2/c^2)(y-mz_0)+(l^2+m^2\kappa^{-2})(|z|+lz_0)-lUt].$$

Substituting (59) in (49), (55) and (56) (or the two previous equations if $w = c$), we get the required potentials. As in the case of a magnetic pole, these expressions are in general too complicated to be of physical interest, but the case when the electric charge moves perpendicularly to the sheet is comparatively simple and the solution of some interest. Here $l = 1$ and $m = 0$, so that we have

$$\alpha = x, \quad \beta = y, \quad \gamma = \kappa(|z|+z_0-Ut).$$

We then find

$$\phi^{(s)} = -\frac{w'}{U+w'}\,\bar{\phi}^{(i)}, \tag{60}$$

an expression which may be compared with (I), § 8 (63). Using (59), this may be written

$$\phi^{(s)} = -\frac{w'}{U+w'} \cdot \frac{1}{\kappa^2 \bar{r}}, \tag{61}$$

where [as in (I), § 8],

$$\bar{r}^2 = \kappa^{-2}(x^2+y^2)+(|z|+z_0-Ut)^2.$$

Some simple cases may be considered: if $U \to 0$, then $\phi^{(s)} \to -\bar{\phi}^{(i)}$, as should be the case. If $U \to c$, $\kappa^{-2} \to 0$, so that $\phi^{(s)} \to 0$ also and the field becomes vanishingly small (cf. I, § 8). Finally (cf. I, § 8), if $U/c \ll 1$, then

$$\phi^{(s)} \sim -\frac{w'}{U+w'} [x^2+y^2+(|z|+z_0-Ut)^2]^{-\frac{1}{2}},$$

so that should $w' \ll U$, the (negative) image will be much dimmed; this will be the case if the sheet is a poor conductor.

Finally, the vector potential in this case is given by

$$\mathbf{A}^{(s)} = \frac{U/c}{\kappa^2(1+U/w')} \frac{\partial}{\partial \alpha} \log\left(\gamma+\sqrt{(\alpha^2+\beta^2+\gamma^2)}\right); \tag{62}$$

the magnetic field $\mathbf{H}^{(s)}$ due to the sheet therefore has no z-component, the x- and y-components being given by

$$H_x^{(s)} = -\frac{U/c}{\kappa^2(1+U/w')} \cdot \frac{y}{(x^2+y^2+\gamma^2)^{\frac{3}{2}}}, \tag{63}$$

$$H_y^{(s)} = +\frac{U/c}{\kappa^2(1+U/w')} \cdot \frac{x}{(x^2+y^2+\gamma^2)^{\frac{3}{2}}}. \tag{64}$$

The magnetic lines of force due to the induced currents are thus circles whose planes are parallel to the sheet and whose centres lie along the path traced by the electric charge (produced indefinitely). This is otherwise obvious.

PART II.

Theory of an infinite plane current-sheet (the flow of currents being without resistance).

8. The problem of the flow of currents without resistance occasionally arises in geomagnetism, when we are dealing with (neutral) ionized gases whose density is sufficiently low for collisions to be neglected, but whose

volume is great enough for the electromagnetic inertia to be considerable. Under certain conditions, which will be considered below, the constitutive equation (which replaces Ohm's Law) is found to be

$$\boldsymbol{\sigma} = -(ne^2/mc^2)\,\mathbf{A}, \tag{65}$$

where $\boldsymbol{\sigma}$ is the current density, $\mathbf{A}$ the vector potential, and n, m, $-e$ the number density, mass and charge of the electrons which carry (or are the chief agents in carrying) the current. (We suppose that the gas is neutral, there being ionic charges present equal and opposite to the electronic charges, but on account of their much greater mass, ineffective as current carriers.)

The relation (65) is identical with that adopted in the theory of super-conductors by F. and H. London*, and the following paragraphs therefore relate also to the case of the induction of currents in an infinite plane super-conducting sheet.

We shall suppose in the present paragraph that we are concerned with a conducting medium in which the current is carried entirely by a number of electrons in it, the motion of the positive ions being supposed negligible. We shall suppose further that the inducing electromagnetic field is of the type considered in § 3, so that at an infinitely remote epoch the induced currents vanish, and that the velocity of the current-carrying electrons in the sheet is small compared with the speed of light. The theory is therefore only approximate.

Let F_α ($\alpha = 1, 2, 3$) denote the components of a field vector F in a system $S(x_1, x_2, x_3)$ of Cartesian axes, so that

$$\mathbf{F} = (F_1, F_2, F_3). \tag{66}$$

Let (x_α) denote the position vector of an electron relative to S, supposed fixed in the medium considered. Then the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are given by

$$\mathbf{E} = -\dot{\mathbf{A}}/c - \operatorname{grad}\phi, \quad \mathbf{H} = \operatorname{curl}\mathbf{A}, \tag{67}$$

in terms of the scalar and vector potentials ϕ and $\mathbf{A}$.

The equation of motion of an electron is

$$m\,d\mathbf{v}/dt = -e\mathbf{E} + e\mathbf{v} \wedge H/c,$$

* *Proc. Royal Soc.* (A), 149 (1935), 71.

and, introducing (67), the α-component may be written

$$(68) \qquad \frac{m}{e}\frac{\partial^2 x_\alpha}{\partial t^2} = \frac{1}{c}\frac{\partial A_\alpha}{\partial t} + \frac{\partial \phi}{\partial x_\alpha} + \frac{1}{c}\left(\sum_\sigma \frac{\partial x_\sigma}{\partial t}\cdot\frac{\partial}{\partial x_\sigma}\right) A_\alpha - \frac{1}{c}\sum_\sigma \frac{\partial x_\sigma}{\partial t}\cdot\frac{\partial A_\sigma}{\partial x_\alpha}.$$

Also, if d/dt be the time derivative following the motion of the electron, of any variable depending on the coordinates (x_α) and the time t, then

$$(69) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_\sigma \frac{\partial x_\sigma}{\partial t}\frac{\partial}{\partial x_\sigma};$$

since $\partial x_\alpha/\partial t = dx_\alpha/dt$, (68) becomes

$$(70) \qquad \frac{d}{dt}\left(\frac{m}{e}\frac{dx_\alpha}{dt} - \frac{1}{c}A_\alpha\right) = \frac{\partial \phi}{\partial x_\alpha} - \frac{1}{c}\sum_\sigma \frac{\partial x_\sigma}{\partial t}\cdot\frac{\partial A_\sigma}{\partial x_\alpha}.$$

We now follow the Lagrangian method used in hydrodynamics*; we introduce as new independent variables the initial position vector (a_α) of the electron and the time t.

Multiplying (70) by $\partial x_\sigma/\partial a_\alpha$ and summing over σ we find

$$(71) \qquad \sum_\sigma \frac{d}{dt}\left(\frac{m}{e}\frac{dx_\sigma}{dt} - \frac{1}{c}A_\sigma\right)\frac{\partial x_\sigma}{\partial a_\alpha} = \frac{\partial \phi}{\partial a_\alpha} - \frac{1}{c}\sum_\sigma \frac{\partial x_\sigma}{\partial t}\cdot\frac{\partial A_\sigma}{\partial a_\alpha} \quad (\alpha = 1, 2, 3).$$

Integrating this equation with respect to the time, following the path of the electron, from $t = t_0$ to t, we find, on integrating by parts, that the left-hand side becomes

$$\sum_\sigma \left(\frac{m}{e}\frac{dx_\sigma}{dt} - \frac{1}{c}A_\sigma\right)\frac{\partial x_\sigma}{\partial a_\alpha} - \sum_\sigma \int_{t_0}^{t}\left(\frac{m}{e}\frac{dx_\sigma}{dt} - \frac{1}{c}A_\sigma\right)\frac{\partial^2 x_\sigma}{\partial t\,\partial a_\alpha}\,dt,$$

if we suppose (A_α) and $(\partial x_\alpha/\partial t)$ to vanish for $t = t_0$. The right-hand side of the integrated equation may be written

$$\frac{\partial}{\partial a_\alpha}\int_{t_0}^{t}\phi\,dt - \frac{1}{c}\sum_\sigma \int_{t_0}^{t}\left[\frac{\partial}{\partial a_\alpha}\left(A_\sigma\frac{\partial x_\sigma}{\partial t}\right) - A_\sigma\frac{\partial^2 x_\sigma}{\partial a_\alpha\,\partial t}\right]dt.$$

Hence (71) becomes

$$(72) \qquad \sum_\sigma \left(\frac{m}{e}\frac{dx_\sigma}{dt} - \frac{1}{c}A_\sigma\right)\frac{\partial x_\sigma}{\partial a_\alpha} = \frac{\partial P}{\partial a_\alpha},$$

where

$$(74) \qquad P = \int_{t_0}^{t}\left[\phi + \tfrac{1}{2}\left(\frac{m}{e}\right)\sum_\alpha\left(\frac{\partial x_\alpha}{\partial t}\right)^2 - \frac{1}{c}\sum_\alpha\left(A_\alpha\frac{\partial x_\alpha}{\partial t}\right)\right]dt.$$

* H. Lamb, *Hydrodynamics* (5th ed., 1924), §§ 13, 15, 146.

Let (p_α) be a vector defined by

$$p_\alpha = m\frac{\partial x_\alpha}{\partial t} - \frac{e}{c}A_\alpha; \tag{75}$$

the components of (75) are then

$$\Sigma\, p_\sigma \frac{\partial x_\sigma}{\partial a_\alpha} = e\frac{\partial P}{\partial a_\alpha} \quad (\alpha = 1, 2, 3). \tag{76}$$

Eliminating P by differentiating any pair of equations in (76) we find

$$\sum_\sigma \left(\frac{\partial p_\sigma}{\partial a_\gamma}\cdot\frac{\partial x_\sigma}{\partial a_\beta} - \frac{\partial p_\sigma}{\partial a_\beta}\cdot\frac{\partial x_\sigma}{\partial a_\gamma}\right) = 0. \tag{77}$$

Using the identity

$$\frac{\partial p_\alpha}{\partial a_\gamma} = \sum_\sigma \frac{\partial p_\alpha}{\partial x_\sigma}\cdot\frac{\partial x_\sigma}{\partial a_\beta},$$

and rearranging we find

$$\sum_{\beta,\,\gamma}\left(\frac{\partial p_\beta}{\partial x_\gamma} - \frac{\partial p_\gamma}{\partial x_\beta}\right)\cdot\frac{\partial(x_\beta, x_\gamma)}{\partial(a_\beta, a_\gamma)} = 0, \tag{78}$$

and two similar equations in which a_α, a_β, a_γ are interchanged cyclically.

If we defined the vector (ζ_α) by

$$\zeta_\alpha = \frac{\partial p_\gamma}{\partial x_\beta} - \frac{\partial p_\beta}{\partial x_\gamma}, \tag{79}$$

then (78) may be written

$$\zeta_\alpha\frac{\partial(x_\beta, x_\gamma)}{\partial(a_\beta, a_\gamma)} + \zeta_\beta\frac{\partial(x_\gamma, x_\alpha)}{\partial(a_\beta, a_\gamma)} + \zeta_\gamma\frac{\partial(x_\alpha, x_\beta)}{\partial(a_\beta, a_\gamma)} = 0, \tag{80}$$

and two like equations obtained from it by interchanging the a's cyclically. These three equations form a system of homogeneous linear equations in (ζ_α); since the vector components (x_α) are independent, the only solution of these equations is

$$\zeta_1 = \zeta_2 = \zeta_3 = 0$$

or

$$(\zeta_\alpha) = 0. \tag{81}$$

Since by (79),

$$\boldsymbol{\zeta} = \operatorname{curl}\mathbf{p},$$

it follows from (81) that

$$\mathbf{p} = \operatorname{grad}\psi, \tag{82}$$

where ψ is any arbitrary function of position and time.

By (75) we therefore have finally

$$m\mathbf{v} - \frac{e}{c}\mathbf{A} = \operatorname{grad}\psi. \tag{83}$$

This result was first deduced* in connection with the theory of superconductors; as has been pointed out in (I), these investigations are incomplete as they do not seem to take into account the action of the magnetic field on the motion of the electrons. For this reason we have thought it worth while to reconsider this problem.

The theorem expressed by the equation (81) is analogous to the theorem in hydrodynamics that if the motion of a fluid is initially irrotational, it will always remain so; remembering that $\mathbf{H} = \operatorname{curl} A$, this equation may be written

$$\operatorname{curl}\mathbf{v} = \frac{e}{mc}\mathbf{H},$$

so that the condition expresses the fact that the vorticity of the electron stream is $\frac{1}{2}eH/mc$; this is one-half the vorticity of an electron moving in a uniform magnetic field. The vortex lines thus coincide with the magnetic lines of force and the spin is such that the electron stream tends to behave as a diamagnetic gas.

9. *The constitutive equations.*

Since ψ in (83) is a function of position and time only, the relations (83) may be written in a still simpler form; for the four-vector $(\mathbf{A}, \phi)$ may be replaced by the four-vector $(A + \operatorname{grad}\psi, \phi - \dot{\psi}/c)$, provided that ψ satisfies the wave equation; we may therefore omit altogether the function ψ in (83) without loss of generality.

The current density is given by

$$\sigma = -ne\mathbf{v}/c, \tag{84}$$

where n is the number of electrons carrying the current; hence by (83), putting $\psi = 0$, we find (65), namely

$$\sigma = -ne^2\mathbf{A}/mc^2.$$

* H. Grayson Smith, *Univ. Toronto Studies, Low Temp. Series*, No. 76 (1935); F. and H. London, *loc. cit.*, § 1, p. 72.

Let ρ denote the space-charge density; then from the equations

$$\operatorname{div}\sigma+\dot{\rho}/c=0$$

$$\operatorname{div}\mathbf{A}+\dot{\phi}/c=0$$

we find

$$\rho=-ne^2\phi/mc^2, \tag{85}$$

since initially $\phi=0$; (84) and (85) are the equations adopted by F. and H. London (*loc. cit.*) in their theory of conduction in super-conductors. It is to be noted, however, that whereas these equations are always true for a super-conductor, in the case of an ionized gas in which collisions are negligible, the equations are true only if initially the inducing electromagnetic field vanishes.

Finally, substituting (83) in (72) we find P to be a function of the time only, so that $\frac{1}{2}mu^2-e\phi$ depends solely on the time.

10. *Boundary conditions.*

We now return to consider the case when the conductor is in the form of an infinite plane-sheet; in (84) and (85), the variables now refer to surface values.

It will be convenient to denote by M and C respectively the four vectors $(\mathbf{A}, \phi)$ and (σ, ν) respectively, where we now write ν for ρ, as in § 2; the constitutive equations may then be written simply as a four-vector equation:

$$C=-kM, \tag{86}$$

where

$$k=ne^2/mc^2; \tag{87}$$

also (12) and (13) may be written in a single four-vector equation

$$\frac{\partial M^{(s)}}{\partial|z|}=-2\pi C \quad (z=0)$$

whence, by (86), the required boundary condition is

$$\frac{\partial M^{(s)}}{\partial|z|}=2\pi kM \quad (z=0), \tag{88}$$

where

$$M=M^{(s)}+M^{(i)}. \tag{89}$$

These equations replace (16) and (17) for a sheet conducting according to Ohm's Law; they differ from the latter equations in the absence of the time derivatives of the field variables, but also in being symmetrical in the components of the four vector M. This is not the case with (16) and (17).

11. *The differential equations of the problem.*

Denoting by $\bar{M}^{(i)}$ the conjugate functions to $M^{(i)}$ as in § 3, we consider the function u defined by

$$u = \frac{\partial M^{(s)}}{\partial |z|} - 2\pi k(M^{(s)} + \bar{M}^{(i)});$$

following the argument already given in § 3, it can be shown that $u = 0$ throughout the region $z \geqslant 0$, provided that $r^n u$ vanishes at infinity for $n > 1$; hence the required equation is

$$\frac{\partial M^{(s)}}{\partial |z|} = 2\pi k(M^{(s)} + \bar{M}^{(i)}); \tag{90}$$

we remark (as has been mentioned already in § 8) that this equation ceases to be true when the speed of the electrons in the sheet becomes comparable with that of light.

12. *Solution of the equation.*

Considering first the case when n, and hence k, is infinite; it is clear from (90) that the only solution which is regular everywhere is

$$M^{(s)} = -\bar{M}^{(i)}; \tag{91}$$

in this case, as in the case of a perfectly conducting sheet (§ 6), the electromagnetic field of the inducing system is completely reflected.

When k is finite, the formal solution can be shown to be

$$M^{(s)} = -\int_0^\infty e^{-\sigma} \bar{M}_\sigma^{(i)} \, d\sigma, \tag{92}$$

where $\bar{M}_\sigma^{(i)}$ is the function derived from $\bar{M}^{(i)}$ by writing $|z| + \sigma/2\pi k$ for $|z|$. This is the general solution: for any other solution differs from (92) by the general solution of the equation

$$\frac{\partial M^{(s)}}{\partial |z|} = 2\pi k M^{(s)};$$

the only solution of this equation which is finite and regular everywhere, including infinity, is identically zero.

By integration by parts, the equation (92) becomes

$$M^{(s)} = -\bar{M}^{(i)} - \sum_1^{\mu} \frac{1}{(2\pi k)^{\mu}} \frac{\partial^{\mu} \bar{M}^{(i)}}{\partial |z|^{\mu}} + \frac{1}{(2\pi k)^{\mu}} \int_0^{\infty} e^{-\sigma} \frac{\partial^{\mu+1} \bar{M}_{\sigma}^{(i)}}{\partial |z|^{\mu+1}} \, d\sigma ; \tag{93}$$

the first term on the right-hand represents the negative image of the inducing field $M^{(i)}$; the remaining terms may be regarded as harmonics derived from it and having axial symmetry with respect to the z-axis.

13. *A magnetic system in uniform motion.*

As an example we consider the case of a magnetic system moving towards the sheet with constant velocity U; it will suffice to consider the case of a single magnetic pole for the reason stated in (I), § 7. The field $M^{(i)}$ in this case has been considered in (I); we neglect the relativity correction since there is no point in considering this in an approximate solution. Defining α, β, γ as in (I), § 7, we have approximately

$$\alpha = x, \quad \beta = y - mz_0 - mUt, \quad \gamma = |z| + lz_0 - lUt. \tag{94}$$

The vector and scalar potentials of the inducing field $M^{(i)}$ are given by

$$\begin{cases} \bar{A}_x^{(i)} = \dfrac{\partial}{\partial \beta} \log \left(\gamma + \sqrt{(\alpha^2+\beta^2+\gamma^2)}\right), \\ \bar{A}_y^{(i)} = -\dfrac{\partial}{\partial \alpha} \log \left(\gamma + \sqrt{(\alpha^2+\beta^2+\gamma^2)}\right), \\ \bar{\phi}_y^{(i)} = \dfrac{mU}{c} \bar{A}_y^{(i)}. \end{cases} \tag{95}$$

By (3)–(5) and (92) it then follows that

$$\mathbf{H}^{(s)} = -\mathrm{grad}_{\alpha\beta\gamma} \Omega^{(s)}, \tag{96}$$

where

$$\Omega^{(s)} = \int_0^{\infty} e^{-\sigma} (\alpha^2+\beta^2+\gamma_{\sigma}^2)^{-\frac{1}{2}} \, d\sigma, \tag{97}$$

and γ_{σ} is obtained by replacing $|z|$ by $|z|+(\sigma/2\pi k)$ in γ.

By (93), we may write

$$\Omega^{(s)} = \frac{1}{\bar{r}} + \sum_1^{\mu} \frac{1}{(2\pi k)^{\mu}} \frac{\partial^{\mu}}{\partial |z|^{\mu}} \cdot \left(\frac{1}{\bar{r}}\right) - \frac{1}{(2\pi k)^{\mu}} \int_0^{\infty} e^{-\sigma} \frac{\partial^{\mu+1}}{\partial |z|^{\mu+1}} \left(\frac{1}{\bar{r}_{\sigma}}\right) d\sigma, \tag{98}$$

where $\bar{r}^2 = \alpha^2+\beta^2+\gamma^2$ and $\bar{r}_{\sigma}$ is obtained by writing γ_{σ} for γ in $\bar{r}$.

Since $$\left\{\frac{\partial^\mu}{\partial|z|^\mu}\right\}\frac{1}{\bar{r}} = \frac{(-1)^\mu \mu!}{\bar{r}^{\mu+1}} P_\mu,$$

where P_μ is the Legendre polynomial of order μ, the series (98), if continued indefinitely, is divergent.

Thus, when k (*i.e.* n) is numerically large, the series (98) continued to infinity represents an asymptotic series for $\Omega^{(s)}$; (97) is in fact the Borel sum of this divergent series.

In the case when k is large the series (98) may be interpreted as follows: the magnetic field due to the currents induced in the sheet may be approximately represented by the positive image of the pole in the sheet together with a dipole, quadrupole, etc., also situated at the position of the image in the sheet, whose strengths are to the strength of the pole as $\frac{1}{2}\pi k$, $2!/(2\pi k)^2$, ..., the axes of these images being directed along the negative portion of the z-axis. The series of images must, however, be discontinued after a certain number of them. Thus in the case being considered, the series must be discontinued after, say, the s-th image, s being the integral part of πkr, and r being the distance of the pole from the sheet. The density of the electric polarisation of the sheet is given by

$$\nu = \frac{mU}{2\pi c} \dot{H}_x^{(s)};$$

as in the case considered in (I), this vanishes if the pole moves perpendicularly to the sheet (when $m = 0$).

The lines of equidensity in the case when $k = \infty$ and $m = 1$ are the same as for a perfectly conducting sheet (see § 6).

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THE CRITICAL CONCOMITANT OF BILINEAR FORMS

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1. The object of the present communication is to connect the invariant theory of bilinear and quadratic forms with that of their elementary divisors. It will be shown that, when the matrix A of such forms has non-linear elementary divisors, certain concomitants vanish identically, and that, in all cases, a certain non-zero *critical* concomitant exists which suffices to detect the exact nature of the elementary divisors. This critical concomitant can always be resolved into factors, simple or repeated, which are linear in the variables and their compounds (§ 7, Theorem IV). The same concomitant will also be shown to be closely connected with the determinant $|H|$ of the transformation which reduces the matrix A to a rational canonical form $B = H^{-1}AH$. The bearing of this upon the theory of pencils of bilinear forms and singular forms is suggested in § 11.

2. *The bilinear form.*

Consider the bilinear form

$$\phi = \Sigma u_i a_{ij} x_j \quad (i, j = 1, 2, \ldots, n) \tag{1}$$

in two sets u, x of contragredient variables. In matrix notation the variables are

$$u = [u_1, u_2, \ldots, u_n], \quad x = \{x_1, x_2, \ldots, x_n\}, \tag{2}$$

where u is a single row, and x a single column of n variables. A non-singular matrix $H = [h_{ij}]$, which transforms x to Hx and u to uH^{-1}, will then transform ϕ, whose matrix is $A = [a_{ij}]$, to a bilinear form whose matrix is HAH^{-1}. We are concerned with this group of transformations and with the projective concomitants of ϕ to which it gives rise.

A complete system of concomitants, in terms of which every rational integral concomitant can be rationally and integrally expressed, has been

evaluated (1). It requires several sets of variables $u, v, w, \ldots$ and also $x, y, z, \ldots$, cogredient with u and x respectively. This complete system consists* of:

(1) Absolute concomitants,

(2) Bilinear forms,

(3) Invariants,

(4) Rank concomitants,

(5) Q-concomitants.

To deal with these we use the symbolic notation. Let the coefficients be expressed as

$$a_{ij} = \alpha_i a_j = \beta_i b_j = \text{etc.}, \tag{3}$$

in terms of symbols $\alpha, \beta, \gamma, \ldots$ cogredient with x, and $a, b, c, \ldots$ cogredient with u. Then the bilinear form is symbolized by

$$\phi = u_\alpha a_x = u_\beta b_x = \text{etc.} \tag{4}$$

These symbols are equivalent in pairs, α and a being interchangeable with β and b respectively, but not α with a, nor α, a with b, β respectively.

The *trace* of the matrix $A = [a_{ij}]$ is Σa_{ii}, whose symbolic form is

$$a_\alpha = a_1 \alpha_1 + \ldots + a_n \alpha_n, \tag{5}$$

an inner product of the sets of symbols a and α. Compound inner products can also be formed, and in particular we need

$$M_r = (ab \ldots c \,|\, \alpha\beta \ldots \gamma) = \begin{vmatrix} a_\alpha b_\alpha \ldots c_\alpha \\ \vdots \\ a_\gamma b_\gamma \ldots c_\gamma \end{vmatrix} \quad (r = 1, 2, \ldots, n), \tag{6}$$

which is an r-th compound of the outer products $ab \ldots c$ and $\alpha\beta \ldots \gamma$, expressed as an r-rowed determinant of symbolic inner products $a_\alpha, a_\beta, \ldots$. For short we write

$$A_r = ab \ldots c, \quad A_r' = \alpha\beta \ldots \gamma, \quad M_r = (A_r | A_r'). \tag{7}$$

Alternatively, by the Binet-Cauchy theorem, we have

$$M_r = \Sigma (ab \ldots c)_{ij \ldots k} (\alpha\beta \ldots \gamma)_{ij \ldots k} \tag{8}$$

* 1, p. 20.

summed for the $\binom{n}{r}$ combinations of r different suffixes $i, j, \ldots, k$ chosen from the first n positive integers.

Next let

$$\psi_{r-1} = (ub \ldots c \,|\, \beta \ldots \gamma x) = \begin{vmatrix} u_\beta b_\beta \ldots c_\beta \\ \vdots \\ u_x b_x \ldots c_x \end{vmatrix} \tag{9}$$

be the analogous r-th compound, consisting of $r-1$ symbols $b, \ldots, c$ and one variable u, together with the complementary $\beta, \ldots, \gamma$ and x. Then ψ_{r-1} is a bilinear form in u and x. In M_r and ψ_{r-1} the suffix r can range from 1 to n inclusive. If $r = n$, the forms factorize into

$$M_n = (ab \ldots c)(\alpha\beta \ldots \gamma), \quad \psi_{n-1} = (ub \ldots c)(\beta \ldots \gamma x), \tag{10}$$

and if $r > n$ they vanish identically.

Lastly, let

$$X_r = xy \ldots z, \quad U_r = uv \ldots w, \tag{11}$$

denote outer products of r variables of one or other type, so that

$$(A_r U_{n-r}) = (ab \ldots uv \ldots) \tag{12}$$

is an n-rowed determinant (whose rows are $a, b, \ldots, u, v, \ldots$) while

$$(A_r' \,|\, U_r) = (\alpha\beta \ldots | uv \ldots) \tag{13}$$

is an r-th compound inner product. Similarly, let

$$\left.\begin{aligned} (A_r' X_{n-r}) &= (\alpha\beta \ldots xy \ldots), \\ (A_r | X_r) &= (ab \ldots | xy \ldots). \end{aligned}\right\} \tag{14}$$

The complete system can now be given as follows:

(1) *Absolute concomitants.*

$$(uv \ldots w) = |u_1 v_2 \ldots w_n|, \quad u_x = (u|x) = u_1 x_1 + \ldots + u_n x_n,$$

$$(xy \ldots z) = |x_1 y_2 \ldots z_n|.$$

(2) *Bilinear forms.*

$$\psi_0 = u_x, \quad \psi_1 = (ua \,|\, ax), \quad \psi_2 = (uab \,|\, a\beta x), \quad \ldots,$$

$$\psi_{n-1} = (uA_{n-1} \,|\, A_{n-1}' x) = (uA_{n-1})(A_{n-1}' x).$$

(3) *Invariants.*

$$M_1 = a_\alpha, \quad M_2 = (ab|\alpha\beta), \ \ldots, \quad M_n = (A_n)(A_n').$$

(4) *Rank concomitants.*

$$(A_r U_{n-r})(A_r'|U_r), \quad (A_r' X_{n-r})(A_r|X_r), \quad r = 1, 2, \ldots, n-1.$$

(5) *Q concomitants.*

$$Q = (A_{q_1} U_{n-q_1})(A'_{q_1}|A_{q_2} U_{q_1-q_2})(A'_{q_2}|\ldots)\ldots(\ldots|A_{q_\nu} U_{q_{\nu-1}-q_\nu})(A'_{q_\nu}|U_{q_\nu}),$$

$$Q' = (A'_{q_1} X_{n-q_1})(A_{q_1}|A'_{q_2} X_{q_1-q_2})(A_{q_2}|\ldots)\ldots(\ldots|A'_{q_\nu} X_{q_{\nu-1}-q_\nu})(A_{q_\nu}|X_{q_\nu}),$$

where $n > q_1 > q_2 \ldots > q_\nu > 0$.

Polarized forms are also admitted; that is, we may replace any variable within a group X_r by a cogredient variable which is not already involved in this group. Similarly for U_r. Also we may replace any X_r by U_{n-r}. Thus the ground form ϕ occurs in the above list under (4) with $r = 1$. It occurs in ψ, also, since

$$\psi_1 = (ua|\alpha x) = u_\alpha a_x - u_x a_\alpha = \phi - u_x . M_1.$$

The list therefore is not strictly irreducible, although it must include all irreducible concomitants. The inequalities $n > q_1 > \ldots > 0$ are necessary for irreducibility.

The above rank concomitants are essentially those compound bilinear forms which are generated by the original matrix A. Let $\phi^{(r)}$ denote the r-th compound of the original form ϕ. Then, from the definition of a compound by means of r-rowed minors of the determinant

$$|A| = \frac{1}{n!}(\alpha\beta\ldots)(ab\ldots),$$

we have

$$\left.\begin{aligned} \phi^{(1)} &= uAx = u_\alpha a_x, \\ \phi^{(2)} &= u^{(2)}A^{(2)}x^{(2)} = \frac{1}{2!}(uv|\alpha\beta)(ab|xy), \\ \phi^{(3)} &= u^{(3)}A^{(3)}x^{(3)} = \frac{1}{3!}(uvw|\alpha\beta\gamma)(abc|xyz), \end{aligned}\right\} \tag{16}$$

and so on. Hence $\phi^{(r)} = \frac{1}{r!}(U_r|A_r')(A_r|X_r)$. Here $u^{(r)}$ denotes the r-th compound variables derived from u, $A^{(r)}$ the r-th compound matrix, and $x^{(r)}$ the r-th compound of the variables x.

A set ϕ_r of concomitants, alternative to these rank concomitants, can be formed by taking powers A^r of the matrix A. Thus

$$\left.\begin{aligned} \phi_0 &= ux \quad = u_x, \\ \phi_1 &= uAx \quad = u_\beta a_x, \\ \phi_2 &= uA^2x = u_\alpha a_\beta b_x, \\ \phi_3 &= uA^3x = u_\alpha a_\beta b_\gamma c_x, \end{aligned}\right\} \tag{17}$$

and so on. These are connected with the above bilinear forms ψ_r by means of relations of the type

$$\psi_r = r!\, u(A^r + p_1 A^{r-1} + p_2 A^{r-2} + \ldots + p_{r-1} A + p_r I)x, \tag{18}$$

where the coefficients p_i are those of the characteristic equation

$$|\lambda I - A| \equiv f(\lambda) \equiv \lambda^n + p_1' \lambda^{n-1} + p_2 \lambda^{n-2} + \ldots + p_n = 0, \tag{19}$$

satisfied by the matrix A. Cf. **1**.

These coefficients p_i are the invariants already quoted, and in fact

$$M_i = (-)^i\, i!\, p_i. \tag{20}$$

3. *The Q concomitants.*

These concomitants Q and Q' which differ in their variables are essentially the same sets of functions of the coefficients. We shall work with the form Q' which involves the variables $x, y, \ldots$ only, and our immediate object is to express it as a determinant involving successive sets of image points.

Let x denote a *point* of $[n-1]$ space, and u a *prime* or $[n-2]$ subspace, so that u_x vanishes whenever the point lies in the prime. Then Ax will denote the *image* point of x in the collineation Γ, whose matrix is A and whose equation is $a_x u_\alpha = 0$. Accordingly let $x, Ax, A^2x, \ldots$ denote a chain or succession of such image points, say $x^0, x^1, x^2, \ldots$, originating in the point x. Symbolically we have

$$x^0 = x, \quad x^1 = a_x \alpha, \quad x^2 = a_{x^1} \alpha = a_\beta b_x \alpha, \quad \ldots, \tag{1}$$

where a_x, a_β, b_x are scalar symbolic factors, and α is a symbolic vector. (The i-th of the n components of x^1 is $a_x \alpha_i$.)

From these points, according to the definition in § 2 (11), we may derive compounds, x^0x^1, $x^0x^1x^2$, ..., which denote a line, plane, ... through the first two, first three, ..., points, respectively. Evidently the lines x^0x^1, x^1x^2, x^2x^3, are successive images, and similarly for higher spaces.

Suppose now that r independent points $x, y, \ldots, z$ are taken. Then their images, which are

$$x^1 = a_x \alpha, \quad y^1 = b_y \beta, \quad \ldots, \quad z^1 = c_z \gamma, \tag{2}$$

produce an r-th compound

$$x^1 y^1 \ldots z^1 = a_x b_y \ldots c_z \alpha\beta \ldots \gamma$$

$$= \frac{1}{r!} (ab \ldots c | xy \ldots z) \alpha\beta \ldots \gamma \tag{3}$$

on interchanging the r pairs of equivalent symbols in all possible ways and adding the results. Here $\alpha\beta \ldots \gamma$ stands for an outer product of r symbolic vectors, so that the interchange of any two α, β changes the sign of the product. This is an r-th compound vector of $\binom{n}{r}$ components.

Hence, if $X_r = xy \ldots z$, $A_r = ab \ldots c$, $A_r' = \alpha\beta \ldots \gamma$, then

$$X_r^1 = x^1 y^1 \ldots z^1 = \frac{1}{r!} (A_r | X_r) A_r'. \tag{4}$$

On putting $y = x^1, \ldots, z = x^{r-1}$ we have at once

$$x^1 x^2 \ldots x^r = \frac{1}{r!} (A_r | x^0 x^1 \ldots x^{r-1}) A_r'. \tag{5}$$

Similarly

$$x^1 x^2 \ldots x^{r-1} = \frac{1}{(r-1)!} (A_{r-1} | x^0 x^1 \ldots x^{r-2}) A_{r-1}',$$

so that

$$x^1 x^2 \ldots x^r = \frac{1}{r!} \frac{1}{(r-1)!} (A_r | x A_{r-1}')(A_{r-1} | x^0 x^1 \ldots x^{r-2}) A_r',$$

where A_r, A_{r-1} involve, in all, $2r-1$ different equivalent symbols $a, b, \ldots$, and $x = x^0$. Similarly, step by step, this yields

$$\frac{1}{r!} \frac{1}{(r-1)!} \cdots \frac{1}{2!} A_r' (A_r | x A_{r-1}')(A_{r-1} | x A_{r-2}') \ldots (A_2 | x a) a_x, \tag{6}$$

where $0 < r \leqslant n$.

When $r > n$ the points $x^0, x^1, \ldots, x^{r-1}$ are necessarily linearly dependent and their outer product vanishes. When $r = n$ we obtain the n-th compound, a single determinant; and, by (6),

$$(x^0 x^1 \ldots x^{n-1}) = \frac{1}{(n-1)!!} (x A_{n-1}')(A_{n-1} | x A_{n-2}') \ldots (A_2 | x a) a_x, \tag{7}$$

where $(n-1)!! = (n-1)!\,(n-2)! \ldots 2!\,1!$.

Now consider the general form Q', which involves the ν pairs of symbols A'_{q_1}, A_{q_1}, A'_{q_2}, A_{q_2}, ..., A'_{q_ν}, A_{q_ν}, that is two sets of Σq_i symbols, $a, b, \ldots$ and their complementary $\alpha, \beta, \ldots$. We shall call ν the *extent* of Q'.

On writing $X^1_{q_\nu} = \frac{1}{q_\nu!}(A_{q_\nu}|X_{q_\nu})A'_{q_\nu}$ we may replace Q' by $q_\nu!$ times a similar form of extent $\nu-1$, wherein X_{q_ν} has become $X^1_{q_\nu}$, and A_{q_ν}, A'_{q_ν} have disappeared. The final factor of the form is now

$$(A_{q_{\nu-1}}|X^1_{q_\nu}X_{q_{\nu-1}-q_\nu}).$$

Repeating the process we have $q_{\nu-1}!$ times a Q' form of extent $\nu-2$, wherein the variables of this final factor have become $X^2_{q_\nu}X^1_{q_{\nu-1}-q_\nu}$ and the symbols $A_{q_{\nu-1}}$, $A'_{q_{\nu-1}}$ have disappeared. Continuing this, step by step, we infer that

$$Q' = q_1!\,q_2!\ldots q_\nu!\,(X_{q_\nu}X^{\nu-1}_{q_{\nu-1}-q_\nu}\ldots X^1_{q_1-q_2}X_{n-q_1}), \tag{8}$$

which is thus a numerical multiple of the n-rowed determinant

$$\Delta = (X^\nu_{r_{\nu+1}}X^{\nu-1}_{r_\nu}\ldots X^1_{r_2}X_{r_1}), \tag{9}$$

where $r_1 = n-q_1$, $r_2 = q_1-q_2$, ... are the first differences of the descending integer sequence $n, q_1, q_2, \ldots, q_\nu$. So we have proved

THEOREM I. *Each Q' concomitant is expressible as an n-rowed determinant whose columns denote the n points given by the variables X_i occurring in Q', or their images; for points within the $(i+1)$-th factor of Q' the i-th images are taken.*

For example, one such Q' in quaternary forms is

$$(a\beta\gamma x)(abc|\,a'\beta'y)(a'b'|zt)$$

where a, b, c, a', b' are five equivalent symbols. This is equal to

$$2!\,(a\beta\gamma x)(abc|z^1t^1y) = 3!\,2!\,(z^2t^2y^1x).$$

Similar remarks apply to the Q forms, which can be expressed as determinants in terms of the primes $u, v, \ldots$ and their images.

COROLLARY 1. *There is exactly one irreducible covariant and one such contravariant among the Q forms.*

For if x alone enters into the form, it must be

$$(x^0x^1x^2\ldots x^{n-1})$$

as given by (7). The corresponding contravariant is

$$(u^0 u^1 u^2 \dots u^{n-1}).$$

Otherwise at least two of the indices for x (or for u) are equal and the determinant vanishes identically.

COROLLARY 2. *The whole system of concomitants can now be expressed in terms of invariants together with the three original types of forms* $(u\,v \dots w)$, u_x, *and* $(x\,y \dots z)$, *where, however, each variable* $u, \dots, w, x, \dots, z$ *can be an original or else an image of a prime or point.*

For the above arguments apply to forms of types (2) and (4) above as well as to the Q and Q' forms, on using § 2 (17).

Since the coefficients of the bilinear form now only occur implicitly by way of linear symbols, this reduces the problem of finding any irreducible system of concomitants of a bilinear form (and also of two quadratic forms, § 10) to that of a system of linear forms, including compounds, only. This would give a starting point for finding the Clebsch system of such forms (cf. **1**, p. 2).

4. *Standard forms.*

On resolving each X_r that occurs in Q' into its constituent point variables $x, y, \dots$, we see by inspection that exactly n of these occur. Also the same variables occur in the form Δ of § 3 (9) as in the original form of Q', but with indices attached. The determinant Δ may now be rearranged by grouping together all repetitions of the same point x, placed with ascending indices, followed by similar groups for any further repetitions. Hence the general form can be written

$$\pm(x^i x^j \dots y^k y^l \dots).$$

By a *standard form* is meant a Q' form in which each set of indices $i, j, \dots$, belonging to the same variable, is consecutive and starts at zero. We write such a form as

$$Q_s' = (x^0 x^1 x^2 \dots x^{p_1} y^0 y^1 \dots y^{p_2} \dots z^0 z^1 \dots z^{p_h}), \tag{1}$$

where $$p_1 + p_2 + \dots + p_h + h = n,$$

and $$p_1 \geqslant p_2 \geqslant \dots \geqslant p_h \geqslant 0.$$

By §3 (8) this Q_s' is a Q' form, apart from a non-zero numerical multiple. The covariant already noticed is such a standard form, with $h = 1$. The inequalities among the p's are admissible by suitable arrangement of the columns in the determinant.

By a *Clebsch form* Q_c' is meant any Q' in which each compound variable X_r contains the first r of the point variables $x, y, \ldots, z, \ldots, t$ in this specified order. Every standard form is a Clebsch form, but the converse is not true. For example, with $n = 6$,

$$(A_5'x)(A_5|A_3'xy)(A_3|A_2'x)(A_2|xy) = -5!\,3!\,2!\,(x^0x^1x^2x^3y^1y^3)$$

is a Clebsch form but not a standard form since y^2 is missing; on the other hand,

$$(A_4'xy)(A_4|A_2'xy)(A_2|ax)\,a_x = 48(x^0x^1x^2x^3y^0y^1)$$

is both.

We may conveniently denote these forms by A_{532} and A_{421} respectively, where the suffixes indicate the nature of the symbolic factors A_i', A_i in the order in which they occur within the form, as inspection at once shows. Evidently there is one Clebsch form for each set of suffixes q in Q', namely

$$A_{q_1 q_2 \ldots q_\nu}, \quad n > q_1 > q_2 > \ldots q_\nu > 0. \tag{2}$$

Now let a graph or tableau of the integers n and N be made, where

$$N = q_1 + q_2 + \ldots + q_\nu, \tag{3}$$

by using a row of n marks ○, followed by ν rows with q_i marks in the i-th row:

$$T \qquad \begin{array}{ll} ○\ ○\ \ldots\ldots\ ○ & n \text{ entries} \\ \times\ \times\ \ldots\ldots\ \times & q_1 \quad ,, \\ \times\ \times\ \ldots\ldots\ \times & q_2 \quad ,, \\ \ldots\ldots & \ldots\ldots \\ \times\ \times\ \ldots\ \times & q_\nu \quad ,, \end{array} \tag{4}$$

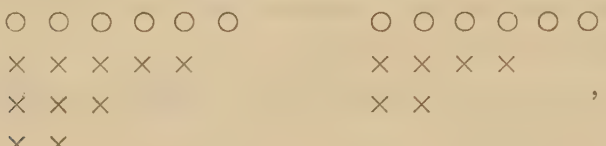
Let each row start in the first column and contain no gaps. The lengths of the rows will be unequal and will diminish downwards. From the crosses $\times$ we obtain a partition $\{q\}$ of N by counting the entries by rows, and the *conjugate* partition $\{\nu\}$ by columns: let us say,

$$\{q\} = \{q_1 q_2 \ldots q_\nu\} = \{\nu_1 \nu_2 \ldots \nu_{q_1}\}' = \{\nu\}', \quad \{\nu\} = \{q\}'. \tag{5}$$

Evidently $\nu_1 = \nu$, and the sequence of ν's must consist of the consecutive integers $\nu, \nu-1, \ldots, 2, 1$ with possible repetitions.

Such a graph T characterizes a Clebsch form by means of its integers q_i. But it also yields the indices of the corresponding point elements. For we may agree to consider the first and longest column as belonging to x, the second column to y, the third to z, and so on until all the longest columns are exhausted. As soon as a shorter column is reached we begin again with x and continue with the variables $x, y, \ldots$, in the agreed order, always starting each new set with x.

For example, the tableaux for A_{532} and A_{42} when $n = 6$ are



where the columns belong to $xyxxyx$ and $xyxyxy$ respectively. We now give the number of crosses in a column as index to its variable and obtain $x^3y^3x^2x^1y^1x^0$ and $x^2y^2x^1y^1x^0y^0$, which give the determinant Δ explicitly.

This is a graphical method of indicating in detail how the variables occurring in Q' are absorbed step by step into the form Δ. Consequently it holds in general, and we can write the result as

$$A_{q_1 q_2 \ldots q_\nu} = (x^{\nu_1}\eta^{\nu_2}\ldots\zeta^{\nu_{q_1}}), \tag{6}$$

where $\eta = x$ or y according as $\nu_2 \neq \nu_1$ or $\nu_2 = \nu_1$: and so on.

The following properties are evident: (i) A set of k equal columns needs k distinct variables $x, y, \ldots$. (ii) In a Clebsch form the indices of x, the leading variable, are always consecutive. (iii) In a *standard tableau*, corresponding to a standard form, each new set of equal columns, reckoning from left to right, has at least as many columns as its predecessor. That is

$$q_\nu \leqslant q_{\nu-1} - q_\nu \leqslant \ldots \leqslant q_1 - q_2 \leqslant n - q_1. \tag{7}$$

(iv) In a standard form the indices p_i are given by

$$p_1 = \nu, \quad p_i = \nu_j,$$

where ν_j is the index belonging to the first set of equal columns (from the left) which contains at least i columns.

Thus, *for a standard form to be irreducible, not only must* $n, q_1, \ldots, 0$ *be strictly monotonic, but their first differences must be (not strictly) monotonic also.*

5. *Standard matrices.*

The matrix of n rows and columns,

$$H = [x^0, x^1, \ldots, x^{p_1}, y^0, y^1, \ldots, y^{p_2}, \ldots, z^0, z^1, \ldots, z^{p_h}],$$

whose determinant $|H|$ has already been introduced as Q_s', will be called a *standard matrix* if it is non-singular, that is, if the n points x^i, etc., are linearly independent. Now such a matrix is employed* in reducing any $n \times n$ matrix A to a rational canonical form B, where

$$B = H^{-1}AH.$$

According to the usual theory (when stated in terms of columns rather than rows of the matrices), to construct such an H any point x is taken for which the value of p_1 is a maximum, that is such that x^{p_1+1} is the first image point to be linearly dependent upon the earlier x's. Then if $p_1 < n-1$ a new point y is taken for which p_2 is maximum, such that y^{p_2+1} is the first image of y to be linearly dependent upon the earlier x's and y's. And so on. The matrix H is then formed as shown above. The columns of H are also given by $x, Ax, A^2x, \ldots, y, Ay, \ldots, z, Az, \ldots$, in this order.

In the ordinary case of distinct latent roots for A (or B), $p_1 = n-1$ and no y is required. In the *regular* case, which includes the ordinary case, again $p_1 = n-1$, and there is said to be one invariant-factor, with possibly repeated latent roots. In the general case there are exactly h invariant-factors. The h indices p_i which are absolute numerical invariants of a given matrix A, when increased by unity give the exponents of the h invariant-factors.

Evidently they can be discovered directly from a knowledge of the complete system of a bilinear form. For suppose that all the distinct partitions of the integer n are arranged in lexical order

$$n, \ \{n-1, 1\}, \ \ldots, \ \{p_1' p_2' \ldots p_h'\}, \ \ldots, \ \{1\,1 \ldots 1\}$$

where $p_i' = p_i + 1$, so that the h integers within each partition are in descending order, as well as the partitions themselves. This arrangement

* 2, p. 49.

at once puts the standard forms Q_s' into the same lexical order, led by the covariant n-ic in x already noticed. Next suppose that

$$Q' = Q\{p_1 \dots p_h\}$$

is the first standard form in this sequence which does not vanish identically. Then not only will the corresponding matrix H be non-singular, but its indices p_i will be maximum in the sense required above for the canonical form. Let this be called the *critical form* Q' relative to a given ground form A. Thus we have proved

THEOREM II. *When the Q' concomitants are arranged in descending lexical order the h indices p_i (when increased by unity) of the first such form which does not vanish identically—the critical form—give the exponents of the h invariant-factors of the matrix of the ground form. Conversely, if there are exactly h invariant factors of A and if their indices are the p_i, then all the Q' concomitants preceding this critical concomitant vanish.*

Reference to the original definition in § 2 shows that each Q' form is a form multilinear in ν sets of compound variables X_{n-q_1}, ..., X_{q_ν} involving exactly n different points x, y, ..., or else is a polar of such a form. In general a multilinear form does not factorize; but it will next be shown that the critical form can always be factorized into forms (repeated if necessary) each of which is linear in one of the ν sets of compound variables.

Evaluation of the critical form.

§ 6. First consider the regular case, that is the case of one invariant-factor. Let the corresponding characteristic function, which is necessarily of degree n, be

$$|\lambda I - A| \equiv (\lambda-\alpha)^{e_1}(\lambda-\beta)^{f_1} \dots (\lambda-\delta)^{g_1}, \tag{1}$$

so that $e_1+f_1+\dots+g_1 = n$, where α, β, ..., δ, the latent roots, are distinct. There is then a classical canonical form*

$$C = \text{diag}\,(C_{e_1}, C_{f_1}, \dots, C_{g_1}), \tag{2}$$

of the matrix A, such that $A = KCK^{-1}$, where each *latent matrix* such as C_{e_1} is of the type $\alpha I_{e_1} + U_{e_1}$, when expressed in terms of its corresponding

* 2, p. 62.

latent root a, a unit matrix I_{e_1} and an auxiliary unit matrix U_{e_1} each of orders $e_1 \times e_1$.

For example, with $n = 4$, $e_1 = 3$, $f_1 = 1$ and therefore two distinct latent roots a, β, we should have

$$K^{-1}AK = C = \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & \beta \end{bmatrix} = \text{diag}\,(C_3, C_1).$$

The *Segre characteristic* for the regular case is the single row of exponents

$$[e_1, f_1, \ldots, g_1], \tag{3}$$

and is [3, 1] for this example. The determinants $|\lambda I_{e_1} - C_{e_1}| \equiv (\lambda - a)^{e_1}$, etc., are the corresponding *elementary divisors* of A. Their product is equal to $|\lambda I - A|$, which is also $|\lambda I - C|$, owing to the cancelling of the determinants of K and K^{-1}. This is indeed the single invariant-factor possessed by the matrix in the regular case.

Apart from a factor which depends solely upon the linear transformation whereby the canonical bilinear form is obtained, any concomitant can be calculated from the simpler C rather than from A.

Now consider the critical form. Since $h = 1$ in the regular case this form must be

$$Q' = (x^0 x^1 x^2 \ldots x^{n-1}) \neq 0, \tag{4}$$

where $x = x^0$ is any existent suitable point. Also if x^1, x^2, ... are successive images of $x = x^0$ in C, we find at once that, for the [3, 1] example,

$$\left.\begin{aligned} x^0 &= \{x_1,\ x_2,\ x_3,\ x_4\}, \\ x^1 &= \{ax_1 + x_2,\ ax_2 + x_3,\ ax_3,\ \beta x_4\}, \\ x^2 &= \{a^2 x_1 + 2ax_2 + x_3,\ a^2 x_2 + 2ax_3,\ a^2 x_3,\ \beta^2 x_4\}, \end{aligned}\right\} \tag{5}$$

and so on. On using x_j^i for the j-th component of x^i we thus obtain the following relation between three matrices:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = \begin{bmatrix} 1 & . & . & 1 \\ a & 1 & . & \beta \\ a^2 & 2a & 1 & \beta^2 \\ a^3 & 3a^2 & 3a & \beta^3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & . \\ x_2 & x_3 & . & . \\ x_3 & . & . & . \\ . & . & . & x_4 \end{bmatrix}. \tag{6}$$

The three corresponding determinants are $(x^0 x^1 x^2 x^3)$, $\Delta(aaa\beta)$ and $-(x_3)^3 x_4$ respectively, where the second is the confluent alternant, which

is equal to $(\beta-\alpha)^3$. Hence

$$(x^0 x^1 x^2 x^3) = (\alpha-\beta)^3 (x_3)^3 x_4 ; \tag{7}$$

this expresses the critical form as the product of two linear forms x_3 and x_4, one of which is repeated thrice, and a non-zero factor depending upon the differences of the latent roots.

The corresponding confluent alternant in the case for Q' above is evidently given by

$$\Delta \equiv \Delta(\alpha^{e_1} \beta^{f_1} \dots \delta^{g_1}) = |\Delta_{e_1} \Delta_{f_1} \dots \Delta_{g_1}|, \tag{8}$$

where Δ_{e_1} consists of e_1 columns c, $\frac{\partial c}{\partial \alpha}$, $\frac{1}{2!}\frac{\partial^2 c}{\partial \alpha^2}$, $\frac{1}{3!}\frac{\partial^3 c}{\partial \alpha^3}$, etc., with $c = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$, and where the remaining columns, for $f_1, \dots, g_1$ are formed similarly. The corresponding third determinant is that of the matrix

$$\text{diag}\,(T_{e_1}, T_{f_1}, \dots, T_{g_1}), \tag{9}$$

where

$$T_{e_1} = \begin{bmatrix} x_1 & x_2 & \dots & x_{e_1} \\ x_2 & & & \cdot \\ \dots & & \iddots & \\ x_{e_1} & \cdot & & \cdot \end{bmatrix}$$

is a triangular persymmetric $e_1 \times e_1$ matrix containing the first e_1 components x_i, while T_{f_1} is another such matrix containing the next f_1 components, and so on. On forming the general identity corresponding to (6) and taking determinants, we have at once

$$(x^0 x^1 \dots x^{n-1}) = \begin{vmatrix} 1 & . & \dots & 1 & . & \dots \\ \alpha & 1 & & \beta & 1 & \\ \vdots & \vdots & & \vdots & & \\ \alpha^{n-1} & (n-1)\alpha^{n-2} & \dots & \beta^{n-1} & \dots & \dots \end{vmatrix} \cdot \begin{vmatrix} T_{e_1} & . & . \\ . & T_{f_1} & . \\ & & \ddots \\ . & . & T_{g_1} \end{vmatrix},$$

whence

$$Q' = \pm\Delta \,.\, \xi_1^{e_1} \xi_2^{f_1} \dots \xi_k^{g_1}, \tag{10}$$

with $\xi_1 = x_{e_1}$, $\xi_2 = x_{e_1+f_1}$, $\xi_k = x_{e_1+f_1+\dots+g_1}$, where the upper indices of the ξ denote exponents. The value* of the confluent determinant is the product

$$\Delta = \Pi(\beta-\alpha)^{e_1 f_1} \neq 0$$

* 2, p. 63.

for all $\binom{k}{2}$ such pairs of latent roots α, β and for index, the product of their corresponding exponents $e_1 f_1$. Hence we have

THEOREM III. *In the regular case, when there is a single invariant-factor with k distinct latent roots repeated $e_1, f_1, \ldots$ times respectively, the critical concomitant, which is an n-ary n-ic covariant in point coordinates x, breaks up into n linear factors of which exactly k are distinct and their repetitions are those of the latent roots.*

COROLLARY I. *When all the n latent roots are distinct the critical form is* $\pm\left(\Pi(\alpha-\beta)\right) x_1 x_2 \ldots x_n$.

It is clear from the explicit form of (2) that there are exactly k latent points, one for each latent matrix. They are the points

$$x = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

belonging to the simplex to which C is referred, where the single non-zero component occurs at positions 1, e_1+1, e_1+f_1+1, etc., as follows from the general form of x^1 in (5). There are also exactly k latent primes, which are given by the equations $x_r = 0$, where r takes the k values e_1, e_1+f_1, ... Hence we have

COROLLARY II. *The critical covariant form factorizes into all k latent primes of the bilinear form. Dually the critical contravariant form*

$$Q' \equiv (u^0 u^1 \ldots u^{n-1})$$

must factorize into all k of the latent points.

7. *The general case.*

To express the critical form in general it is convenient to use bideterminants of m rows and $2n$ columns. A bideterminant is a form such as M_r in § 2, and therefore is capable of two determinantal forms, by the Binet-Cauchy theorem, as shown in § 2 (6) and (8).

For three rows and four columns a bideterminant is

$$(ab\,|\,xy) \equiv \begin{vmatrix} a_1 & b_1 & x_1 & y_1 \\ a_2 & b_2 & x_2 & y_2 \\ a_3 & b_3 & x_3 & y_3 \end{vmatrix}, \qquad \ldots (1)$$

which is equal to each of the following expressions

$$\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}, \quad \Sigma (ab)_{ij} (xy)_{ij}. \tag{2}$$

Let $(x^0 x^1 \dots x^{p_1} y^0 y^1 \dots y^{p_2} \dots z^0 z^1 \dots z^{p_h})$ be the critical form, so that the matrix A has h invariant-factors, and $p_1 \geqslant p_2 \geqslant \dots \geqslant p_h$. The Segre characteristic* may consequently be written

$$\mathcal{S} = \begin{bmatrix} e_1 & f_1 & \cdots & g_1 \\ e_2 & f_2 & \cdots & g_2 \\ \cdots & \cdots & \cdots & \cdots \\ e_h & f_h & \cdots & \end{bmatrix} \tag{3}$$

where the elements are positive or zero integers adding up to n, so that $\Sigma e + \Sigma f + \dots + \Sigma g = n$. There are k columns, belonging in order to the k distinct latent roots $\alpha, \beta, \dots, \delta$. In each column the integers are arranged in decreasing value, so that $e_1 \geqslant e_2 \geqslant \dots \geqslant e_h \geqslant 0$, etc. Also at least one of the elements in the h-th row is non-zero. Furthermore

$$e_i + f_i + \dots + g_i = p_i + 1$$

for $i = 1, 2, \dots, h$.

The critical form will now be expressed as a bideterminant

$$(\mathbf{A} \mid \mathbf{X}) \tag{4}$$

consisting of m rows, and two wings each of n columns, where

$$m = (p_1 + 1) h \geqslant n, \quad h > 1. \tag{5}$$

As the structure is complicated it is best seen by an example.

Let $(\lambda - \alpha)^3 (\lambda - \alpha)^2 (\lambda - \alpha) (\lambda - \beta)^3 (\lambda - \beta)$ show the five elementary divisors of the characteristic function of a certain ten-rowed matrix A, so that

$$\begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 1 \\ 1 & \end{bmatrix}, \qquad \begin{aligned} &m = 18, \quad n = 10, \quad h = 3, \quad k = 2, \\ &p_1 = 5, \quad p_2 = 2, \quad p_3 = 0, \quad \alpha \neq \beta. \end{aligned}$$

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* 2, p. 70.

Then the critical form can be written explicitly as

e_1+f_1						e_2+f_2			e_3	e_1			e_2		e_3	f_1				f_2	
1	α	α^2	α^3	α^4	α^5					x_1	x_2	x_3	x_4	x_5	x_6						e_1
.	1	2α	$3\alpha^2$	$4\alpha^3$	$5\alpha^4$					x_2	x_3		x_5								
.	.	1	3α	$6\alpha^2$	$10\alpha^3$					x_3											
1	β	β^2	β^3	β^4	β^5											x_7	x_8	x_9	x_{10}		f_1
.	1	2β	$3\beta^2$	$4\beta^3$	$5\beta^4$											x_8	x_9				
.	.	1	3β	$6\beta^2$	$10\beta^3$											x_9					
						1	α	α^2		y_1	y_2	y_3	y_4	y_5	y_6						e_1
						.	1	2α		y_2	y_3		y_5								
						.	.	1		y_3											
						1	β	β^2								y_7	y_8	y_9	y_{10}		f_1
						.	1	2β								y_8	y_9				
						.	.	1								y_9					
									1	z_1	z_2	z_3	z_4	z_5	z_6						e_1
									.	z_2	z_3		z_5								
									.	z_3											
									1							z_7	z_8	z_9	z_{10}		f_1
									.							z_8	z_9				
									.							z_9					

(6)

where the border numbers e or f simply indicate how many rows or columns are involved in the corresponding block.

By multiplying each of the ten columns of the left wing into each of the ten on the right we obtain a ten-rowed determinant, in which it is agreed that the i-th right and j-th left columns provide the ij-th element. From $j=1$ we thus obtain as first column $\{x_1, x_2, \ldots, x_{10}\}$ which is x. From $j=2$ we obtain the next column $\{\alpha x_1+x_2, \alpha x_2+x_3, \alpha x_3, \ldots, \beta x_{10}\}$ which is x^1, where the image of x has been formed from the matrix

$$C = \begin{bmatrix} \alpha & 1 & & & & & & & & \\ & \alpha & 1 & & & & & & & \\ & & \alpha & & & & & & & \\ & & & \alpha & 1 & & & & & \\ & & & & \alpha & & & & & \\ & & & & & \alpha & & & & \\ & & & & & & \beta & 1 & & \\ & & & & & & & \beta & 1 & \\ & & & & & & & & \beta & \\ & & & & & & & & & \beta \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} e_1 \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\} e_2 \quad ,, \\ e_3 \quad ,, \\ \left.\vphantom{\begin{matrix}\beta\\\beta\\\beta\end{matrix}}\right\} f_1 \quad ,, \\ f_2 \quad ,, \end{matrix} \qquad (7)$$

Proceeding thus for each value of j we find that $\xi^{j+1} = C\xi^j$ for $\xi = x, y, z$; and, on carrying out the whole process, the bideterminant is equal to the ten-rowed determinant

$$(x^0 x^1 x^2 x^3 x^4 x^5 y^0 y^1 y^2 z^0).$$

But this is the critical form for the given example. Hence the bideterminant gives the desired result.

Before evaluating the bideterminant we may notice the structure of its two wings. The left wing consists of three (h) isolated blocks, each of p_1+1 rows, which number is the sum of the indices in the top row of the Segre characteristic. The number of columns in the i-th block is given by the sum of the i-th row in the characteristic, for $i = 1, 2, \ldots, h$. The top block is a square containing the confluent alternant matrix of all the latent roots, with their highest indices $e_1, f_1, \ldots$ among those of the elementary divisors. The lower blocks are truncated repetitions of this block, always starting with the same initial column and continuing until the requisite space is filled. It will be observed that the alternants are written by rows and not as in § 6 (6) by columns.

Thus each of the h blocks falls into k layers of $e_1, f_1, \ldots, g_1$ rows respectively, a total of $m = (p_1+1)h$ rows.

The right wing falls into the same layers, hk in all, of which the first k belong to x, the next to y, and so on. The k layers of x contain k isolated blocks, corresponding in depths to the k different sections of the confluent alternant. The top layer contains a fringe of triangular matrices in descending order of size, say

$$X(e) = \left| \begin{array}{ccccc} T_{e_1} & & T_{e_2} & & \cdots \\ & & & & \end{array} \right\} e_1 \text{ rows} \qquad (8)$$

$$e_1 \quad + e_2 + \quad \cdots \quad + e_h \quad \text{columns}$$

The top of the fringe contains the first p_1+1 components of x in their consecutive order, while persymmetry within the triangles determines each lower row.

Just as this fringe belongs to the indices e_i of the first column in the Segre characteristic, so each subsequent column leads to a further fringe $X(f), \ldots, X(g)$, giving in all k isolated fringes, the total top row of which

is composed of the n components x_r of x in their natural order x_1 to x_n. This pattern is fully repeated in each further block of k layers with $y, \ldots, z$ respectively written for x.

To evaluate the bideterminant we expand it by the Binet-Cauchy theorem as the sum of products of two collateral n-rowed determinants, one from the left and one from the right wing. Manifestly each non-zero determinant from the left must contain the leading block—the confluent determinant $\Delta(\alpha^{e_1}\beta^{f_1}\ldots\delta^{g_1})$. Since in any pair forming a product the same rows are used from right and left wings, it follows that all the rows of $X(e)$ must be used on each occasion. Two typical cases arise: (i) $e_1 > e_2$, (ii) $e_1 = e_2$.

(i) If $e_1 > e_2$, then the leading triangle T_{e_1} of the first fringe $X(e)$ extends by $e_1 - e_2$ rows below all succeeding triangles of the fringe. This tail of the fringe, which belongs to x and therefore must be used, forms an $(e_1 - e_2)$-rowed minor which is isolated upon its rows within the n-rowed determinant of the right wing. Hence no further element from the first $e_1 - e_2$ columns on the right may occur in the expansion. Since all the rows of this tail must be utilized, this causes all the lowest $e_1 - e_2$ rows of each lower e-layer, $y, \ldots, z$ to be blank for the expansion, and therefore also cancels out the same rows on the left. Briefly, we may delete the lowest $e_1 - e_2$ rows of each α-block on the left, except the first.

Apart from sign the value of this minor is evidently $x_{e_1}^{e_1 - e_2}$, since it is a triangular determinant with x_{e_1} repeated throughout its diagonal. (In the illustration this tail is simply x_3, which causes y_3 and z_3 of its column to disappear, and also the unit alongside y_3.)

Let the process be carried out for each index $e_1, f_1, \ldots$ which differs from the index immediately below it in the array $\mathcal{S}$. This process therefore reduces the number of e-rows of the y-block to $e_1 - (e_1 - e_2) = e_2$, of f_1 to f_2, and so on, but leaves any set, g say, unaltered, whenever $g_1 = g_2$. Hence y, and its collateral left wing block, will now have $e_2 + f_2 + \ldots$ rows only, which is also the number of its columns on the left. Hence this block has become a confluent minor alternant, which, like the earlier one for x, must be retained in every term of the expansion.

(ii) If $e_1 = e_2$, either $e_2 > e_3$ or $e_2 = e_3$. First take $e_2 > e_3$. In this case each e-fringe, belonging to x or $y \ldots$ or z, will have two equal tails upon the $t = e_2 - e_3$ lowest rows of the fringe. These tails will occupy $2t$ columns, and will lie upon t rows of the right wing which are otherwise vacant. Here e stands for any one of the indices $e, f, \ldots, g$. Although certain y-rows may have been erased through the previous case (for instance if

$f_1 > f_2$), they will certainly not include any row belonging to the e-fringe of y. Hence each y-row of these tails is collateral with a part of the left wing alternant, which we know must be retained; so that all the rows of the tails for both x and y, that is $2t$ rows, must be retained. But the tails of the x and y fringes form a $2t$-rowed minor which is isolated upon its rows within the n-rowed determinant of the right wing. Hence no further element from the corresponding $2t$ columns on the right may be utilized. As in case (1), the process erases the lowest layer of $t = e_2 - e_3$ rows in any further lower block $Z(e)$ and its collateral α-block.

When this is done for all such indices $e, f, \ldots, (e_2 > e_3, f_2 > f_3, \ldots)$ the third block on the left wing is reduced to a square of $e_3 + f_3 + \ldots$ rows and columns. But, as before, any set g is unaltered whenever $g_1 = g_2 = g_3$.

The value of the $2t$-rowed minor so formed is evidently $(xy)^t_{rs}$, which is the t-th power of a two-rowed minor formed from the tips x_r, x_s, y_r, y_s of the four tails. Here $r = e_1$, $s = 2e_1$, for the minor so formed from the e-block.

Next, if $e_1 = e_2 = e_3$, take $e_3 > e_4$. This likewise leads to three equal tails upon $e_3 - e_4$ lowest rows of a fringe, reduces the fourth block on the left wing to a square of $e_4 + f_4 + \ldots$ rows and columns, and segregates a minor $(xyz)^t$ with $t = e_3 - e_4$. This process continues until the last possibility $e_1 = e_2 = \ldots = e_h$ is considered, which will isolate one or more h-rowed minors $(xy \ldots z)$. Therefore the fringes can be shorn of further tails systematically in ascending order by repeating the process until the top row of each surviving fringe is dealt with in the final h-th stage.

The bideterminant has been reduced in this way to $\Sigma e + \Sigma f + \ldots = n$ rows and $2n$ columns, so that it can at once be factorized into two n-rowed determinants. The erasures, from below upwards, on the left have reduced the e_1-rows collateral with y to a set e_2, the e_1 of z to e_3, and so on, for all indices $e, f, \ldots$. The result is a confluent alternant at each level, namely

$$\begin{aligned} &\Delta(\alpha^{e_1}\beta^{f_1}\ldots)\,.\,\Delta(\alpha^{e_2}\beta^{f_2}\ldots)\,.\,\Delta(\alpha^{e_3}\ldots)\,.\ldots \\ &\quad = \Pi(\beta-\alpha)^{e_1 f_1}\,\Pi(\beta-\alpha)^{e_2 f_2}\,.\ldots \\ &\quad = \prod_{\alpha \neq \beta} (\beta-\alpha)^{(e|f)}, \end{aligned} \tag{8}$$

taken for each of $\binom{k}{2}$ pairs of latent roots α, β, where

$$(e|f) = e_1 f_1 + e_2 f_2 + \ldots + e_h f_h$$

is the inner product of the two columns of indices in $\mathcal{S}$ belonging to the particular pair of roots.

On the right each set of columns e or f, etc., furnishes a distinct product of minors. From what has been proved the e-products can be written

$$X(E) \equiv x_{e_1}^{e_1-e_2}(xy)_{e_1,\,e_1+e_2}^{e_2-e_3} \dots (xy\dots z)_{e_1,\,\dots,\,e_1+\dots+e_h}^{e_h} \tag{9}$$

where the exponents are the first differences of the descending indices in the e-column of $\mathcal{S}$, and the suffixes are the consecutive sums of these indices. Hence the whole bideterminant is

$$(\mathbf{A}\,|\,\mathbf{X}) \equiv \pm\Pi(\beta-\alpha)^{(e|f)}.\,X(E).\,X(F).\dots.\,X(G), \tag{10}$$

where the suffixes for F, etc., incorporate previous sums.

The value of (6) is therefore

$$(\beta-\alpha)^{11}\,x_3(xy)_{35}(xyz)_{356}(x_9)^2(xy)_{9,\,10}, \tag{11}$$

where the index $11 = 3\times3+2\times1+1\times0$, and those of the other factors are the first differences of indices in $\mathcal{S}$, formed columnwise, while the suffixes are the sums, by columns, carried forward. The critical form has now been completely factorized, since the factors in the general case must agree with those of the canonical form. Hence we have the following result:

THEOREM IV. *When the bilinear form has exactly h invariant-factors and k distinct latent roots, the critical concomitant is a mixed form in first, second, up to h-th compound variables, which breaks up into factors each of which is linear in one or other such set of variables. The factors form k completely distinct groups. Within a group the factors form a nest of latent spaces, each factor being repeated δ times, where δ is a first difference of the column indices in the Segre characteristic of the bilinear form.*

Reference to the canonical form and its characteristic (3) shows that the distinct factors x_{e_1}, (xy), ... of $X(E)$ denote that system of latent spaces $[n-2]$, $[n-3]$, ... which is associated with one particular latent root. In such a *nest* of spaces, each is contained in its predecessor. They are only of consecutive dimension, starting with a prime $[n-2]$, when all e of the e-indices differ. Which are present or absent is plain from the formula (9).

COROLLARY. *The critical Q form dual to Q' factorizes into k nests of latent spaces given in terms of the latent points.*

As regards the latent nests of spaces in the case (11) above, any prime of the net $\lambda x_3+\mu x_5+\nu x_6=0$ is latent, as also of the pencil $\lambda x_9+\mu x_{10}=0$. A prime is here an 8-fold in 9 dimensions. The first nest consists of a prime, a 7-fold within it and a 6-fold within both. The second nest is a distinct prime and a 7-fold within it.

The dual form Q factorizes into two nests: (1) a point, a line and a plane, and (2) a point and line. Each point of these lines and plane is latent. The geometrical study of such systems is well known [4].

8. *Summary for the bilinear form.*

We may sum up these results as follows: from any bilinear form ϕ a complete set of concomitants can be constructed without recourse to special coordinate systems. The $n+1$ irreducible invariants are the coefficients of the binary n-ic which furnish the latent roots α, β, ... with their precise repetitions, say E, F, ..., so that $\Sigma E=n$. On arranging the Q (or Q') forms of the complete system in lexical order of standard forms, an order which merely depends on the dimensions in various prime or point variables, the first non-vanishing form (the critical form) is selected. This form always factorizes into linear factors in nests of compound variables. These furnish all the exponents of the elementary divisors by means of the repetitions among the linear factors which give explicitly the first differences of the exponents belonging to a separate latent root.

10. *Two quadratic forms.*

The complete system of two quadratic forms also possesses Q concomitants [3] from which a set of standard forms and a critical form can be selected, and thus the elementary divisors of two symmetric matrices can be determined by the above methods. The only restriction is that one of the two quadratics must be non-singular. There is an *a priori* reason for this, since any non-symmetric matrix may be resolved into two symmetric matrix factors, one of which is non-singular, according to a theorem of Frobenius*. Consequently any collineation can be constructed by successive polarization in two quadrics.

* 2, p. 152.

Let $\Sigma a_{ij}x_ix_j$ and $\Sigma r_{ij}x_ix_j$ be two quadratics, given symbolically by

$$f = a_x^2 = b_x^2 = \ldots, \quad g = r_x^2 = s_x^2 = \ldots, \tag{1}$$

where g is assumed to be non-singular, so that its reciprocal

$$\Gamma = u_\rho^2 = u_\sigma^2 = \ldots$$

is also non-singular. From a point x its polar prime in f can be formed, $a_x a$, whose pole in Γ is the point $a_x a_\rho \rho$. Thus a point to point collineation Γf is established, which is compounded of the two quadrics in this order. As in § 3 we may form a sequence of image points

$$x^0 = x, \quad x^1 = a_x a_\rho \rho, \quad x^2 = a_{x^1} a_\rho \rho = a_\rho a_\sigma b_x b_\sigma \rho, \quad \ldots, \tag{2}$$

where a, b are equivalent symbols for f, and ρ, σ for Γ.

The q-th compound is

$$x^1 y^1 \ldots z^1 = a_x b_y \ldots c_z a_\rho b_\sigma \ldots c_\tau \rho\sigma \ldots \tau$$

$$= \frac{1}{q!\,q!}\,(ab\ldots c\,|\,xy\ldots z)(ab\ldots c\,|\,\rho\sigma\ldots\tau)\,\rho\sigma\ldots\tau,$$

on interchanging first the q equivalent symbols ρ, and next the q symbols a, and adding the results. This abbreviates to

$$q!^2\, X_q^{\,1} = (A_q|\,X_q)(A_q|\,R_q')\,R_q', \tag{3}$$

where the bilinear symbol A_q' has now become $(A_q|\,R_q')\,R_q'$, the accent denoting a convolution of symbols ρ, σ contragredient to the symbols a, r. The numerical factor is the square of the original factor. Similarly, for $0 < q \leqslant n$,

$$x^1 x^2 \ldots x^q = \frac{1}{q!\,!^2}\,R_q'(A_q|\,R_q')(A_q|\,xR_{q-1}')(A_{q-1}|\,R_{q-1}')\ldots a_\rho a_x, \tag{4}$$

and

$$(x^0 x^1 \ldots x^{n-1}) = \frac{1}{(n-1)!\,!^2}\,(xR_{n-1}')(A_{n-1}|\,R_{n-1}')\ldots a_\rho a_x. \tag{5}$$

This naturally leads to a theory of Q concomitants for two quadrics, with standard and critical forms which characterize the elementary divisors, just as they did for the single bilinear form. But in order to complete the theory it is necessary to state the results, not only in terms of the symbols ρ, σ, but also, more symmetrically, in terms of the original r, s.

Now $\rho = r_1 r_2 \dots r_{n-1}$ in terms of $n-1$ equivalent symbols r; and the discriminant of g, which is $(r_1 r_2 \dots r_n)^2$ to a numerical factor $1/n!$, is also $r_{i\rho_i}^2$, with ρ_i replacing the $n-1$ other symbols r.

Consider now $(a_1 a_2 \dots a_q | \rho_1 \rho_2 \dots \rho_q) b_{1\rho_1} \dots b_{q\rho_q}$, where each a_i or b_i is a symbol or variable cogredient with a. This expression is equal to

$$\Pi \dot{a}_{i\rho_i} \Pi b_{i\rho_i} = \Omega_R (A_q R_{n-q})(r_2 \rho_2) \dots (r_q \rho_q) \Pi b_{i\rho_i},$$

by the fundamental identity which convolves all a_i in the first factor of the first product, where Ω_R is the determinantal permutation of the $n-1$ symbols r of ρ_1, let us say

$$\rho_1 = R_{n-1} = r_2 r_3 \dots r_q R_{n-q}, \quad A_q = a_1 a_2 \dots a_q.$$

The expression has ρ_1 already convolved within the factor $b_{1\rho_1}$; hence each term of the sum Ω is identically equal, and the whole can be written

$$\frac{(n-1)!}{(n-q)!} (A_q R_{n-q}) \dots \Pi b_{i\rho_i}.$$

On interchanging the n symbols of $(r_i \rho_i)$ in all ways, separately for each value of $i = 2, 3, \dots, q$, we get

$$\frac{(n-1)!}{(n-q)!} \frac{1}{n^{q-1}} (A_q R) . \Pi r_\rho . (b_1 \dot{r}_2 \dots \bar{r}_q R) b_{2\dot{\rho}_2} \dots b_{\bar{q}\bar{\rho}_q},$$

where $R = R_{n-q}$. On convolving $r_i \rho_i$ explicitly, for $i = 2, 3, \dots, q$, we obtain the factor $\Delta = r_\rho^2$ repeated $q-1$ times. The process transfers each b_i to the place of r_i and thus convolves $b_1 b_2 \dots b_q = B_q$. After adding all $q!$ permutations of these b_i on both sides of the identity we finally obtain the result

$$(A_q | R_q')(B_q | R_q') = q! \frac{(n-1)!}{(n-q)!} \frac{1}{n^{q-1}} \Delta^{q-1} (A_q R_{n-q})(B_q R_{n-q}), \qquad (6)$$

identically in all $2q$ symbols of A_q, B_q. This relation may be regarded as one which transforms any symbolic expression given in terms of ρ symbols occurring as q-th compounds (or to currency q) into one which involves the alternative and dual r symbols as $(n-q)$-th compounds.

For example in quinary forms, with $n = 5$, $q = 2$,

$$(\dots \rho\sigma)(\dots \rho\sigma) = \tfrac{8}{5} r_\rho^2 (\dots rst)(\dots rst).$$

Now consider the complete system [3] of two quadrics f and g. When expressed in terms of the variables u, v, w, ... only, it consists of P and Q

forms, where

$$\left.\begin{aligned} P_{ij} &= (A_i R_j U_k)^2, \quad i+j+k=n, \\ Q &= (A_{q_1} U_{n-q_1})(A_{q_1} R_{j_1} U_{k_1})(R_{j_1} A_{q_2} U'') \dots (A_{q_\nu} R_{j_\nu} U^{(2\nu-1)})(R_{j_\nu} U^{(2\nu)}), \end{aligned}\right\} \quad (7)$$

and $\quad n > q_1 > q_2 > \dots > q_\nu > 0 \quad$ and $\quad 0 < j_1 < j_2 < \dots < j_\nu < n.$

Each symbolic factor contains in all n symbols or variables of type a, r or u. Among these Q forms we may single out the Clebsch forms, by taking each compound U that occurs to be one or other of the nest of spaces u, $uv = U_2$, $uvw = U_3$, etc., up to U_{n-1}. Again, those forms wherein no u occurs in the second, fourth and further alternate factors may be called image forms. They are such that

$$q_1 + j_1 = \dots = q_\nu + j_\nu = n. \quad (8)$$

An image form will be shown to possess the property, in common with Q forms of the bilinear system, that it can be expressed as a determinant

$$(UV^1 \dots W^\nu) \quad (9)$$

to a constant factor, where $U, V, \dots, W$ are the compounds U_{n-q_1}, etc., which occur in the $\nu+1$ odd factors of the symbolic product Q in this order, and where the indices denote successive image formations in the collineation $f\Gamma$, reciprocating a prime first in Γ, then in f, and getting an image prime.

Such an image u^1 is given by $u^0 = u$, $u^1 = u_\rho a_\rho a$, etc., so that

$$\begin{aligned} q!^2 U_q^{\,1} &= (R_q' \,|\, U_q)(A_q \,|\, R_q') A_q \\ &= \lambda \Delta^{q-1} (R_{n-q} U)(A_q R_{n-q}) A_q, \end{aligned} \quad (10)$$

by (6), where λ is numerical. If $R_{n-q} = R_j$ and (8) holds, this result applies at once to the Q form, rolling it up from the right, two factors at a time, and finally producing the determinant (9) after the manner of § 3. The repeated factor Δ is the discriminant of the non-degenerate second quadric g.

Similar remarks apply to the forms

$$Q' = (A_{q_1} \,|\, X_{q_1})(A_{q_1} R_{j_1} \,|\, X_{k_1}) \dots,$$

where $n > q_1 > \dots > 0$, $q_i + j_i = k_i$, $0 < j_1 < \dots < n$, which belong to a complete system of two quadrics and point variables $x, y, \dots, z$; and indeed, with fairly obvious modifications, to such complete systems for the ground

forms $f = a_x^2$, $\Gamma = u_\rho^2$, and also for the case when both sets of variables u and x occur simultaneously.

Whereas a Q form possesses two sets of indices q and j, an image form in effect possesses one only, which may be taken to be the q set, owing to the restriction (8). The image form may now be written $A_{q_1 q_2} \ldots$ as before, with all the previous arguments available, so that we can distinguish the *standard forms* and the *critical form* by examining the integer sequence (q_i). This establishes the following result.

THEOREM V. *Those Q (or Q') forms in the complete system of two quadrics, from which variables are absent in the even factors, are characterized by a single integer sequence (q_i) with properties as in the earlier case of the bilinear form. The first standard form among the Q (or Q') forms which does not vanish will factorize into linear compound forms and give the indices of the elementary divisors for a pair of symmetric matrices, one of which is non-singular, by means of the repetitions of the factors.*

The integer ν in (7) has the significance of $h-1$ in the bilinear case, for the expression (9) shows that exactly $\nu+1$ invariant-factors must exist if it refers to the critical form. The linear factors of the critical Q form are the expressions in prime coordinates for the latent nests of the system, which are more naturally looked upon as the self-conjugate frames for the case of two quadrics.

For example, if $\nu = 0$, and all the latent roots differ, the critical Q gives n distinct points, forming the unique self-conjugate polyhedron of two quadrics. But for all further cases the critical form shows exactly how the self-conjugate system must be modified. The dual critical form Q' gives the primes of the self-conjugate system.

For example, the form A_{421} is

$$(R_4' xy)(A_4|R_4')(A_4|R_2' xy)(A_2|R_2')(A_2|\rho x)\, a_\rho\, a_x$$

for senary quadrics a_x^2 and u_ρ^2 when all variables are expressed in points $x, y, \ldots$. This is a Q' form: the first differences of the suffixes of A, which are 2, 1, 0, show that it is a standard form. It simplifies successively to

$$\begin{aligned}
&(R_4' xy) \ \ldots \ \ldots \ \ldots \ (A_2|x^1 x^0) \\
&= 2!^2 (R_4' xy)(A_4|R_4')(A_4|x^2 x^1 x^0 y^0) \\
&= 4!^2\, 2!^2 (x^3 x^2 x^1 y^1 x^0 y^0).
\end{aligned}$$

It is a critical form provided that the pair of quadrics has exactly two invariant-factors of orders 4 and 2 (so that $4+2=6$, the number for the

senary case). These integers are the number of factors x and of y occurring in the image sequences. Such a form will factorize just as in the earlier bilinear case, and determine the elementary divisors of this quartic and quadratic invariant-factor.

Again, the matrix, diag $\left(\begin{bmatrix} . & \alpha+\lambda \\ \alpha+\lambda & 1 \end{bmatrix}, \alpha+\lambda, \beta+\lambda\right)$, is that of two quaternary quadrics f, g whose Segre characteristic is $\big((21)\,1\big)$, when $\alpha \neq \beta$. The matrix is the canonical form for $f+\lambda g$. The polar of the point $\{x_1, x_2, x_3, x_4\}$ in f is $[\alpha x_2, \alpha x_1+x_2, \alpha x_3, \beta x_4]$ whose pole in the quadric g is $\{\alpha x_1+x_2, \alpha x_2, \alpha x_3, \beta x_4\}$ which is now in the classical canonical form of a single non-symmetric quaternary matrix with the same Segre characteristic.

The critical form is

$$(R_2 xy)(A_2 R_2)(A_2 R_3 x)(aR_3)\, a_x$$

in terms of symbols r, s (or alternatively we use R_2' and ρ for R_2 and R_3). This is also $(x^0x^1x^2y^0)$ to a constant factor. It resolves into factors, giving two planes and a line in one of the planes, by means of its bideterminant

$$\left|\begin{array}{cccc|cccc} 1 & \alpha & \alpha^2 & . & x_1 & x_2 & x_3 & . \\ . & 1 & 2\alpha & . & x_2 & . & . & . \\ 1 & \beta & \beta^2 & . & . & . & . & x_4 \\ . & . & . & 1 & y_1 & y_2 & y_3 & . \\ . & . & . & . & y_2 & . & . & . \\ . & . & . & 1 & . & . & . & y_4 \end{array}\right| = (\beta-\alpha)^2\,(xy)_{23}\, x_2 x_4,$$

where $x_2 = 0$, $x_4 = 0$ give the planes and $(xy)_{23} = 0$ the line. Hence the critical form resolves into three such factors, without recourse to the canonical form. This critical form belongs to all pairs of quadratics $\big((31)\big)$, $\big((21)\,1\big)$, with the same type of invariant-factors, one cubic and one linear, as characterized by its image form $(x^0x^1x^2y^0)$, so that it could also factorize into a repeated plane and a line therein $(xy)_{34}x_3^2$, for the $\big((31)\big)$ case, but of course the two quadrics would then have a different canonical specification.

11. *Combinants and pencils of bilinear forms.*

The above treatment could be applied to other types of matrices, Hermitian, skew symmetric and orthogonal. It would be interesting to

derive the minimal indices of a singular matrix pencil by this method. Algebraically this would be a problem in combinants rather than in a single bilinear form or in a pair of quadrics. The singular case first arises through treating the group of matrices $H(\lambda A+\mu B)K$ where H and K are any two independent non-singular matrices, and $\lambda A+\mu B$ is any member of the pencil containing A and B. For these the invariant theory is simpler than the above, since the symbolic factors which arise can be confined to those wherein the product a_ρ, involving contragredient symbols, is excluded, owing to the independence of the transformations H and K of the variables u and x. For this group the complete system of two distinct bilinear forms $u_\alpha a_x$ and $u_\rho r_x$ can be constructed by virtually the same argument as for two quadrics a_x^2 and r_x^2.

In fact any concomitant is necessarily an aggregate of factors $(\alpha \dots r \dots u \dots)$ and $(\alpha \dots \rho \dots x \dots)$, which can be prepared for duplicates A_q, A_q' and arranged as

$$(A_{q_1}\dots)(A'_{q_1}\dots)(A_{q_2}\dots)(A'_{q_2}\dots)\dots, \quad (q_1 \geqslant q_2 \geqslant \dots).$$

Then the sets of symbols $R_j = rs\dots$ and $R_j' = \rho\sigma\dots$ can be systematically paired from right to left. In all this the derangements of elements a, r, u occur in one set (say the odd set) of factors, and those of α, ρ, x occur within the even set. This yields a complete system of forms P_{ij} and $Q_{q\dots j\dots}$, analogous to those of § 10 (7), namely

$$P_{ij} = (A_i R_j U_k)(A_i' R_j' X_k) \quad (i+j+k=n),$$

$$Q = (A_{q_1} U_{n-q_1})(A'_{q_1} R'_{j_1} X_{k_1})(R_{j_1} A_{q_2} U'') \dots (R_{j_\nu} X^{(\nu)})$$

$$(n > q_1 > \dots > q_\nu > 0, \quad 0 < j_1 < \dots < j_\nu < n).$$

Those P forms for which k vanishes are manifestly the invariants, $n+1$ in number, which form the coefficients of powers of $\lambda : \mu$ in the characteristic equation $|\lambda A+\mu B| = 0$. The singular case arises when, and only when, all these invariants vanish.

(*Note. Added* 9 *March*, 1945.) Since writing the above I have only recently discovered that Dr. John Williamson, 5, had already written on the same theme in 1934, when he discussed the covariants of two quadrics, and showed explicitly the need of a critical form for the class of concomitants to which he confined the investigation. This class

corresponds in the present work to the graph or tableau consisting of one row together with one column of crosses. Dr. Williamson gave a full account of the regular case for the two quadrics.

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LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (I)

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Last year, after earlier special results by H. Davenport†, L. J. Mordell extended Minkowski's classical work on lattice points in convex bodies‡ to certain two-dimensional non-convex domains. Then, very recently, he has found very general results for finite or infinite non-convex domains§.

His method depends on an application of Minkowski's theorem on linear forms. He thus obtains point sets abutting on the given domain and containing at least one lattice point. By forming other lattice points from these so obtained, he succeeds in finding the lattices of smallest determinant that contain no inner points of the given domain, except the origin of the coordinate system.

In the present paper I consider the same problem for a general class of two-dimensional domains which I call *simple star domains*. A simple star domain is a bounded and closed set that contains the origin of the coordinate system as an inner point and is symmetrical in the origin‖. Further, its boundary is a Jordan curve that is composed of a finite number of analytical arcs, and which is intersected in just *one* point by every radius vector from the origin. For these domains I derive a finite algorithm to find the smallest determinant for which lattices exist containing no inner points of the set except the origin. This method is a generalization of one due to Minkowski¶. In the second part of this paper I shall apply the algorithm

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† *Proc. London Math. Soc.* (2), 44 (1938), 412–431, and 45 (1939), 98–125.

‡ *Geometrie der Zahlen* (Leipzig und Berlin, 1910).

§ *Proc. London Math. Soc.* (2), 48 (1943), 198–228 and 339–390; *Journal London Math. Soc.*, 17 (1942), 107–115.

‖ The restriction to *symmetrical* domains can easily be removed (§ 16).

¶ *Ges. Abh.*, II, 3–42 (Leipzig und Berlin, 1911).

to some special domains. The method can probably be extended to more than two dimensions; but the work required will be excessive. In special cases the method can be applied to infinite domains.

I should like to express my gratitude to Prof. Mordell for his criticisms and his help with the manuscript.

CHAPTER I. THE GENERAL THEORY.

1. *Lattices in the* (x, y)*-plane.*

Let an arbitrary rectangular coordinate system in the plane be given. The unit square with vertices at the points

$$(0, 0),\ (1, 0),\ (1, 1),\ (0, 1)$$

is therefore of area unity.

We identify the point of coordinates (x, y) with the vector of components (x, y), and apply the usual vector notation. If

$$R: (\alpha, \beta) \quad \text{and} \quad S: (\gamma, \delta)$$

are two vectors, u, v two real numbers, then $P = uR+vS$ denotes the vector of components

$$x = u\alpha+v\gamma, \quad y = u\beta+v\delta.$$

Further

$$(R, S) = \alpha\delta-\beta\gamma$$

stands for the determinant of the two vectors. Hence

$$A = |(R, S)|,$$

where A is the area of the parallelogram Π with vertices at O, R, $R+S$, S; here $O: (0, 0)$ denotes the origin of the coordinate system.

The points or vectors R and S are called

$$\textit{dependent}, \quad \text{if} \quad (R, S) = 0,$$

$$\textit{independent}, \quad \text{if} \quad (R, S) \neq 0.$$

In the first case the parallelogram Π degenerates into a line segment, and R, S are collinear with O.

Assume now that $(R, S) \neq 0$, and let Λ be the lattice of all points

$$P = uR+vS \qquad (u, v = 0, \pm 1, \pm 2, \ldots).$$

The positive number

$$d = d(\Lambda) = |(R, S)|$$

is called the *determinant* of the lattice, and R, S are called a *basis* of Λ. If

$$R^* = u_1 R + v_1 S \quad \text{and} \quad S^* = u_2 R + v_2 S$$

are any two points of Λ, then the non-negative integer

$$n = \text{ind}\,(R^*, S^*) = |u_1 v_2 - u_2 v_1| = \frac{|(R^*, S^*)|}{|(R, S)|}$$

is called the *index* of R^*, S^*. This index vanishes if, and only if, R^* and S^* are dependent. If $n \neq 0$, then the area A^* of the parallelogram Π^* with vertices at O, R^*, R^*+S^*, S^* is given by

$$A^* = |(R^*, S^*)| = nA.$$

In the special case $n = 1$, the two points R^*, S^* form a new basis of Λ, and the equality $A^* = A$ holds.

From the definition of n,

$$|(R^*, S^*)| = d(\Lambda)\,\text{ind}\,(R^*, S^*). \tag{1}$$

We shall apply this simple relation repeatedly.

We require the following lemma†‡:

The lattice Λ has at least one reduced basis. *This is characterized by the property that no diagonal of Π^* is smaller than the sides of this parallelogram. Hence all angles of Π^* lie between* 60° *and* 120°, *and the area $A^* = d(\Lambda)$ of Π^* satisfies the inequalities*

$$\sqrt{\tfrac{3}{4}} \cdot \overline{OR^*} \times \overline{OS^*} \leqslant d(\Lambda) \leqslant \overline{OR^*} \times \overline{OS^*}. \tag{2}$$

A point

$$P = uR + vS$$

of Λ is called *primitive* when the greatest common divisor $Gcd(u, v) = 1$. The point P is primitive if, and only if, there is no lattice point on the line segment connecting O with P except the endpoints O and P. It is well known† that corresponding to every primitive lattice point $R^* = P$ there exist lattice points S^* such that R^*, S^* form a basis of Λ.

† Bachmann, *Quadratische Formen*, 2 (Leipzig und Berlin, 1923), Kap. 5.

‡ $\overline{PQ}$ denotes the distance between the two points P and Q.

2. *Star domains.*

A *star domain* K in the (x, y)-plane is a point set with the following properties:

(*a*) *K is a bounded and closed set.*

(*b*) *The origin O is an inner point of K.*

(*c*) *If K contains a point P, K also contains the point $-P$, which is symmetrical to P in the origin.*

(*d*) *If C denotes the boundary† of K, then every radius vector from the origin intersects C in just one point.*

From this definition some simple properties of K follow at once. For an arbitrary $r > 0$, let Γ_r be the circle

$$x^2+y^2 \leqslant r^2.$$

By (*a*) and (*b*) there exist two numbers P and ρ with

$$0 < \rho \leqslant \mathrm{P}$$

such that Γ_{P} contains K, and that K contains Γ_ρ. We call Γ_{P} an *outer circle* and Γ_ρ an *inner circle* of K‡.

Further, by (*d*), corresponding to every angle ϕ there is exactly one point

$$P(\phi) : (x, y)$$

of coordinates

$$x = r(\phi)\cos\phi, \quad y = r(\phi)\sin\phi,$$

such that

$$r(\phi) > 0,$$

and that $P(\phi)$ lies on C. From (*c*) and (*d*), the point symmetrical to $P(\phi)$ also lies on C and is given by

$$P(\phi+\pi) = -P(\phi).$$

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† *I.e.* the set of all those points of K which are not inner points.

‡ Instead of using Γ_{P} and Γ_ρ, we may compare K with other circumscribed and inscribed convex domains. This would improve some of the later inequalities.

Obviously for all angles ϕ,

$$\rho \leqslant r(\phi) \leqslant \mathrm{P}. \tag{3}$$

THEOREM 1. *$r(\phi)$ is a continuous function of ϕ.*

Proof. Suppose the assertion is false. Then there is an angle ϕ_0 and an infinite sequence of angles

$$\phi_1, \phi_2, \phi_3, \ldots, \tag{4}$$

with limit ϕ_0, such that

$$\lim_{\nu \to \infty} r(\phi_\nu)$$

either does not exist, or exists and is different from $r(\phi_0)$. By (3), we can find an infinite subsequence

$$\phi_{\nu_1}, \phi_{\nu_2}, \phi_{\nu_3}, \ldots$$

of (4) for which

$$\lim_{k \to \infty} \phi_{\nu_k} = \phi_0, \quad \text{but} \quad \lim_{k \to \infty} r(\phi_{\nu_k}) = r^* \neq r(\phi_0).$$

By property (a), the point P^* of coordinates

$$x = r^* \cos \phi_0, \quad y = r^* \sin \phi_0$$

belongs to K. It also belongs to C, since it is the limit of the boundary points

$$P(\phi_{\nu_k}) \qquad (k = 1, 2, 3, \ldots).$$

Hence there are two different points $P(\phi_0)$ and P^* of C on the same radius vector from O, in contradiction to (d).

By Theorem 1, C is a *Jordan curve*. It therefore divides the plane into two separate parts: its interior which contains only points of K, and its exterior which contains no points of K.

Hence with a point P also all points tP, where $0 \leqslant t \leqslant 1$, belong to K. Further, if P_1 and P_2 are two points on C such that the whole line segment from P_1 to P_2 belongs to K, then the triangle with vertices at O, P_1, P_2 consists only of points of K.

3. *The lattice problem.*

A lattice Λ is called *admissible* with respect to the star domain K, say, for shortness, K-admissible, if none of its points except O is an inner point

of K. The lattice may, however, contain points on the boundary C of K.

There are admissible lattices. For the lattice generated by

$$(P, 0) \quad \text{and} \quad (0, P)$$

is Γ_P-admissible, and so is also K-admissible.

Hence the lower bound

$$\Delta(K) = \text{l. b. } d(\Lambda)$$

extended over all K-admissible lattices, exists and is a finite non-negative number.

THEOREM 2. *If the star domain K_1 is contained in a second star domain K_2, then*

$$\Delta(K_1) \leqslant \Delta(K_2).$$

Proof. Every K_2-admissible lattice is also K_1-admissible.

THEOREM 3. *For a circle Γ_r,*

$$\Delta(\Gamma_r) = \sqrt{\tfrac{3}{4}} \cdot r^2.$$

Proof. The assertion is equivalent to the classical result on the minimum of a positive definite binary quadratic form†.

THEOREM 4. *Let Γ_P be an outer circle and Γ_ρ an inner circle of the star domain K. Then*

$$0 < \sqrt{\tfrac{3}{4}} \cdot \rho^2 \leqslant \Delta(K) \leqslant \sqrt{\tfrac{3}{4}} \cdot P^2.$$

Proof. Evident from Theorems 2 and 3.

The problem now is to determine $\Delta(K)$ for a given star domain K. I generalize a method of Minkowski for convex domains, and show how the problem can be reduced to a finite number of elementary extremal problems.

4. *Critical lattices.*

A K-admissible lattice Λ is called a *critical lattice* if

$$d(\Lambda) = \Delta(K),$$

i.e. if the lower bound is actually attained by its determinant. I now prove a theorem fundamental for what follows. It should be remarked that it need not hold for *infinite* domains.

† Bachmann, *Quadratische Formen*, 2 (Leipzig und Berlin, 1923), Kap. 5.

THEOREM 5. *Corresponding to every star domain K, there is at least one critical lattice.*

Proof. By the definition of $\Delta(K)$, there exists an infinite sequence of admissible lattices†

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

such that

$$\lim_{\nu\to\infty} d(\Lambda_\nu) = \Delta(K).$$

By Theorem 4, we may assume that, for all indices ν,

$$d(\Lambda_\nu) \leqslant \mathrm{P}^2. \tag{5}$$

Let R_ν, S_ν, for $\nu = 1, 2, 3, \ldots$, be a *reduced basis* of Λ_ν (§ 1). Then by formula (2) of § 1,

$$d(\Lambda_\nu) \geqslant \sqrt{\tfrac{3}{4}} \cdot \overline{OR}_\nu \times \overline{OS}_\nu.$$

Further, since Λ_ν is Γ_ρ-admissible,

$$\overline{OR}_\nu \geqslant \rho, \quad \overline{OS}_\nu \geqslant \rho.$$

Therefore, by (5),

$$\overline{OR}_\nu \leqslant \frac{2}{\sqrt{3}} \frac{\mathrm{P}^2}{\rho} \quad \text{and} \quad \overline{OS}_\nu \leqslant \frac{2}{\sqrt{3}} \frac{\mathrm{P}^2}{\rho}.$$

Hence reduced basis points R_ν and S_ν of all lattices Λ_ν lie at a *bounded* distance from the origin.

It is therefore possible to choose an infinite subsequence

$$\Lambda_{\nu_1}, \Lambda_{\nu_2}, \Lambda_{\nu_3}, \ldots$$

of the Λ's such that R_{ν_k} tends to a limit point R^*, and S_{ν_k} to a limit point S^*. Denote by Λ^* the lattice of basis R^*, S^*. This lattice is K-admissible. It is also critical, since

$$d(\Lambda^*) = \lim_{k\to\infty} d(\Lambda_{\nu_k}),$$

and since, by hypothesis, this limit is equal to $\Delta(K)$.

It may be mentioned that there are star domains with any finite or infinite number of critical lattices.

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† These lattices need not be all different.

5. *Regular and singular lattices.*

By property (c) and (d) of a star domain, two different points on its boundary are collinear with O if, and only if, they are symmetrical in this point. Hence three different points on the boundary C cannot all be collinear with O.

THEOREM 6. *Every critical lattice contains at least two independent points P_1 and P_2 on C.*

This theorem is an immediate consequence of the

THEOREM 7. *Let Λ be a K-admissible lattice which has either no points or only one pair of symmetrical points P, $-P$ on C. Then there exists a K-admissible lattice Λ^* such that $d(\Lambda^*) < d(\Lambda)$.*

Proof. First assume that no points of Λ lie on C. Denote by R, S an arbitrary basis of Λ and by ϑ a sufficiently small positive number. The lattice Λ^* of basis R and $(1-\vartheta)S$ is evidently K-admissible; but its determinant $d(\Lambda^*) = (1-\vartheta)d(\Lambda)$ is smaller than $d(\Lambda)$.

Secondly, assume that Λ has only the two points P and $-P$ on C. Then P is a primitive lattice point (§ 1); hence there exists a second lattice point S which with $R = P$ forms a basis of Λ. Let again ϑ be a sufficiently small positive number. The lattice Λ^* of basis R and $(1-\vartheta)S$ still contains only the two points P and $-P$ on C, and is therefore admissible; again $d(\Lambda^*) < d(\Lambda)$.

It is useful to make the following distinction.

A critical lattice of K is called *singular* or *regular*, according as it has *four points* or *more than four points* on the boundary C of K.

6. *The index of two lattice points on C.*

Let Λ be a K-admissible lattice which contains two independent points P_1 and P_2 on C. Upper bounds for the index of these two points (§ 1) are given by the following theorems.

THEOREM 8. $$\operatorname{ind}(P_1, P_2) \leqslant \frac{2}{\sqrt{3}}\left(\frac{P}{\rho}\right)^2.$$

Proof. Let R, S be a basis of Λ. By § 1,

$$\operatorname{ind}(P_1, P_2) = \frac{|(P_1, P_2)|}{|(R, S)|}.$$

Since K lies entirely in Γ_P, obviously

$$|(P_1, P_2)| \leqslant P^2.$$

Further, since K contains Γ_ρ, Λ is Γ_ρ-admissible. Hence, by Theorem 3,

$$d(\Lambda) = |(R, S)| \geqslant \Delta(\Gamma_\rho) = \sqrt{\tfrac{3}{4}} \cdot \rho^2.$$

The assertion follows at once from these two inequalities.

THEOREM 9. *Let ϑ with $0 < \vartheta \leqslant 1$ be a number such that all points of the parallelogram Π_0 with vertices at*

$$\vartheta(P_1+P_2), \quad \vartheta(P_1-P_2), \quad \vartheta(-P_1-P_2), \quad \vartheta(-P_1+P_2)$$

belong to K. Then

$$\operatorname{ind}(P_1, P_2) \leqslant \vartheta^{-2}.$$

Proof. Let R, S be a basis of Λ. Every lattice point P can be written as $P = uR+vS$ with integral u, v; in particular

$$P_1 = u_1 R+v_1 S, \quad P_2 = u_2 R+v_2 S.$$

When we eliminate R and S, P takes the form

$$P = \frac{\omega_1 P_1+\omega_2 P_2}{u_1 v_2-u_2 v_1}, \quad \text{where} \quad \omega_1 = v_2 u-u_2 v, \ \omega_2 = v_1 u+u_1 v.$$

By § 1, the denominator has absolute value

$$n = |u_1 v_2-u_2 v_1| = \operatorname{ind}(P_1, P_2).$$

Except for the sign, n is the determinant of the two linear forms ω_1 and ω_2 in u and v. By Minkowski's theorem on linear forms, integral values of u and v not both zero exist such that

$$|\omega_1| \leqslant n^{\frac{1}{2}}, \quad |\omega_2| \leqslant n^{\frac{1}{2}}.$$

Hence there is a lattice point

$$P = \lambda_1 P_1+\lambda_2 P_2 \neq O$$

with coefficients

$$\lambda_1 = \frac{\omega_1}{u_1 v_2-u_2 v_1}, \quad \lambda_2 = \frac{\omega_2}{u_1 v_2-u_2 v_1}$$

which satisfy the inequalities

$$|\lambda_1| \leqslant n^{\frac{1}{2}-1} = n^{-\frac{1}{2}}, \quad |\lambda_2| \leqslant n^{\frac{1}{2}-1} = n^{-\frac{1}{2}}.$$

If $n^{-\frac{1}{2}} < \vartheta$, then P would be an inner point of Π_0, hence also an inner point of K; this is impossible since Λ is K-admissible. Therefore

$$n^{-\frac{1}{2}} \geqq \vartheta \quad i.e., \quad n \leqq \vartheta^{-2},$$

as was to be proved.

THEOREM 10. *Let ϑ with $0 < \vartheta \leqq 1$ be a number such that all points of the parallelogram with vertices at*

$$\vartheta P_1,\ \vartheta P_2,\ -\vartheta P_1,\ -\vartheta P_2$$

belong to K. Then

$$\text{ind}\,(P_1, P_2) \leqq 2\vartheta^{-2}.$$

I omit the proof, which is similar to that of Theorem 9.

THEOREM 11. *If the two points P_1 and P_2 of Λ on C have the property that all points of the line segment joining P_1 to P_2 are inner points of K, then*

$$\text{ind}\,(P_1, P_2) = 1,$$

i.e., P_1 and P_2 form a basis of Λ.

Proof. Assume that the assertion is false; then the lattice Λ^* generated by P_1 and P_2 is only a sub-lattice of Λ. Hence there is at least one point Q of Λ which does not belong to Λ^*. We can find a point P of Λ^* such that $P + \epsilon Q$, where $\epsilon = 1$ or $\epsilon = -1$, belongs to the triangle T with vertices at O, P_1, P_2. The point $P + \epsilon Q$ so found is an element of Λ, but not of Λ^*, and is therefore different from the vertices of T. It cannot be an inner point of K since Λ is admissible. This forms a contradiction to the hypothesis, since, by § 2, all points of T different from P_1 and P_2 are inner points of K.

7. *Lattices with at least four points on C.*

Let Λ be an arbitrary lattice which contains two *primitive* and *independent* points P_1 and P_2 on the boundary C of K. Then there is a lattice point S^* which together with P_1 generates Λ. Further, P_1 and $S = gP_1 + \epsilon S^*$ also form a basis of Λ, where g is an arbitrary integer and $\epsilon = \pm 1$. The second point P_2 can be written as

$$P_2 = u^* P_1 + v^* S^* = (u^* - \epsilon g v^*) P_1 + \epsilon v^* S$$

with two integers u^* and v^*. Here

$$|v^*| = \text{ind}\,(P_1, P_2) > 0,$$

since P_1 and P_2 are independent. Determine ϵ and g such that

$$v_2 = \epsilon v^* > 0 \quad \text{and} \quad 0 \leqslant u_2 = u^* - \epsilon g v^* < |v^*| = v_2.$$

Then $$P_2 = u_2 P_1 + v_2 S,$$

where

$$v_2 = \text{ind}\,(P_1, P_2), \quad 0 \leqslant u_2 < v_2, \quad Gcd(u_2, v_2) = 1;$$

the third formula holds, since P_2 is primitive. For given v_2, u_2 is one of $\phi(v_2)$ different integers, where $\phi(n)$ is Euler's function.

An arbitrary point P of Λ can be written as

$$P = uP_1 + vS$$

with integral coefficients u and v. On eliminating S,

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \omega_1 = uv_2 - u_2 v, \quad \omega_2 = v.$$

The two coefficients ω_1 and ω_2 are integers satisfying the congruence

$$\omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

but are otherwise arbitrary. Obviously

$$|\omega_1| = \text{ind}\,(P, P_2), \quad |\omega_2| = \text{ind}\,(P, P_1).$$

Now, if Λ is an *admissible* lattice and P lies on C, then by Theorem 8,

$$(6) \qquad 0 < v_2 \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_1| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_2| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2.$$

Hence there are only a *finite* number of possibilities for the coefficients v_2, ω_1, ω_2. Better approximations for these coefficients are obtained if Theorems 9–11 can be applied.

The important question now arises: *When is the lattice Λ of all points*

$$(7) \qquad P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

admissible, where u_2 and v_2 are two arbitrary integers such that

$$0 \leqslant u_2 < v_2, \quad Gcd\,(u_2, v_2) = 1\,?$$

An answer is given by

THEOREM 12. *The lattice Λ is admissible if no point $P \neq O$ of Λ, with*

$$|\omega_1| \leqslant \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}, \quad |\omega_2| \leqslant \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}, \tag{8}$$

is an inner point of K.

Proof. It suffices to show that a point P for which at least one of the inequalities (8) is false does not lie in Γ_{P}, for then the point cannot belong to K.

Suppose then that, say,

$$|\omega_1| > \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}. \tag{9}$$

Since P_2 lies on C and therefore in Γ_{P},

$$\overline{OP}_2 \leqslant \mathrm{P}.$$

By § 1, $|(P_1, P_2)|$ is the area of the parallelogram with vertices at O, P_1, P_1+P_2, P_2. Hence the distance of P_1 from the line L through O and P_2 is

$$\frac{|(P_1, P_2)|}{\overline{OP}_2} \geqslant \frac{|(P_1, P_2)|}{\mathrm{P}}.$$

The distance of

$$P = \lambda_1 P_1 + \lambda_2 P_2, \qquad \lambda_1 = \frac{\omega_1}{v_2}, \quad \lambda_2 = \frac{\omega_2}{v_2},$$

from L is the same as that of the point $\lambda_1 P_1$ from this line, and so by the last inequality and by (9) is not less than

$$|\lambda_1| . \frac{|(P_1, P_2)|}{\mathrm{P}} > \mathrm{P}.$$

Hence P lies at a distance greater than P from the origin and does not belong to Γ_{P}.

It is useful to remark that the inequalities (8) in Theorem 12 can be replaced by

$$|\omega_1| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_2| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2,$$

provided that

$$v_2 \leqslant \frac{2}{\sqrt{3}} \frac{|(P_1, P_2)|}{\rho^2};$$

and if this last condition is not satisfied, then the lattice is not admissible, by the proof of Theorem 8.

8. *Simple star domains.*

Henceforth we consider the special class of star domains defined below.

A bounded rectifiable curve A in the (x, y)-plane is called an *analytical arc*, if it has the following properties:

(*a*) *To every point P on A there exists an infinity of vectors*

$$P', P'', P''', \ldots$$

not all zero such that every point P^* of A sufficiently near to P can be written as

$$P^* = P + P' \frac{s}{1!} + P'' \frac{s^2}{2!} + P''' \frac{s^3}{3!} + \ldots,$$

where s is a real parameter, and where this power series converges if $|s|$ is sufficiently small†.

(*b*) *The coefficient P' is different from 0 except at most in the two end points of A.*

It follows that in every point of A there is a unique tangent and an osculating circle.

THEOREM 13. *The common points of two analytical arcs are either finite in number, or they form a third analytical arc.*

Proof. The statement follows from the classical properties of regular analytical functions.

A star domain K is called *simple* if its boundary C consists of a *finite* number $A_1, A_2, \ldots, A_l$ of analytical arcs. Let

$$\Sigma = \{Q_1, Q_2, \ldots, Q_l\}$$

be the set of end points of these arcs, where the notation is such that

$$A_\lambda \text{ and } A_{\lambda+1} \text{ meet in } Q_\lambda \quad (\lambda = 1, 2, \ldots, l; \ A_{l+1} = A_1).$$

† By the Heine-Borel theorem, a finite number of these series suffices to represent the whole arc.

A straight line T is called an *inner* or an *outer tac-line* of K at a point P of C, according as *all* or *no* points of T sufficiently near to, but different from, P belong to K. There is evidently no tac-line at a point of inflexion, and there may not be one at the points of Σ. Excluding these points, there is just one tac-line at every other point of C, namely the tangent. In a point of Σ, the tac-lines fill up a complete angle if the two arcs of C meeting there have different tangents. When, however, these tangents coincide, there are three possibilities. Either there is *no* tac-line, or just *one* tac-line, or *all lines through this point except the tangent* are tac-lines.

Excluding the points of C which have no tac-line, the remaining points will be called *convex* or *concave*, according as they have an *outer* or *inner* tac-line.

Then C can be divided into a finite number of open arcs, consisting of only convex, or concave, points. The corresponding arcs are also called *convex* or *concave*.

It is clear that there are at most a finite number of tac-lines of K which have a given direction.

9. *Properties of the singular lattices.*

Let Λ be a singular lattice of K, having the four points $\pm P_1$ and $\pm P_2$ on C. Then for the two integers u_2 and v_2 associated with these points, all points of Λ are of the form

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Denote by P_1^* and P_2^* any two points of C sufficiently near to P_1 and P_2, respectively. Form the new lattice Λ^* of all points

$$P^* = \frac{\omega_1 P_1^* + \omega_2 P_2^*}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 = 0 \pmod{v_2}.$$

This lattice evidently is also admissible. Since Λ is critical, Λ^* must satisfy the inequality

$$d(\Lambda^*) \geqq d(\Lambda). \tag{10}$$

We may assume without loss of generality that

$$(P_1, P_2) > 0, \text{ and therefore also } (P_1^*, P_2^*) > 0.$$

By formula (1) in § 1,

$$(P_1, P_2) = v_2 d(\Lambda), \quad (P_1^*, P_2^*) = v_2 d(\Lambda^*),$$

and so (10) takes the simpler form

$$(P_1^*, P_2^*) \geqslant (P_1, P_2). \tag{11}$$

THEOREM 14. *Let $\pm P_1$ and $\pm P_2$ be the four points of the singular lattice Λ on C. Then the straight lines through $\pm P_1$ parallel to the vector P_2, and the straight lines through $\pm P_2$ parallel to the vector P_1, are inner tac-lines of K.*

Proof. If the assertion is false, we may assume, without loss of generality, that a point P_2^{**} arbitrarily near to P_2 lies on the line through P_2 parallel to the vector P_1 and does not belong to K. The lattice Λ^{**} through P_1 and P_2^{**} and defined by the integers u_2 and v_2 is still admissible, but has on C only the lattice points O, P_1 and $-P_1$. It is therefore not critical. It has the same determinant as Λ, and so, by Theorem 7, there exists an admissible lattice of smaller determinant.

Theorem 14 enables us to find the singular lattices containing points of Σ; these lattices are *finite* in number.

We may assume from now on that neither P_1 nor P_2 belongs to Σ. Hence the points P_1^* and P_2^* above can be written as

$$P_1^* = P_1 + P_1' \frac{s}{1!} + P_1'' \frac{s^2}{2!} + \ldots,$$

$$P_2^* = P_2 + P_2' \frac{t}{1!} + P_2'' \frac{t^2}{2!} + \ldots,$$

where $|s|$, $|t|$ are sufficiently small; also by (b) of § 8,

$$P_1' \neq 0, \quad P_2' \neq 0.$$

When we substitute these series for P_1^* and P_2^*, (11) takes the form

$$\{(P_1', P_2)s + (P_1, P_2')t\} + \tfrac{1}{2}\{(P_1'', P_2)s^2 + 2(P_1', P_2')st + (P_1, P_2'')t^2\} + \ldots \geqslant 0.$$

This inequality is to hold if $|s|$ and $|t|$ are sufficiently small. Hence, first,

$$(P_1', P_2) = (P_1, P_2') = 0, \tag{12}$$

i.e. the tangents to C at P_1, P_2 are parallel to the vectors P_2, P_1 respectively. This result is contained in Theorem 14.

Secondly,

$$(13) \quad (P_1'', P_2)s^2+2(P_1', P_2')st+(P_1, P_2'')t^2 \geqslant 0 \quad \text{for all real } s, t.$$

If the quadratic form on the left-hand side is positive definite, then (11) holds in the stronger form

$$(P_1^*, P_2^*) > (P_1, P_2)$$

for all pairs of points P_1^*, P_2^* sufficiently near to P_1, P_2, except for the pair $P_1^* = P_1$, $P_2^* = P_2$.

From (12) and (13) we can find all pairs of points P_1, P_2 not in Σ through which a singular lattice may be drawn. In general, the equations (12) have only a *finite* number of solutions, since they reduce to a finite number of pairs of equations whose left-hand sides are analytical functions.

An exceptional case may arise when the two equations (12) lead to an *infinity* of points P_1, P_2. Then both equations are satisfied if P_1 lies anywhere on an arc B_1 of C, and P_2 determined by P_1 lies on an arc B_2 of C†. This is the case if B_2 is the envelope of the straight lines

$$uy-vx = \kappa,$$

where $\kappa \neq 0$ is a constant and $P_1 : (u, v)$ runs over all points of B_1. Geometrically, B_2 is obtained by rotating through a right angle about the origin the polar reciprocal of B_1 with respect to the circle

$$x^2+y^2 = \kappa.$$

Evidently for all points P_1 and P_2 of these arcs associated by (12),

$$(P_1, P_2) = \kappa.$$

Hence, even if there is an infinity of admissible lattices, their determinants can only have one of the *finite* number of values

$$d(\Lambda) = \frac{\kappa}{v_2}.$$

Hence Theorem 14 and the formulae (12), (13) enable us to determine a finite or infinite set

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$$

† The two arcs may have common points or even be identical.

of lattices amongst which the singular lattices will be found. Even if the set is infinite, their determinants

$$d(\Lambda_1),\ d(\Lambda_2),\ d(\Lambda_3),\ \ldots$$

have only a *finite* number of different values. It is therefore possible to decide which is the *smallest* determinant.

10. *An example of a star domain with a singular lattice.*

The results of the last paragraph do not assert that there exist domains with singular lattices, and so I give an example of a simple star domain with this property.

Let θ be a number in the interval

$$\tfrac{3}{4} < \theta < 1.$$

Then K, the set of all points (x, y) which lie in at least one of the two rectangles

$$|x| \leqslant \theta,\ |y| \leqslant \tfrac{1}{2} \quad \text{and} \quad |x| \leqslant \tfrac{1}{2},\ |y| \leqslant \theta,$$

is a simple star domain; it has the form of a cross.

Let Λ be a critical lattice of K. If Λ is singular, then by Theorem 14 it must be the lattice Λ_0 which passes through

$$P_1\colon\ (\tfrac{1}{2}, -\tfrac{1}{2}) \quad \text{and} \quad P_2\colon\ (\tfrac{1}{2}, \tfrac{1}{2}).$$

Hence, by Theorem 11, P_1 and P_2 form a basis of Λ_0 which then consists of all points $P = \omega_1 P_1 + \omega_2 P_2$ with integral ω_1 and ω_2. The determinant of Λ_0 is

$$d(\Lambda_0) = \tfrac{1}{2}.$$

We show now that no regular lattice exists; therefore any critical lattice must be singular and so identical with Λ_0.

A regular lattice would contain at least three pairs of symmetrical points $\pm P_1$, $\pm P_2$, $\pm P_3$ on the boundary C of K. Hence, from the symmetry of the figure, we may assume without loss of generality that there are at least two independent lattice points P_1 and P_2 on that part of C formed by the three line segments

$$L_1\colon\ x = \theta,\ -\tfrac{1}{2} \leqslant y \leqslant \tfrac{1}{2};\quad L_2\colon\ \tfrac{1}{2} \leqslant x \leqslant \theta,\ y = \tfrac{1}{2};\quad L_3\colon\ \tfrac{1}{2} \leqslant x \leqslant \theta,\ y = -\tfrac{1}{2}.$$

By Theorem 11,

$$\operatorname{ind}(P_1, P_2) = 1,$$

and we may also assume that $(P_1, P_2) > 0$; therefore

$$d(\Lambda) = (P_1, P_2).$$

We now show that

$$(P_1, P_2) > \tfrac{1}{2}$$

by distinguishing the following four cases:

(*a*) *Both P_1 and P_2 lie on L_1.* Since $P_1 - P_2$ is not an inner point of K, P_1 and P_2 must lie at least a distance θ apart. Then, for admissible lattices, (P_1, P_2) is a minimum for an obvious infinity of lattices; for example,

$$P_1 : (\theta, -\tfrac{1}{2}\theta), \quad P_2 : (\theta, \tfrac{1}{2}\theta), \quad \text{and then} \quad (P_1, P_2) = \theta^2 > \tfrac{1}{2}.$$

(*b*) *P_1 lies on L_1, P_2 on L_2,* say

$$P_1 : (\theta, \eta), \quad P_2 : (\xi, \tfrac{1}{2}).$$

Since $P_1 - P_2$ is not an inner point of K, necessarily

$$-\tfrac{1}{2} \leqslant \eta \leqslant \tfrac{1}{2} - \theta.$$

This condition is easily proved to be also sufficient for the lattice to be admissible. If Λ is critical, then $P_1 - P_2$ must lie on C, and we have $\eta = \frac{1}{2} - \theta$. Hence

$$(P_1, P_2) = \tfrac{1}{2}\theta - \xi(\tfrac{1}{2} - \theta),$$

and is a minimum for $\xi = \frac{1}{2}$, when

$$(P_1, P_2) = \theta - \tfrac{1}{4} > \tfrac{1}{2}.$$

A similar proof holds when P_1 lies on L_3 and P_2 on L_1.

(*c*) *P_1 lies on L_3, P_2 on L_2.* It is easily seen that no admissible lattice of this kind has more than *two* pairs of symmetrical points on C.

(*d*) *Both P_1 and P_2 lie on L_2, or both lie on L_3.* It is obvious that no such lattice is admissible.

Hence Λ_0 is critical and so is singular. There are no other critical lattices. Hence*

$$\Delta(K) = d(\Lambda_0) = \tfrac{1}{2}.$$

* For another example of a star domain with a singular lattice, see the last part of this paper, p. 182.

11. *Determination of the regular lattices* (1).

We obtained in § 9 a method of finding all existing singular lattices. In this and the following paragraphs we give a similar method for the determination of the regular lattices. We construct a finite or infinite set of lattices

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$$

which must contain every existing regular lattice. This set has the important property that the set

$$\{d(\Lambda_1), d(\Lambda_2), d(\Lambda_3), \ldots\}$$

of determinants contains only a *finite* number of different values.

Let Λ be an arbitrary regular lattice of K and let

$$\pm P_1, \pm P_2, \ldots, \pm P_q \qquad (q \geqslant 3)$$

be the different pairs of symmetrical points of Λ which lie on C; then any two of these points with different indices are linearly independent. Every point P of Λ can be written as

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Here u_2, v_2 are the two integers associated with P_1 and P_2, and ω_1 and ω_2 are integers. In particular

$$P_\kappa = \frac{\Omega_{\kappa 1} P_1 + \Omega_{\kappa 2} P_2}{v_2}, \quad \Omega_{\kappa 1} + u_2 \Omega_{\kappa 2} \equiv 0 \pmod{v_2} \quad (\kappa = 3, 4, \ldots, q).$$

Since $$|\Omega_{\kappa 1}| = \operatorname{ind}(P_2, P_\kappa), \quad |\Omega_{\kappa 2}| = \operatorname{ind}(P_1, P_\kappa),$$

these integers are bounded, by Theorem 8.

We choose the notation so that $(P_1, P_2) > 0$, and hence by (1),

$$(P_1, P_2) = v_2 d(\Lambda),$$

and remark that these formulae remain satisfied if P_1 and P_2 are replaced by P_2 and $-P_1$.

It is now convenient to distinguish several types of lattices.

FIRST TYPE. *At least two of the points*

(14) $$P_1, P_2, \ldots, P_q$$

belong to the set Σ (§ 8).

Since Σ has only a finite number of points and u_2, v_2 are bounded, there exist only a finite number of lattices of this kind.

SECOND TYPE. *One, and only one, of the points* (14) *belongs to* Σ.

Call this now the point P_2. Denote by C^* the curve in the (x, y)-plane described by a point

$$P_3^* = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2}{v_2},$$

where the point P_1^* runs over the whole boundary of K. Since both P_1 and P_3 lie on C, the point P_3 must be a point of intersection of C and C^*. The two curves C and C^* consist of a finite number of analytical arcs, and so, by Theorem 13, have in general only a finite number of points of intersection. Then there exist at most a finite number of admissible lattices which contain P_2 and two independent points P_1, P_3 on C.

It is possible, however, that the curves C and C^* have an infinity of points and so an arc B in common. Then, when P_3^* runs over B, P_1^* describes an arc A on C, and *vice versa*. From the lattice Λ defined by

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

we derive the lattice Λ^* given by

$$P^* = \frac{\omega_1 P_1^* + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Then if we vary Λ^* by making P_1^* run over the arc A, each point P^* of Λ^* describes a curve.

Consider a point P of Λ independent of P_2; *i.e.* $\omega_1 \neq 0$. The corresponding point P^* is then a point of K if P_1^* is a point of the set $S(\omega_1, \omega_2)$ of all points

$$P_1^* = \frac{v_2 P^* - \omega_2 P_2}{\omega_1},$$

where P^* runs over all points of K. Evidently, since $S(\omega_1, \omega_2)$ is a domain similar and similarly situated to K, its boundary consists of a finite number of analytical arcs, and so has in common with A a finite number of points† and a finite number of arcs†.

† This, of course, includes the case when there are none.

Suppose now that Λ^* is an admissible lattice. If the domain $S(\omega_1, \omega_2)$ has points in common with K, then by Theorem 12 both $|\omega_1|$ and $|\omega_2|$ are less than a number depending only on K, *i.e.* at most a finite number of sets $S(\omega_1, \omega_2)$ have points in common with A. Hence the set of those points of A which are not inner points of any $S(\omega_1, \omega_2)$ consists of a finite number of points† and a finite number of sub-arcs $A_1, A_2, \ldots, A_h$ of A†. Let P_1^* run over the whole extent, say of A_1. The determinant $d(\Lambda^*)$ of Λ^* is a minimum for critical lattices; hence the area of the parallelogram $O, P_1^*, P_1^*+P_2, P_2$ is a minimum and so P_1^* is as near as possible to the line through O and P_2. Then either there is only a finite number of points P_1^* of A_1, or A_1 contains segments s of straight lines parallel to the vector P_2, and then $d(\Lambda^*)$ is constant however P_1^* is chosen on s.

Hence, in the second type, there exists a finite or infinite number of possible regular lattices, and in either case their determinants have only a finite number of different values.

12. *Determination of the regular lattices* (2).

We suppose from now on that no point of Λ belongs to Σ. Hence, if P_κ is an arbitrary point of the set (14), then every point P_κ^* on C sufficiently near to P_κ can be defined by a convergent power series

$$P_\kappa^* = P_\kappa + P_\kappa' \frac{s_\kappa}{1!} + P_\kappa'' \frac{s_\kappa^2}{2!} + \ldots \qquad (\kappa = 1, 2, \ldots, q), \tag{15}$$

where s_κ is a real parameter and $|s_\kappa|$ is small.

We further assume, without loss of generality, that

$$(P_1', P_2) \neq 0. \tag{16}$$

For, by hypothesis (§ 8), $P_1' \neq 0$. Hence, if $(P_1', P_2) = 0$, then $(P_1', P_3) \neq 0$, since P_3 is independent of P_2. We then can interchange P_2 and P_3.

The inequality (16) implies that corresponding to every point P_1^* on C in the neighbourhood of P_1 there is just *one* point P_2^* on C in the neighbourhood of P_2 such that

$$(P_1^*, P_2^*) = (P_1, P_2). \tag{17}$$

† This, of course, includes the case when there are none.

For on substituting the series (15) for P_1^* and P_2^*, this equation takes the form

$$\{(P_1', P_2)s_1+(P_1, P_2')s_2\}$$

$$+\tfrac{1}{2}\{(P_1'', P_2)s_1^2+2(P_1', P_2')s_1s_2+(P_1, P_2'')s_2^2\}+\ldots=0,$$

where the series converges for all sufficiently small $|s_1|$, $|s_2|$. By (16), this equation can be solved with respect to s_1 as a power series

$$(18)\quad s_1=-\frac{(P_1, P_2')}{(P_1', P_2)}s_2$$

$$-\frac{(P_1'', P_2)(P_1, P_2')^2-2(P_1', P_2')(P_1, P_2')(P_1', P_2)+(P_1, P_2'')(P_1', P_2)^2}{(P_1', P_2)^3}\frac{s_2^2}{2}$$

$$+\ldots,$$

convergent for sufficiently small $|s_2|$. We write s instead of s_2, and substitute the series (18) for s_2 in the expression (15) for P_1^*. Then we have the parametric representation

$$(19)\quad \begin{cases} P_1^*=P_1+P_1^{(1)}\dfrac{s}{1!}+P_1^{(2)}\dfrac{s^2}{2!}+\ldots, \\ P_2^*=P_2+P_2'\dfrac{s}{1!}+P_2''\dfrac{s^2}{2!}+\ldots, \end{cases}$$

for points P_1^* and P_2^* on C sufficiently near to P_1 and P_2 and satisfying (17). In the first series the coefficient $P_1^{(1)}$ is given by

$$(20)\quad P_1^{(1)}=-\frac{(P_1, P_2')}{(P_1', P_2)}P_1'.$$

$P_1^{(1)}$ has the same direction as P_1', unless $(P_1, P_2')=0$ when $P_1^{(1)}$ becomes the null vector. Then the series for P_1^* may reduce to its first term. This happens if, and only if, the part of C in the neighbourhood of P_2 is a line segment parallel to the vector P_1.

Let u_2, v_2 be the integers associated with P_1 and P_2. For every pair of points P_1^*, P_2^* on C sufficiently near to P_1, P_2 and connected by (17), we form the lattice Λ^*,

$$P^*=\frac{\omega_1P_1^*+\omega_2P_2^*}{v_2},\quad \text{where}\quad \omega_1+u_2\omega_2\equiv 0\pmod{v_2}.$$

In particular, let

$$P_\kappa^{**}=\frac{\Omega_{\kappa 1}P_1^*+\Omega_{\kappa 2}P_2^*}{v_2}\qquad(\kappa=3, 4, \ldots, q)$$

be the points of this lattice in the neighbourhood of P_κ. By (19), these points P_κ^{**} can be written as power series in s. From (17),

$$d(\Lambda^*) = d(\Lambda).$$

THEOREM 15. *If $|s|$ is sufficiently small, then for each s at least one of the $q-2$ points $P_3^{**}, P_4^{**}, \ldots, P_q^{**}$ belongs to K.*

Proof. Suppose the assertion to be false; then, for some s, arbitrarily small, none of the points $P_3^{**}, P_4^{**}, \ldots, P_q^{**}$ belongs to K. Hence, for these values of s, the lattice Λ^* contains only the four points $\pm P_1^*, \pm P_2^*$ on C and is admissible. From (16), it is not a singular lattice, and so it is not a critical lattice. Hence there is a critical lattice of determinant less than $d(\Lambda^*) = d(\Lambda)$, which is impossible.

We now apply this theorem to the types when no points of (14) belong to Σ. We distinguish two cases, according as $q = 3$ or $q > 3$.

13. *Determination of the regular lattices* (3).

THIRD TYPE ($q = 3$). *The regular lattice Λ has exactly six points $\pm P_1, \pm P_2, \pm P_3$ on C.*

By theorem 15, the point

$$P_3^{**} = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2^*}{v_2} \tag{21}$$

belongs to K for sufficiently small $|s|$. When we substitute the power series (19) for P_1^* and P_2^*, (21) takes the form

$$P_3^{**} = P_3 + P_3^{(1)} \frac{s}{1!} + P_3^{(2)} \frac{s^2}{2!} + \ldots,$$

where, by (20),

$$P_3^{(1)} = \frac{\Omega_{31} P_1^{(1)} + \Omega_{32} P_2'}{v_2} = \frac{-\Omega_{31}(P_1, P_2') P_1' + \Omega_{32}(P_1', P_2) P_2'}{v_2(P_1', P_2)}.$$

Since P_1 and P_2 are independent, $(P_1, P_2) \neq 0$. Hence

$$\Omega_{31} = -\frac{v_2(P_2, P_3)}{(P_1, P_2)}, \quad \Omega_{32} = \frac{v_2(P_1, P_3)}{(P_1, P_2)},$$

since $$P_3 = \frac{\Omega_{31} P_1 + \Omega_{32} P_2}{v_2}.$$

Hence the coefficient $P_3^{(1)}$ can be written as

$$P_3^{(1)} = \frac{(P_2, P_3)(P_1, P_2')\, P_1' + (P_1, P_3)(P_1', P_2)\, P_2'}{(P_1, P_2)(P_1', P_2)}.$$

On noting the terms in $P_3^{(1)}$, we have

$$(P^{(1)}, P_3') = \frac{(P_2, P_3)(P_1, P_2')(P_1', P_3') + (P_1, P_3)(P_1', P_2)(P_2', P_3')}{(P_1, P_2)(P_1', P_2)}$$

i.e., with a more symmetrical numerator,

$$(22)\quad (P_3^{(1)}, P_3') = -\frac{(P_1, P_2')(P_2, P_3')(P_3, P_1') + (P_1', P_2)(P_2', P_3)(P_3', P_1)}{(P_1, P_2)(P_1', P_2)}.$$

This follows easily from the well-known identity

$$(Q_1, Q_2)(Q_3, Q_4) = (Q_1, Q_3)(Q_2, Q_4) - (Q_1, Q_4)(Q_2, Q_3).$$

Suppose first that $P_3^{(1)} \neq 0$. For sufficiently small values of $|s|$, P_3^{**} describes a curve passing through P_3; from the series for P_3^{**}, its tangent at P_3 is parallel to the vector $P_3^{(1)}$. Since all points of this curve sufficiently near to P_3 belong to K, its tangent at P_3 coincides with that of C which is parallel to the vector P_3'. Hence we have a vector equation

$$(23)\quad P_3^{(1)} = \mu P_3',$$

with a scalar μ. Suppose, secondly, that $P_3^{(1)} = 0$; then this formula remains true with $\mu = 0$.

From (23), $$(P_3^{(1)}, P_3') = 0,$$

and so by (22),

$$(24)\quad (P_1, P_2')(P_2, P_3')(P_3, P_1') + (P_1', P_2)(P_2', P_3)(P_3', P_1) = 0$$

for all regular lattices of the third type.

The argument leading to (23) shows further that the three points P_1, P_2, P_3 satisfy an *inequality* condition arising from terms of the second order and corresponding to the sign of the second derivative in minima problems. I omit this condition since I have not succeeded in expressing it in a simple form similar to (24).

We can now determine all regular lattices of the third type. For given integers v_2, Ω_{31}, Ω_{32}, we have to find all sets of three points P_1, P_2, P_3 on C which satisfy the equations (24) and

$$\Omega_{31} P_1 + \Omega_{32} P_2 - v_2 P_3 = 0. \tag{25}$$

Since the boundary C consists of a finite number of analytical arcs, (24) and (25) are in general independent and so have only a finite number of solutions; to these correspond only a finite number of possible regular lattices.

It may happen that only two of the three equations given by (24) and (25) are independent. Then one of the three points, say P_1, may be taken arbitrarily on an arc A of C, and then P_2 and P_3, being determined by P_1, lie on other arcs of the boundary C. It is clear from the construction of P_3 that all the lattices Λ so obtained have the same determinant. By using Theorem 12, we can decide which of these lattices are admissible, and then which are critical.

Finally, we note that the three equations given by (24) and (25) never reduce to *one* independent equation. For then P_1, P_2 would have arbitrary positions on C, and so the area of the parallelogram O, P_1, P_1+P_2, P_2 would not be constant.

14. *Determination of the regular lattices* (4).

FOURTH TYPE. *The regular lattice* Λ *has* $2q \geqslant 8$ *points*

$$\pm P_1, \pm P_2, \ldots, \pm P_q$$

on C.

In the equations

$$\Omega_{\kappa 1} P_1 + \Omega_{\kappa 2} P_2 - v_2 P_\kappa = 0 \quad (\kappa = 3, 4, \ldots, q), \tag{26}$$

the coefficients v_2, $\Omega_{\kappa 1}$, $\Omega_{\kappa 2}$ are bounded integers, by Theorem 8. There are, in general, only a finite number of sets of points P_1, P_2, ..., P_q on C satisfying (26), since the number $2q-4$ of conditions in (26) is at least equal to the number q of parameters of the p points. For these points, Theorem 15 gives a necessary inequality condition.

In exceptional cases there may exist an infinity of lattices with at least eight points on C. Since the boundary C consists of a finite number of analytical arcs, this can occur only if P_1 runs over an arc B_1 of C, while P_2, P_3, ..., P_q, as functions of P_1, describe other arcs of C. The determinants of the lattices through these points depend on the positions

of P_1; for a critical lattice, the determinant is a minimum. This leads to the following considerations which might have been applied exactly the same way to find the condition (24) for the third type.

Let Λ be one of these critical lattices, let Λ^* be a neighbouring lattice, and denote by $\pm P_1, \pm P_2, \ldots, \pm P_q$ and $\pm P_1^*, \pm P_2^*, \ldots, \pm P_q^*$ the points on C of these two lattices. Then

$$P_3 = \frac{\Omega_{31} P_1 + \Omega_{32} P_2}{v_2}, \quad P_3^* = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2^*}{v_2}. \tag{27}$$

By the last paragraph,

$$\Omega_{31} = -\frac{v_2(P_2, P_3)}{(P_1, P_2)}, \quad \Omega_{32} = \frac{v_2(P_1, P_3)}{(P_1, P_2)}.$$

The equations (27) can then be written as

$$(P_2, P_3) P_1 + (P_3, P_1) P_2 + (P_1, P_2) P_3 = 0, \tag{28}$$

$$(P_2, P_3) P_1^* + (P_3, P_1) P_2^* + (P_1, P_2) P_3^* = 0. \tag{29}$$

On substituting the power series

$$P_\kappa^* = P_\kappa + P_\kappa' \frac{s_\kappa}{1!} + \ldots \qquad (\kappa = 1, 2, 3)$$

in (29), we get by (28) the vector equation

$$Q \equiv (P_2, P_3) P_1' s_1 + (P_3, P_1) P_2' s_2 + (P_1, P_2) P_3' s_3 + \ldots = 0,$$

where the dots denote terms in s_1, s_2, s_3 of the second and higher degrees. Since $Q \equiv 0$, $(Q, P_3') = 0$, *i.e.*

$$(P_2, P_3)(P_1', P_3') s_1 + (P_3, P_1)(P_2', P_3') s_2 + \ldots = 0. \tag{30}$$

Suppose first that (P_1', P_3') and (P_2', P_3') are not both zero. Let, *e.g.*, $(P_2', P_3') \neq 0$; then the equation (30) can be solved for s_2 as a power series in s_1,

$$s_2 = -\frac{(P_2, P_3)(P_1', P_3')}{(P_3, P_1)(P_2', P_3')} s_1 + \ldots.$$

The lattice Λ^* contains therefore points P_1^*, P_2^* given by

$$P_1^* = P_1 + P_1' s_1 + \ldots, \quad P_2^* = P_2 + \frac{(P_2, P_3)(P_3', P_1')}{(P_3, P_1)(P_2', P_3')} P_2' s_1 + \ldots.$$

By hypothesis, Λ is critical; hence

$$(P_1^*, P_2^*) \geqslant (P_1, P_2),$$

that is $$c' s_1 + c'' \frac{s_1^2}{2!} + \ldots \geqslant 0,$$

where $$c' = \frac{(P_2, P_3)(P_3', P_1')(P_1, P_2')}{(P_3, P_1)(P_2', P_3')} + (P_1', P_2).$$

This inequality must hold for sufficiently small $|s_1|$. Hence $c' = 0$ or

$$(P_2, P_3)(P_1, P_2')(P_1', P_3') + (P_1, P_3)(P_1', P_2)(P_2', P_3') = 0,$$

a condition which clearly also holds if $(P_1', P_3') = (P_2', P_3') = 0$.

Hence the points P_1, P_2, P_3 satisfy equation (24) of the last paragraph. Similar equations hold with P_3 replaced by P_4, ..., P_q.

Therefore, as in § 13, the infinity of lattices Λ^* have all the same determinant.

Such cases actually occur. For, if $R > 0$ is sufficiently large, then the simple star domain

$$|xy| \leqslant 1, \quad x^2 + y^2 \leqslant R^2$$

has an infinity of regular lattices, and these, by suitable choice of R, will contain an arbitrary large number of lattice points on C.

15. *Conclusion.*

Let us now collect our results for simple star domains. We have developed a method of constructing a finite or infinite set of lattices containing all critical lattices. This set has the remarkable property that the determinants of its elements assume only a *finite* number of different values, the smallest of which is the minimum $\Delta(K)$.

The members of the set are:

1. *The singular lattices with at least one point in* Σ. (§ 9.)

2. *The singular lattices with no points in* Σ. These are obtained by solving the equations (12). (§ 9.)

3. *The regular lattices with either one point in* Σ *or with two independent points in* Σ. (§ 11.)

4. *The regular lattices with no points in* Σ *and having exactly six points on* C. These are obtained by solving the equations (24) and (25). (§ 13.)

5. *The regular lattices with no points in* Σ *and having at least eight points on* C. (§ 14.)

6. The set may further contain non-critical lattices either satisfying the conditions for stationary solutions, or having a certain number of points in Σ or on C, or possessing tac-lines of a certain direction. (§§ 9, 11, 13, 14.)

The method of the paper reduces the problem of finding $\Delta(K)$ to a finite number of elementary questions no longer involving number theory, namely, the solving of sets of equations in a small number of unknowns. I show in further parts of this paper that the method is practicable.

16. *Unsymmetrical star domains.*

We have considered so far only star domains K which are symmetrical in the origin. This restriction can now be easily removed.

Let H be a closed set in the (x, y) plane containing the origin as an inner point and bounded by a Jordan curve J of the following kind:

(*a*) *J consists of a finite number of analytical arcs.*

(*b*) *Every radius vector from the origin intersects J in a single point.*

If H is not symmetrical in O, then denote by K the set of all points P for which at least one of the two points P, $-P$ belongs to H. Then K obviously is a simple star domain (§ 8).

As in §§ 3 and 4, we define H-admissible lattices, the lower bound $\Delta(H)$, and critical lattices with respect to H. Then a lattice which contains at least one point $P \neq 0$, and so both points P and $-P$ of K, contains also at least one point $\pm P$ of H, and *vice versa*. Therefore the H-admissible lattices are identical with the K-admissible lattices, the critical lattices of H are identical with those of K, and

$$\Delta(H) = \Delta(K). \tag{31}$$

Hence there is no difference between the lattice problem for H and the lattice problem for K.

17. *The use of sub-domains.*

By Theorem 2,

$$\Delta(K') \leqslant \Delta(K),$$

when the star domain K' is contained in the star domain K. Hence if there exists at least one critical lattice of K' which is admissible and therefore critical with respect to K, then

$$\Delta(K') = \Delta(K).$$

This remark enables us to derive from known results new ones for larger domains. For instance, since

$$\Delta(\Gamma_r) = \sqrt{\tfrac{3}{4}} \cdot r^2,$$

we find without difficulty for h, a regular hexagon or a regular star hexagon of side a,

$$\Delta(h) = \frac{3\sqrt{3}}{8} a^2.$$

This method of extension applies in particular to *infinite star domains*. These are defined in the following way. Let S be an arbitrary point set in the (x, y)-plane, and $S(r)$ the subset of all points of S which lie in the circle Γ_r. Then S is called an infinite star domain if it is not bounded, and if $S(r)$ for every $r > 0$ is a simple star domain†.

There need not be admissible lattices with respect to an infinite star domain. If there are such lattices, then we can again define $\Delta(S)$. It does not now follow that critical lattices exist. But for certain infinite star domains, critical lattices may exist, and these may be already critical for all subsets $S(r)$ where r is sufficiently large. Then, as above, we can consider $S(r)$ and find the value of $\Delta(S)$. As examples, I mention the domains

$$|F(x, y)| \leqslant 1,$$

where $F(x, y)$ is a cubic binary form; these were considered for the first time by L. J. Mordell‡.

Addition (*March* 1945). In the three years since this paper was written, I have extended the theory of star domains to bounded or unbounded star bodies in any number of dimensions, and have proved a number of existence theorems. I have also evaluated $\Delta(K)$ for some special domains, *e.g.* the domain

$$|x+y| \leqslant \sqrt{5}, \quad |xy| \leqslant 1.$$

† By the preceding paragraph, it suffices to consider infinite domains which are *symmetrical* in the origin.

‡ *Proc. London Math. Soc.* (2), 48 (1943), 198–228,

These results are contained in the following papers:

(1) *Journal London Math. Soc.* 18 (1943), 233–238.

(2) *Proc. Cambridge Phil. Soc.*, 40 (1944), 107–120.

(3) "On lattice points in a cylinder", *Quarterly Journal of Math.*, 17 (1946), 16–18.

(4) "On lattice points in n-dimensional star bodies, I, II".

My student, Mrs. K. Ollerenshaw, D.Phil., has evaluated $\Delta(K)$ for some very interesting classes of star domains; see *Journal London Math. Soc.*, 19 (1944), 178–184; *Proc. Cambridge Phil. Soc.*, 41 (1945), 77–96; and a paper to appear in the *Quarterly Journal of Math.* One of the domains considered by her possesses a continuous infinity of singular lattices.

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LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (II)

By KURT MAHLER.

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In this second part I give, in Chapter II, some general results on symmetrical or unsymmetrical convex domains. In Chapter III, $\Delta(H)$ is determined for any *circle*, *square*, or *triangle* in the (x, y)-plane which contains the origin as an inner point.

CHAPTER II. CONVEX STAR DOMAINS.

18. *A restriction.*

In this chapter we consider only symmetrical or unsymmetrical star domains whose boundaries do *not* include segments of straight lines. Results for the excluded domains can be found by using the method of the first chapter, and are more complicated.

19. *Convex domains symmetrical in the origin.*

We recall that a bounded and closed point set in the (x, y)-plane is called *convex* if with any two points P_1 and P_2 it contains all points

$$tP_1+(1-t)P_2, \quad \text{where} \quad 0 \leqslant t \leqslant 1.$$

The convex set is *symmetrical in a point* Q if with a point P it contains also the point $2Q-P$. We consider only convex domains of which O is an *inner* point, and whose boundary consists of a finite number of analytical arcs; *i.e.* these domains are symmetrical or unsymmetrical simple star domains (§§ 8 and 16).

Let K be a convex domain symmetrical in O. By Theorem 14 and by the restriction in § 18 there are no singular lattices. Therefore every critical lattice of K contains at least three pairs of points $\pm P_1$, $\pm P_2$, $\pm P_3$

on C. The triangle with vertices at P_1, P_2, P_3 lies entirely in K. Hence, by Theorem 11,

$$\text{ind}\,(P_1, P_2) = \text{ind}\,(P_1, P_3) = \text{ind}\,(P_2, P_3) = 1.$$

Hence we may choose P_1 and P_2 as a basis of the lattice, and then, without loss of generality,

$$P_3 = P_1 - P_2.$$

Therefore those admissible lattices are critical which contain three points of this special kind on C and have minimum determinant. This is exactly Minkowski's result†, as it ought to be since our method is derived from his.

The results of the first chapter enable us to determine $\Delta(K)$ for convex domains which are *not* symmetrical in the origin. For a special class of these unsymmetrical convex domains, particularly simple results are obtained in the next paragraph.

20. *Strongly unsymmetrical convex domains.*

Let H be an unsymmetrical convex star domain and let $-H$ be the set of points P for which $-P$ lies in H, and let K be the set of points which belong to at least one of these two domains. We say that H is *strongly unsymmetrical* if the boundaries of H and $-H$ have at most a finite number of points of intersection.

For instance, *every convex star domain H which is symmetrical in a point $Q \neq O$ is strongly unsymmetrical.* For assume that the assertion is false. Then, by Theorem 13, the boundaries of H and $-H$ have an arc A in common, and so also the arc $-A$ symmetrical to A in O. We can find a point P on A which is different from the symmetrical points $-P$ on $-A$ and $2Q-P$ on $2Q-A$, and at which there exists a tangent to the boundary of H. Then H has just one tac-line at each of the three points P, $-P$, and $2Q-P$, and by symmetry these three lines are parallel. This, however, is impossible, since a convex domain has only two tac-lines of given direction.

Let now H be an arbitrary strongly unsymmetrical convex domain, and denote by C_+ and C_- those parts of the boundary C of K which belong to H and $-H$, respectively. The two curves C_+, C_- have only a finite number of common points, say the set Σ^*. The elements of Σ^* can be arranged in pairs P, $-P$ of points; include one of these points in C_+, and the other in C_-.

---

† *Diophantische Approximationen* (Leipzig und Berlin, 1907), Kapitel II, § 14.

It is evident from Theorem 14 that the four points of a *singular* lattice of K which lie on the boundary C belong to Σ^*. Since Σ^* has only a finite number of elements, no difficulty arises in finding which of these lead to singular lattices.

Consider now a regular lattice Λ, and let $\pm P_1$, $\pm P_2$, ..., $\pm P_q$, so that $q \geqslant 3$, be its points on C. Let the notation be such that the points with positive signs belong to C_+, and those with negative signs to C_-. The line segments joining any two of the points P_1, P_2, ..., P_q lie entirely in H and so also in K. Hence, by Theorem 11,

$$\text{ind}\,(P_\mu, P_\nu) = 1 \quad (\mu, \nu = 1, 2, \ldots, q;\ \mu \neq \nu). \tag{32}$$

We take P_1 and P_2 as a basis of Λ; by our notation, $-P_1$ and $-P_2$ belong to C_-. From (32), the lattice points P_3, P_4, ..., P_q on C are of one of the four types

$$\pm P_1 \pm P_2.$$

Suppose first that $q > 3$, so that at least two of these four points, say P_3 and P_4, belong to C_+. Then $P_3 + P_4 \neq O$, since otherwise one of these points would belong to C_-. Hence, by the form of P_3 and P_4,

$$P_5 = \frac{P_3 + P_4}{2}$$

is also a lattice point different from O. By the convexity of H, P_5 belongs to H and therefore lies on its boundary. The three points P_3, P_4, P_5 on the boundary of H are collinear; hence the line segment joining P_3 and P_4 forms part of this boundary, contrary to the restriction in § 18.

We have, then, $q = 3$, and there is only one further lattice point $P_3 = \pm P_1 \pm P_2$ on C. We may, if necessary, interchange the indices of P_1, P_2, P_3; therefore either

$$P_3 = P_1 - P_2 \quad \text{or} \quad P_3 = -P_1 - P_2.$$

We have thus proved

THEOREM 16. *Let Λ be a regular lattice of a strongly unsymmetrical domain H. Then the set C_+ on the boundary of H contains exactly three lattice points, and these lattice points are of one of the two types,*

$$P_1, P_2, P_1 - P_2, \quad \text{or} \quad P_1, P_2, -P_1 - P_2.$$

CHAPTER III. THE UNSYMMMETRICAL CIRCLE, SQUARE, AND TRIANGLE.

By means of the results so far obtained, I determine the value of $\Delta(H)$ for every circle, square, or isosceles right-angled triangle in the (x, y)-plane which contains the origin O as an inner point. A suitable affine trans-

formation extends these results to arbitrary ellipses, parallelograms, or triangles containing O as inner points.

I am much indebted to Mrs. W. R. Lord for the simple geometrical proof in § 21.

21. *The circle.*

Let

$$0 < a < r \tag{33}$$

and let H be the circle

$$(x-a)^2+y^2 \leqslant r^2 \tag{34}$$

of radius r with centre at the point $P: (a, 0)$. By (33), H contains O as an inner point, and is not symmetrical in O.

The domain $-H$ symmetrical to H in O is defined by

$$(x+a)^2+y^2 \leqslant r^2. \tag{35}$$

The boundaries of H and $-H$ intersect at the two points

$$Q': \left(0, \sqrt{(r^2-a^2)}\right) \quad \text{and} \quad Q'': \left(0, -\sqrt{(r^2-a^2)}\right),$$

which are symmetrical in O. Denote by C_+ that arc of the boundary of H for which

$$x \geqslant 0, \tag{36}$$

and by C_- the arc of the boundary of $-H$ symmetrical to C_+ in O.

Since the boundaries of H and $-H$ intersect at two points only, there are no *singular lattices*. Therefore every critical lattice Λ of H contains three points P_1, P_2, P_3 on C_+; the symmetrical points $-P_1$, $-P_2$, $-P_3$ lie on C_-. By (36),

$$P_1+P_2+P_3 \neq O;$$

hence, without loss of generality, by Theorem 16, we may take

$$P_1+P_3 = P_2. \tag{37}$$

The problem now is to find three points P_1, P_2, P_3 on C_+ satisfying (37) and generating a lattice Λ of minimum determinant. This is equivalent to finding a parallelogram Π of minimum area J with vertices at O, P_1, P_2, P_3, where P_1, P_2, P_3 lie on C_+.

The geometrical method of Mrs. Lord proceeds as follows:
Let Q be the centre of Π, that is, the point

$$Q = \frac{P_2}{2} = \frac{P_1+P_3}{2}.$$

Further, denote by u, v, w the distances

$$u = \overline{OQ} = \overline{QP}_2, \quad v = \overline{PQ}, \quad w = \overline{P_1Q} = \overline{QP}_3,$$

and by α the angle OQP. Since QP is perpendicular to P_1P_3, α is the complement of the angle between the diagonals. With our notation,

$$r = \overline{PP}_2.$$

Also by a well-known property of the medians of a triangle

$$2(u^2+v^2) = a^2+r^2. \tag{38}$$

Therefore, by the cosine theorem,

$$r^2 = u^2+v^2+2uv\cos\alpha = \frac{a^2+r^2}{2}+2uv\cos\alpha,$$

and

$$\cos\alpha = \frac{r^2-a^2}{4uv}.$$

Hence

$$J = 2uw\cos\alpha = \frac{(r^2-a^2)w}{2v} = \frac{r^2-a^2}{2}\sqrt{\left(\frac{r^2}{v^2}-1\right)},$$

since

$$v^2+w^2 = r^2.$$

By (38), this expression for J can be written as

$$J = \frac{r^2-a^2}{2}\sqrt{\left(\frac{2r^2}{a^2+r^2-2u^2}-1\right)}.$$

By this formula, J is an increasing function of u, so that its minimum is attained for the smallest possible value of u. Now, if P_2 runs over C_+, then P_1 and P_3 vary on C_+ in the same direction as P_2. Therefore the minimum of J is assumed if either $P_1 = Q''$ or $P_3 = Q'$. We obtain the same lattice in both cases, namely that with the following four points on C_+,

$$\begin{cases} \left(0, \sqrt{(r^2-a^2)}\right), \quad \left(0, -\sqrt{(r^2-a^2)}\right), \\ \left(\frac{2a+\sqrt{(a^2+3r^2)}}{2}, \frac{\sqrt{(r^2-a^2)}}{2}\right), \quad \left(\frac{2a+\sqrt{(a^2+3r^2)}}{2}, \frac{-\sqrt{(r^2-a^2)}}{2}\right). \end{cases} \tag{39}$$

This lattice Λ_0 is therefore critical; its determinant is

$$d(\Lambda_0) = \frac{\sqrt{(r^2-a^2)}}{2}\{2a+\sqrt{(a^2+3r^2)}\},$$

and there are no further critical lattices. Hence we have

THEOREM 17. *Let a and r satisfy* (33), *and let H be the circle* (34). *Then*

$$\Delta(H) = \frac{\sqrt{(r^2-a^2)}}{2}\{2a+\sqrt{(a^2+3r^2)}\}. \tag{40}$$

There is only one critical lattice Λ_0; this contains the four points (39) *on C_+.*

By continuity, the formula (40) remains true if $a=0$; in this limiting case we obtain Theorem 3. In the excluded case $a=r$, there are admissible lattices of arbitrary small positive determinant.

22. *The square.*

Let

$$0<a<r \quad \text{and} \quad 0<b<r, \tag{41}$$

and let H be the square

$$|x-a|\leqslant r, \quad |y-b|\leqslant r \tag{42}$$

with centre at $P:(a, b)$ and with sides of length $2r$ parallel to the coordinate axes. By (41), H contains O as an inner point, and is not symmetrical in O.

The domain $-H$ symmetrical to H in O is defined by

$$|x+a|\leqslant r \quad \text{and} \quad |y+b|\leqslant r. \tag{43}$$

The boundaries of H and $-H$ intersect at the two points

$$Q':(a-r, r-b) \quad \text{and} \quad Q'':(r-a, b-r)$$

which are symmetrical in O. Denote by C_+ that part of the boundary of H between Q' and Q'' which does not belong to $-H$; and then let C_- be symmetrical to C_+ in O.

Let Λ be a critical lattice. When Λ is singular, then, by Theorem 14, it contains the two points

$$P':(a+r, 0) \quad \text{and} \quad P'':(0, b+r)$$

on C. By Theorem 11, these two points form a basis, and so

$$d(\Lambda) = (r+a)(r+b).$$

I show presently that there are admissible lattices of smaller determinant; hence *there are no singular lattices.*

Let, next, Λ be a regular lattice of H, and let P_1, P_2, P_3 be the three points of Λ on C_+; the symmetrical points $-P_1$, $-P_2$, $-P_3$ lie on C_-. It is again possible to choose the notation so that

$$P_1+P_3=P_2. \tag{44}$$

The problem is to find three points of this kind such that the area J of the parallelogram Π with vertices at O, P_1, P_2, P_3 is a minimum. The following is a list of all parallelograms which have to be considered:

(*a*) $r-a\leqq\xi\leqq r+a$. The vertices of Π are at

$$P_1\colon(\xi,\ b-r);\quad P_2\colon(r+a,\ 2b);\quad P_3\colon(r+a-\xi,\ r+b),$$

and its area is

$$J=J_1(\xi)=2b\xi+(r+a)(r-b).$$

This area becomes a minimum for $\xi=r-a$, namely

$$J_1=r^2+(a+b)r-3ab=(r+a)(r+b)-4ab.$$

(*b*) $b-r\leqq\eta\leqq 0$. The vertices of Π are at

$$P_1\colon(a+r,\ \eta);\quad P_2\colon(r+a,\ r+b+\eta);\quad P_3\colon(0,\ r+b),$$

and the area is

$$J=J_2=(r+a)(r+b)>J_1.$$

(*c*) $a-r\leqq\xi\leqq 0$. The vertices of Π are at

$$P_1\colon(r+a,\ 0);\quad P_2\colon(r+a+\xi,\ r+b);\quad P_3\colon(\xi,\ r+b),$$

and the area is

$$J=J_3=(r+a)(r+b)>J_1.$$

(*d*) $r-b\leqq\eta\leqq r+b$. The vertices of Π are at

$$P_1\colon(r+a,\ r+b-\eta);\quad P_2\colon(2a,\ r+b);\quad P_3\colon(a-r,\ \eta),$$

and the area is

$$J=J_4(\eta)=2a\eta+(r-a)(r+b).$$

This area is a minimum for $\eta=r-b$, namely

$$J_4=J_1.$$

We see from these formulae that there is just one critical lattice Λ_0. This lattice has on C_+ the four points

$$(45) \qquad (r-a, b-r), \quad (r+a, 2b), \quad (2a, r+b), \quad (a-r, r-b),$$

and its determinant is

$$d(\Lambda_0) = r^2+(a+b)r-3ab.$$

Therefore we have

THEOREM 18. *Let a, b, and r satisfy* (41), *and let H be the square* (42). *Then*

$$(46) \qquad \Delta(H) = r^2+(a+b)r-3ab.$$

There is only one critical lattice Λ; this contains the four points (45) *on C_+.*

By continuity, the formula (46) remains true if the signs "$<$" in (41) are replaced by "$=$", provided that the case $a=b=r$ is excluded. In this exceptional case there are admissible lattices of arbitrarily small positive determinant.

23. *The isosceles right-angled triangle.*

Let

$$(47) \qquad a>b>0, \quad 3a+b<2s,$$

and let H be the isosceles right-angled triangle with vertices at

$$(48) \qquad (-a, b), \quad (-a+s, b+s), \quad (-a+s, b-s),$$

and $-H$ the symmetrical triangle with vertices at

$$(a, -b), \quad (a-s, -b-s), \quad (a-s, -b+s).$$

By (47), the origin O is an inner point of the triangle T with vertices at

$$(-a, b), \quad \left(-a+\frac{s}{2}, b-\frac{s}{2}\right), \quad \left(-a+\frac{2s}{3}, b\right).$$

Here the first point is a vertex of H, the second point lies at the centre of a side of H, and the third point is the centre of gravity of H.

Evidently the parallelogram Π with centre at O and vertices at

$$(a-s, -b-s), \quad (s-a, -2a-b+s), \quad (-a+s, b+s), \quad (a-s, 2a+b-s)$$

and of area $$J = 4(s-a)(a+b)$$

is contained in the star domain K formed by combining H and $-H$. By Minkowski's theorem on linear forms,

$$\Delta(\Pi) = (s-a)(a+b).$$

Since $\Delta(H) = \Delta(K)$, we get, by Theorem 2,

$$\Delta(H) \geqslant (s-a)(a+b).$$

But here the sign of equality holds, since the lattice Λ_0 of basis

$$P_1: (s-a, -2a-b+s), \quad P_2: (s-a, b+s)$$

and of determinant

$$d(\Lambda_0) = (s-a)(a+b)$$

is easily seen to be H-admissible, and is therefore critical.

Hence we have

THEOREM 19. *Let a, b, and s satisfy* (47), *and let H be the isosceles right-angled triangle with vertices at the points* (48). *Then*

$$\Delta(H) = (s-a)(a+b). \tag{49}$$

By continuity, the formula (49) remains valid if one or more of the signs "$<$" and "$>$" in (47) are replaced by "$=$", provided only that the case $a = b = 0$ is excluded. In this exceptional case there are admissible lattices of arbitrarily small determinant.

24. *The general ellipse, parallelogram, and triangle.*

From the theorems in §§ 21–23 we easily obtain the following more general ones, by applying a suitable affine transformation of the (x, y)-plane. I omit the rather trivial proofs.

THEOREM 20. *Let H be an ellipse of area $J\pi$ which contains O as an inner point. Let the concentric, similar, and similarly situated ellipse through the origin be of area $J_0\pi$. Then*

$$\Delta(H) = \tfrac{1}{2}\left(\sqrt{(J-J_0)}\right)\left(2\sqrt{(J_0)}+\sqrt{(3J+J_0)}\right).$$

THEOREM 21. *Let H be a parallelogram which contains O as an inner point. Let the lines through O parallel to its sides divide H into four parallelograms of areas J_1, J_2, J_3, J_4, where the indices are chosen so that*

$J_1 \leqslant J_2 \leqslant J_3 \leqslant J_4$. *Then*

$$\Delta(H) = J_2 + J_3 - J_1.$$

THEOREM 22. *Let H be a triangle which contains the origin O as an inner point. Let the lines through O parallel to any two of its sides together with the third side form triangles of areas J_1, J_2, J_3, where the notation is such that $J_1 \leqslant J_2 \leqslant J_3$. Then*

$$\Delta(H) = 2\sqrt{(J_2 J_3)}.$$

For symmetrical ellipses and parallelograms, Theorems 20 and 21 reduce to classical results. Theorem 22, however, seems to be the first lattice point property of *triangles* to have been stated.

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LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (III)

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If $$f(x, y) = ax^2+2bxy+cy^2$$ is a positive definite binary quadratic form of determinant

$$ac-b^2 = 1,$$

and E denotes the domain

$$f(x, y) \leqslant 1,$$

bounded by the ellipse $f(x, y) = 1$, then by a classical result†,

$$\Delta(E) = \sqrt{\tfrac{3}{4}}.$$

There exists a continuous infinity of critical lattices Λ. Every such lattice contains just six points $\pm P_1$, $\pm P_2$, $\pm P_3$ on the boundary of E. It is possible to choose the notation such that

$$P_1+P_2+P_3 = O.$$

Conversely, six arbitrary boundary points of this type generate a critical lattice, any two independent points among them forming a basis.

The present fourth chapter of this paper deals with the more complicated domain K obtained by combining two concentric ellipses each of area π. An algorithm is developed for determining $\Delta(K)$, which turns out to be a rather complicated function of the simultaneous invariant of the two ellipses.

A similar method can be applied to all domains obtained by combining two convex domains with centre at O, *e.g.* the star-shaped octagon investigated by Prof. Mordell.

† Bachmann, *Quadratische Formen*, II (Leipzig und Berlin, 1923), Kap. 5.

CHAPTER IV. THE DOMAIN BOUNDED BY TWO ELLIPSES.

25. *The invariant J.*

Let

$$(50)\quad f_1(x, y) = a_1x^2+2b_1xy+c_1y^2 \quad \text{and} \quad f_2(x, y) = a_2x^2+2b_2xy+c_2y^2$$

be two positive definite binary quadratic forms of determinants

$$(51)\quad a_1c_1-b_1^2 = a_2c_2-b_2^2 = 1.$$

Further, let

$$(52)\quad J = a_1c_2-2b_1b_2+c_1a_2$$

be the *simultaneous invariant* of these two forms. If an affine transformation of determinant unity,

$$(53)\quad x = \alpha x'+\beta y', \quad y = \gamma x'+\delta y', \quad \text{where} \quad \alpha\delta-\beta\gamma = 1,$$

changes f_1 and f_2 into the new forms

$$f_1'(x', y') = a_1'x'^2+2b_1'x'y'+c_1'y'^2$$

and

$$f_2'(x', y') = a_2'x'^2+2b_2'x'y'+c_2'y'^2,$$

then by the invariantive property of the determinants and of J,

$$a_1'c_1'-b_1'^2 = a_2'c_2'-b_2'^2 = 1, \quad a_1'c_2'-2b_1'b_2'+c_1'a_2' = J.$$

It is always possible to choose the transformation (53) so that f_1' and f_2' take the canonical forms

$$(54)\quad f_1'(x', y') = x'^2+y'^2 \quad \text{and} \quad f_2'(x', y') = \lambda x'^2+\frac{1}{\lambda}y'^2,$$

where λ is a positive number. In this case

$$(55)\quad J = \lambda+\frac{1}{\lambda}.$$

I assume in this chapter that f_1 and f_2, and so also f_1' and f_2', are *not identical*. Hence $\lambda \neq 1$, and therefore, from (55),

$$(56)\quad J > 2.$$

We may further suppose without loss of generality that $\lambda > 1$.

26. *The domain K.*

Let now K be the domain of all points (x, y) satisfying at least one of the two inequalities

$$f_1(x, y) \leqslant 1 \quad \text{and} \quad f_2(x, y) \leqslant 1.$$

Hence K is formed by combining two concentric ellipses each of area π. It is evident that K is a simple star domain; we can then consider the lower bound $\Delta(K)$.

The affine transformation (53) changes K into a domain K' formed by the points (x', y') satisfying at least one of the inequalities

$$f_1'(x', y') \leqslant 1 \quad \text{and} \quad f_2'(x', y') \leqslant 1.$$

Hence K' is of the same type as K.

We can assert that

$$\Delta(K) = \Delta(K'). \tag{57}$$

For (53) changes K-admissible lattices into K'-admissible lattices, and critical lattices of K into critical lattices of K'; and it leaves the determinant of two points and so also the determinant of a lattice invariant.

Choose the transformation (53) so that f_1, f_2 change into the two forms (54). Then K' becomes the set of all points (x', y') for which at least one of the inequalities

$$x'^2+y'^2 \leqslant 1 \quad \text{and} \quad \lambda x'^2+\frac{1}{\lambda}\, y'^2 \leqslant 1$$

holds. Here λ is determined uniquely as a function of J by

$$\lambda = \frac{J+\sqrt{(J^2-4)}}{2}.$$

Hence the lower bound $\Delta(K) = \Delta(K')$ becomes a function of J, say

$$\Delta(K) = D(J). \tag{58}$$

27. *A property of the critical lattices.*

By the last paragraph, we may assume from now on that

$$f_1(x, y) = x^2+y^2, \quad f_2(x, y) = \lambda x^2+\frac{1}{\lambda}\, y^2.$$

The two ellipses $f_1 = 1$ and $f_2 = 1$ intersect at the four points

$$Q_1 : (\mu, \nu), \quad Q_2 : (-\mu, \nu), \quad Q_3 : (-\mu, -\nu), \quad Q_4 : (\mu, -\nu),$$

where

$$\mu = \sqrt{\left(\frac{1}{\lambda+1}\right)}, \quad \nu = \sqrt{\left(\frac{\lambda}{\lambda+1}\right)}.$$

Denote by C_1 and C_2 those arcs of $f_1 = 1$ and $f_2 = 1$, respectively, which together form the boundary $C = C_1 + C_2$ of K. Hence, on describing C in a positive direction, the arc of C

from Q_4 to Q_1 belongs to C_1,

from Q_1 to Q_2 belongs to C_2,

from Q_2 to Q_3 belongs to C_1,

from Q_3 to Q_4 belongs to C_2.

We use the convention of counting every one of the four points Q_1, Q_2, Q_3, Q_4 twice, once in C_1 and once in C_2.

The affine transformation of determinant unity,

$$x \to \lambda^{-\frac{1}{2}} y, \quad y \to \lambda^{\frac{1}{2}} x, \tag{59}$$

evidently transforms K into itself, interchanges the parts C_1 and C_2 of C, and permutes the points Q_1, Q_2, Q_3, Q_4 cyclically, and by the last paragraph it changes critical lattices again into critical lattices. Hence to every critical lattice with just m points on C_1 and n points on C_2 there corresponds a second critical lattice with just n points on C_1 and m points on C_2.

THEOREM 23. *A critical lattice Λ of K has at most six points on C_1. If it contains six points on C_1, then these are of the form $\pm P_1$, $\pm P_2$, $\pm P_3$, where $P_1 + P_2 + P_3 = O$. Further,*

$$\Delta(K) = d(\Lambda) = \sqrt{\tfrac{3}{4}}, \tag{60}$$

and there are also six lattice points of the same type on C_2†.

Proof. The lattice Λ is admissible with respect to the circle $f_1 \leqslant 1$, and so, by the introduction, cannot contain more than six points on its boundary. If it has six points on C_1, then these are of the mentioned form, and the lattice is critical with respect to the circle; hence (60) is satisfied. Then Λ must also be critical with respect to the ellipse $f_2 \leqslant 1$; for otherwise, since $d(\Lambda) = \sqrt{\frac{3}{4}}$, at least one lattice point $P \neq O$ would be an *inner* point of the ellipse and so also an inner point of K. Hence there are also exactly six points of Λ on C_2.

THEOREM 24. *Let Λ be a critical lattice with less than six points on C_1. Then there are just four lattice points $\pm P_1$, $\pm P_2$ on C_1, and four lattice points $\pm P_3$, $\pm P_4$ on C_2†.*

† It is possible for some of the lattice points on C_1 to be identical with lattice points on C_2. This happens when some of the points Q_1, Q_2, Q_3, Q_4 are lattice points

Proof. First, let Λ be a singular lattice. Then, by Theorem 14, its only points on C are Q_1, Q_2, Q_3, Q_4; the assertion is therefore true. Secondly, let Λ be regular; then it has at least six points on C. We may assume, by the last theorem, that there are just four points of Λ on C_1; otherwise we apply the transformation (59) and thus obtain a regular lattice with this property.

Let, then, the four lattice points on C_1 be $\pm P_1$, $\pm P_2$, and assume that there are only two symmetrical lattice points $\pm P_3$ on C_2. Then at most one of the two pairs of symmetrical points Q_1, Q_3 and Q_2, Q_4 belong to Λ. Hence there exists a sufficiently small angle α such that the rotation

$$x \to x \cos\alpha - y \sin\alpha, \quad y \to x \sin\alpha + y \cos\alpha$$

changes Λ into a new lattice Λ^* with only four points $\pm P_1^*$, $\pm P_2^*$ on C_1 and containing no further points $P \neq O$ of K. This lattice is therefore K-admissible, but not critical. Hence there exist lattices of smaller determinants. But this is impossible, since obviously $d(\Lambda^*) = d(\Lambda)$.

By Theorem 11, any two points of Λ on C_1, or any two such points on C_2, form a basis. Hence, if for brevity we write

$$Y = D(J), \quad \text{then} \quad \sqrt{\tfrac{3}{4}} \leqslant Y \leqslant 1. \tag{61}$$

For K contains the circle $f_1 = 1$; further, $|(P, Q)| \leqslant 1$ for any two points P and Q on C_1, or on C_2.

28. *A sufficient condition for admissible lattices.*

The construction of the critical lattices of K makes use of

THEOREM 25. *Suppose that the lattice Λ of determinant*

$$d(\Lambda) \geqslant \sqrt{\tfrac{3}{4}}$$

has a basis consisting of two points P_1, P_2 on $f_1 = 1$, and a second basis consisting of two points P_3, P_4 on $f_2 = 1$. Then Λ is K-admissible.

Proof. It suffices to show that no lattice point $P \neq O$ is an inner point of $f_2 = 1$; the analogous result for $f_1 \leqslant 1$ is proved similarly.

Every point $P : (x, y)$ can be written as

$$P = uP_3 + vP_4, \quad \text{where} \quad u = \frac{(P, P_4)}{(P_3, P_4)}, \quad v = -\frac{(P, P_3)}{(P_3, P_4)}. \tag{62}$$

The new coordinates u, v are integers if, and only if, P is a lattice point. The result of replacing x, y by u, v is that f_2 takes the form

$$f_2(x, y) = f_2^*(u, v) = u^2 + 2suv + v^2, \tag{63}$$

since the two points $u = 1$, $v = 0$ and $u = 0$, $v = 1$ lie on $f_2^* = 1$. By the invariance property of the determinant of a quadratic form,

$$1 - s^2 = (P_3, P_4)^2 = d(\Lambda)^2 \geqslant \tfrac{3}{4}, \tag{64}$$

so that

$$-\tfrac{1}{2} \leqslant s \leqslant \tfrac{1}{2}. \tag{65}$$

Hence f_2^* is a reduced form†. Its minimum for integral u, v not both zero is then 1, as asserted.

Henceforth let $S(J)$ be the set of lattices Λ with the following properties:

(*a*) Λ *has a basis* P_1, P_2 *on* $f_1 = 1$, *and a basis* P_3, P_4 *on* $f_2 = 1$.

(*b*) *The determinant* $d(\Lambda) \geqslant \sqrt{\tfrac{3}{4}}$.

We shall prove later that $S(J)$ *has only a finite number of elements*, say the lattices

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_n.$$

By Theorem 25, these lattices are K-admissible; by Theorems 23 and 24, all critical lattices Λ belong to $S(J)$. Hence

$$D(J) = \min_{\nu = 1, 2, \ldots, n} d(\Lambda_\nu), \tag{66}$$

and so the critical lattices of K are just those elements Λ_ν of $S(J)$ for which $d(\Lambda_\nu)$ assumes the minimum value $D(J)$.

29. *Construction of the set* $S(J)$.

Let Λ be a lattice in $S(J)$. We may assume, without loss of generality, that the two bases

$$P_1 : (x_1, y_1),\ P_2 : (x_2, y_2) \quad \text{and} \quad P_3 : (x_3, y_3),\ P_4 : (x_4, y_4)$$

of Λ satisfy the inequalities

$$(P_1, P_2) > 0 \quad \text{and} \quad (P_3, P_4) > 0; \tag{67}$$

† See footnote †, page 168.

hence

$$(68) \qquad d(\Lambda) = (P_1, P_2) = (P_3, P_4) = x_1y_2 - x_2y_1 = x_3y_4 - x_4y_3.$$

The inequalities (67) remain satisfied if the pair of points P_1, P_2 is replaced by one of the four pairs

$$P_1, P_2, \quad \text{or} \quad P_2, -P_1, \quad \text{or} \quad -P_1, -P_2, \quad \text{or} \quad -P_2, P_1;$$

and if the pair of points P_3, P_4 is replaced by one of the four pairs

$$P_3, P_4, \quad \text{or} \quad P_4, -P_3, \quad \text{or} \quad -P_3, -P_4, \quad \text{or} \quad -P_4, P_3.$$

This gives a set Ω of $4 \times 4 = 16$ pairs of bases of Λ.

By the basis property and by (68), there are four integers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$(69) \qquad P_3 = \alpha_1 P_1 + \beta_1 P_2, \quad P_4 = \alpha_2 P_1 + \beta_2 P_2, \quad \alpha_1\beta_2 - \alpha_2\beta_1 = +1.$$

When the pair of bases P_1, P_2 and P_3, P_4 is replaced by one of the other pairs in Ω, then $\alpha_1, \beta_1, \alpha_2, \beta_2$ undergo certain permutations and changes of signs, for which I refer to the following table.

The 16 *elements of* Ω.

P_1	P_2	P_3	P_4	α_1	β_1	α_2	β_2	X	Y	u	v	s	1
P_1	P_2	P_4	$-P_3$	α_2	β_2	$-\alpha_1$	$-\beta_1$	X	Y	v	$-u$	$-s$	2
P_1	P_2	$-P_3$	$-P_4$	$-\alpha_1$	$-\beta_1$	$-\alpha_2$	$-\beta_2$	X	Y	$-u$	$-v$	s	3
P_1	P_2	$-P_4$	P_3	$-\alpha_2$	$-\beta_2$	α_1	β_1	X	Y	$-v$	u	$-s$	4
P_2	$-P_1$	P_3	P_4	β_1	$-\alpha_1$	β_2	$-\alpha_2$	$-X$	Y	u	v	s	5
P_2	$-P_1$	P_4	$-P_3$	β_2	$-\alpha_2$	$-\beta_1$	α_1	$-X$	Y	v	$-u$	$-s$	6
P_2	$-P_1$	$-P_3$	$-P_4$	$-\beta_1$	α_1	$-\beta_2$	α_2	$-X$	Y	$-u$	$-v$	s	7
P_2	$-P_1$	$-P_4$	P_3	$-\beta_2$	α_2	β_1	$-\alpha_1$	$-X$	Y	$-v$	u	$-s$	8
$-P_1$	$-P_2$	P_3	P_4	$-\alpha_1$	$-\beta_1$	$-\alpha_2$	$-\beta_2$	X	Y	u	v	s	9
$-P_1$	$-P_2$	P_4	$-P_3$	$-\alpha_2$	$-\beta_2$	α_1	β_1	X	Y	v	$-u$	$-s$	10
$-P_1$	$-P_2$	$-P_3$	$-P_4$	α_1	β_1	α_2	β_2	X	Y	$-u$	$-v$	s	11
$-P_1$	$-P_2$	$-P_4$	P_3	α_2	β_2	$-\alpha_1$	$-\beta_1$	X	Y	$-v$	u	$-s$	12
$-P_2$	P_1	P_3	P_4	$-\beta_1$	α_1	$-\beta_2$	α_2	$-X$	Y	u	v	s	13
$-P_2$	P_1	P_4	$-P_3$	$-\beta_2$	α_2	β_1	$-\alpha_1$	$-X$	Y	v	$-u$	$-s$	14
$-P_2$	P_1	$-P_3$	$-P_4$	β_1	$-\alpha_1$	β_2	$-\alpha_2$	$-X$	Y	$-u$	$-v$	s	15
$-P_2$	P_1	$-P_4$	P_3	β_2	$-\alpha_2$	$-\beta_1$	α_1	$-X$	Y	$-v$	u	$-s$	16
1	2	3	4	5	6	7	8	9	10	11	12	13	

Let a new system of rectangular coordinates U, V be defined by

$$x = x_1 U - y_1 V, \quad y = y_1 U + x_1 V, \tag{70}$$

or conversely, since $x_1^2 + y_1^2 = 1$,

$$U = x_1 x + y_1 y, \quad V = -y_1 x + x_1 y. \tag{71}$$

In this system, P_1 and P_2 have the coordinates

$$U_1 = 1,\ V_1 = 0 \quad \text{and} \quad U_2 = X = x_1 x_2 + y_1 y_2,\ V_2 = Y = x_1 y_2 - x_2 y_1;$$

here

$$X^2 + Y^2 = 1, \quad Y = d(\Lambda) > 0. \tag{72}$$

Further, by (69), the coordinates of P_3 and P_4 are given by

$$U_3 = \alpha_1 + \beta_1 X,\ V_3 = \beta_1 Y \quad \text{and} \quad U_4 = \alpha_2 + \beta_2 X,\ V_4 = \beta_2 Y.$$

Finally, if, as in § 28, we introduce u, v by (62), then

$$\begin{aligned} U &= (\alpha_1 + \beta_1 X)\, u + (\alpha_2 + \beta_2 X)\, v, \\ V &= \quad \beta_1 X\ u + \quad \beta_2 Y\ v, \end{aligned}$$

and so, on solving for u and v, we have

$$\begin{cases} Yu = +\beta_2 YU - (\alpha_2 + \beta_2 X)\, V. \\ Yv = -\beta_1 YU + (\alpha_1 + \beta_1 X)\, V. \end{cases} \tag{73}$$

I refer to the last table for the changes of these numbers α_1, β_1, α_2, β_2, X, Y, u, v, when the pair of bases P_1, P_2 and P_3, P_4 is replaced by another pair in Ω.

By § 28, f_2 takes the form (63) in u and v. By (64) and (72),

$$s = \epsilon X, \quad \text{where} \quad \epsilon = \pm 1. \tag{74}$$

An inspection of the table shows that it is always possible to choose the pair of bases P_1, P_2 and P_3, P_4 in Ω so that the following inequalities are satisfied:

$$X \geqslant 0, \quad s \geqslant 0, \quad \alpha_1 \geqslant 0. \tag{75}$$

Therefore, in particular,

$$s = X. \tag{76}$$

Replace u and v by U and V. Then f_2 changes into

$$f_2(x, y) = F_2(U, V) = AU^2 + 2BUV + CV^2, \tag{77}$$

where, by (63), (73), and (76),

$$(78)\quad \begin{cases} A=\beta_1^2-2\beta_1\beta_2 X+\beta_2^2, \\ YB=-\beta_1(\alpha_1+\beta_1 X)+X\{\beta_2(\alpha_1+\beta_1 X)+\beta_1(\alpha_2+\beta_2 X)\}-\beta_2(\alpha_2+\beta_2 X), \\ Y^2C=(\alpha_1+\beta_1 X)^2-2(\alpha_1+\beta_1 X)(\alpha_2+\beta_2 X)X+(\alpha_2+\beta_2 X)^2. \end{cases}$$

Further, since the change from x, y to U, V is an orthogonal transformation,

$$f_1(x, y)=F_1(U, V)=U^2+V^2.$$

Hence the simultaneous invariant

$$J=A+C,$$

so that, by (72) and (78),

$$(79)\quad (\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-J)-2(\alpha_1-\beta_2)(\alpha_2-\beta_1)X \\ -\{2(\alpha_1\beta_2+\alpha_2\beta_1)-J\}X^2=0.$$

For given J, this is a quadratic equation for X. *It does not reduce to an identity*, for then

$$\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2=J, \quad 2(\alpha_1\beta_2+\alpha_2\beta_1)=J;$$

hence

$$(\alpha_1-\beta_2)^2+(\alpha_2-\beta_1)^2=0,$$

and since $\alpha_1\geqq 0$, $\alpha_1\beta_2-\alpha_2\beta_1=1$,

$$\alpha_1=\beta_2=1, \quad \alpha_2=\beta_1=0, \quad J=2.$$

This value of J was, however, excluded by § 25.

By the assumption (b) in § 28, and by (72) and (75),

$$(80)\qquad 0\leqq X\leqq \tfrac{1}{2}.$$

Suppose now, conversely, that (79) has a solution X satisfying these inequalities. Then the coefficients A, B, C of F are given by (78), with

$$(81)\qquad Y=|\sqrt{(1-X^2)}|.$$

We further obtain the (U, V)-coordinates of P_1, P_2, P_3, P_4 from their expressions as functions of $\alpha_1, \beta_1, \alpha_2, \beta_2, X, Y$. There remains the reduction of $F_1(U, V)$ and $F_2(U, V)$ to the normal form (54) by means of an orthogonal transformation (71); this problem is dealt with in the theory of conics. After this reduction, the (x, y)-coordinates of P_1, P_2, P_3, P_4 and so the lattice Λ are known.

Therefore, in order to construct all elements of $S(J)$, it suffices to solve (79) with respect to X. Here the coefficients α_1, β_1, α_2, β_2 must take all integral values with

$$(82)\qquad \alpha_1\geqq 0, \quad \alpha_1\beta_2-\alpha_2\beta_1=1,$$

for which both (79) and (80) can be satisfied.

30. *The finiteness of $S(J)$.*

THEOREM 26. *The set $S(J)$ has only a finite number of elements.*

Proof. It suffices to show that the conditions (79) and (80) are solvable for at most a finite number of sets of integers $\alpha_1, \beta_1, \alpha_2, \beta_2$.

The equation (79) can be written as

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = J, \tag{83}$$

where

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{(\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2)-2(\alpha_1-\beta_2)(\alpha_2-\beta_1)X-2(\alpha_1\beta_2+\alpha_2\beta_1)X^2}{1-X^2}.$$

This expression Φ is a *positive definite* quadratic form in $\alpha_1, \beta_1, \alpha_2, \beta_2$; for it can be written as

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{1}{1-X^2}(\alpha_1-X^2\beta_2-X\alpha_2+X\beta_1)^2+(1+X^2)\left(\beta_2+\frac{X}{1+X^2}\alpha_2-\frac{X}{1+X^2}\beta_1\right)^2 + \frac{1}{1+X^2}(\alpha_2+X^2\beta_1)^2+(1-X^2)\beta_1^2.$$

From this identity, by (80),

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) \geqslant (1-X^2)\beta_1^2 \geqslant \tfrac{3}{4}\beta_1^2.$$

Further, from the definition of Φ,

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \Phi(X;\ \beta_1, \alpha_1, \beta_2, \alpha_2) = \Phi(X;\ \alpha_2, \beta_2, \alpha_1, \beta_1) = \Phi(X;\ \beta_2, \alpha_2, \beta_1, \alpha_1).$$

Hence β_1 may be replaced by $\alpha_1, \beta_1, \alpha_2, \beta_2$ in the last inequality, and so, by (83),

$$\max(\alpha_1^2, \beta_1^2, \alpha_2^2, \beta_2^2) \leqslant \frac{4J}{3}, \tag{84}$$

which proves the assertion.

Let then

$$\Lambda_\nu \qquad (\nu = 1, 2, \ldots, n)$$

be the elements of $S(J)$; let

$$\alpha_1^{(\nu)}, \beta_1^{(\nu)}, \alpha_2^{(\nu)}, \beta_2^{(\nu)} \qquad (\nu = 1, 2, \ldots, n)$$

be the sets of four integers; and let

$$\Phi_\nu(X) = \Phi(X;\ \alpha_1^{(\nu)}, \beta_1^{(\nu)}, \alpha_2^{(\nu)}, \beta_2^{(\nu)}) \quad (\nu = 1, 2, \ldots, n)$$

be the functions belonging to these lattices. The following table contains all functions Φ_ν which represent at least one value of J in $2 \leqslant J \leqslant 25$ for an argument X in $0 \leqslant X \leqslant \frac{1}{2}$.

Table of all functions Φ *which represent* J *for* $J \leqslant 25$.

$\Phi(\frac{1}{2})$	$(1-X^2)\,\Phi(X;\alpha_1,\beta_1,\alpha_2,\beta_2)$	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2
2	$2-2X^2$ †	1	0	0	1												
	$3-4X+2X^2$	0	1	−1	1	0	−1	1	−1	1	−1	1	0				
$\frac{10}{3}$	$2+2X^2$ ‡	0	1	−1	0	0	−1	1	0								
	$3-2X^2$	1	1	0	1	1	0	1	1	1	−1	0	1	1	0	−1	1
	$6-8X+2X^2$	0	1	−1	2	0	−1	1	−2	2	−1	1	0				
	$7-12X+6X^2$	1	−2	1	−1	1	−1	2	−1								
$\frac{22}{3}$	$3+4X+2X^2$	0	1	−1	−1	0	−1	1	1	1	1	−1	0				
	$6-2X^2$	1	2	0	1	1	−2	0	1	1	0	2	1	1	0	−2	1
	$7-6X^2$	1	1	1	2	1	−1	−1	2	2	1	1	1	2	−1	−1	1
	$11-12X+2X^2$	0	1	−1	3	0	−1	1	−3	3	−1	1	0				
	$15-24X+10X^2$	1	−3	1	−2	1	−1	3	−2	2	−3	1	−1	2	−1	3	−1
	$18-32X+14X^2$	1	−2	2	−3	3	−2	2	−1								
14	$6+8X+2X^2$	0	1	−1	−2	0	−1	1	2	2	1	−1	0				
	$11-2X^2$	1	3	0	1	1	−3	0	1	1	0	3	1	1	0	−3	1
	$15-4X-10X^2$	1	2	1	3	1	−1	−2	3	3	1	2	1	3	−2	−1	1
	$18-16X+2X^2$	0	1	−1	4	0	−1	1	4	4	−1	1	0				
	$27-40X+14X^2$	1	−4	1	−3	1	−1	4	−3	3	−4	1	−1	3	−1	4	−1
	$38-72X+34X^2$	2	−3	3	−4	4	−3	3	−2								
$\frac{58}{3}$	$7+12X+6X^2$	1	2	−1	−1	1	1	−2	−1								
	$15+4X-10X^2$	1	−2	−1	3	1	1	2	3	3	2	1	1	3	−1	−2	1
	$18-14X^2$	2	3	1	2	2	−3	−1	2	2	1	3	2	2	−1	−3	2
	$34-48X+18X^2$	2	−5	1	−2	2	−1	5	−2								
	$39-60X+22X^2$	1	−3	2	−5	1	−2	3	−5	5	−3	2	−1	5	−2	3	−1
	$47-84X+38X^2$	3	−5	2	−3	3	−2	5	−3								
$\frac{70}{3}$	$11+12X+2X^2$	0	1	−1	−3	0	−1	1	3	3	1	−1	0				
	$18-2X^2$	1	4	0	1	1	−4	0	1	1	0	4	1	1	0	−4	1
	$27-20X+2X^2$	0	1	−1	5	0	−1	1	−5	5	−1	1	0				
	$27-12X-14X^2$	1	3	1	4	1	−1	−3	4	4	−3	−1	1	4	1	3	1
	$43-60X+18X^2$	1	−5	1	−4	1	−1	5	−4	4	−5	1	−1	4	−1	5	−1
	$66-128X+62X^2$	3	−4	4	−5	5	−4	4	−3								

† Excluded case.
‡ Singular lattices.

As this table shows, there are in general two, three, or four systems of integers $\alpha_1^{(\nu)}$, $\beta_1^{(\nu)}$, $\alpha_2^{(\nu)}$, $\beta_2^{(\nu)}$ belonging to the same function Φ_ν and so also an equal number of lattices Λ_ν†. It is easily seen that if there are different critical lattices belonging to the same function Φ_ν, then these are transformed into one another by the *group G of order* 4 generated by the following two affine transformations:

The symmetry in the y-axis,

$$\text{A}: \qquad x \to -x, \quad y \to y.$$

The interchange of $f_1 = 1$ *and* $f_2 = 1$,

$$\text{B}: \qquad x \to \lambda^{-\frac{1}{2}} y, \quad y \to \lambda^{\frac{1}{2}} x.$$

For A replaces the integers α_1, β_1, α_2, β_2 by

$$\epsilon\beta_2, \ \epsilon\alpha_2, \ \epsilon\beta_1, \ \epsilon\alpha_1,$$

where $\epsilon = \pm 1$ is such that $\epsilon\beta_2 \geqslant 0$, and B replaces them by

$$\alpha_1, \ -\alpha_2, \ -\beta_1, \ \beta_2.$$

From now on, two critical lattices are considered as equivalent if they are related by an element of this group G; equivalent lattices belong to the same function Φ_ν.

31. *The value of* $D(J)$ *for* $2 \leqslant J \leqslant 25$.

By formula (66) in § 28,

$$D(J) = \min_{\nu = 1, 2, \ldots, n} d(\Lambda_\nu).$$

Hence, if

$$Y = D(J), \ X = |\sqrt{(1-Y^2)}|, \quad \text{and} \quad Y_\nu = d(\Lambda_\nu), \ X_\nu = |\sqrt{(1-Y_\nu^2}|,$$

then

$$\Phi_\nu(X_\nu) = J, \quad 0 \leqslant X_\nu \leqslant \tfrac{1}{2}, \tag{85}$$

$$X = \max_{\nu = 1, 2, \ldots, n} X_\nu. \tag{86}$$

† Two systems of integers

$$0, \ 1, \ -1, \ \beta_2^{(\nu)} \quad \text{and} \quad 0, \ -1, \ 1, \ -\beta_2^{(\nu)}$$

are interchanged by elements of Ω (§ 29) and generate the same lattice.

By a study of the last table I find that for every J in $2 \leqslant J \leqslant 25$ and for every Φ_ν there is *at most one* solution X_ν of (85). Further, most of these solutions X_ν can be ignored for the following reasons.

The rows of the table have been arranged in sets of functions

$$(1-X^2)\,\Phi_\nu(X)$$

so that $\Phi_\nu(\frac{1}{2})$ is the same in each set. It was also found possible to arrange the rows according to *increasing values of these functions* for variable values of X; *e.g.*, in the second set,

$$\frac{2+2X^2}{1-X^2} \leqslant \frac{3-2X^2}{1-X^2} \leqslant \frac{6-8X+2X^2}{1-X^2} \leqslant \frac{7-12X+8X^2}{1-X^2} \quad \text{for} \quad 0 \leqslant X \leqslant \tfrac{1}{2}.$$

Hence, for a given value of J in $2 \leqslant J \leqslant 25$, the maximum $X = X_\nu$ belongs to one of those 11 equations

$$\Phi_\nu(X_\nu) = J$$

in which the function Φ_ν is either at the *beginning* or at the *end* of one of the 6 sets of rows of the table. There is no difficulty in deciding which is the largest of these solutions X_ν. The result depends on the value of J, and is given in the following table. This table further contains the minimum determinant

$$D(J) = \Delta(K)$$

and the corresponding critical lattice†.

In the table, the numbers σ_k are defined thus:

$$\sigma_0 = 2, \quad \sigma_1 = \tfrac{10}{3}, \quad \sigma_2 = \tfrac{22}{3}, \quad \sigma_3 = 14, \quad \sigma_4 = \tfrac{58}{3}, \quad \sigma_5 = \tfrac{70}{3};$$

and J_n is defined thus

$$J_1 = \tfrac{34}{15}, \quad J_2 = \frac{3+14\sqrt{3}}{6}, \quad J_3 = 10, \quad J_4 = \frac{178+576\sqrt{14}}{143},$$

$$J_5 = \frac{63+88\sqrt{7}}{14}.$$

---

† If there exist several critical lattices, then they are all equivalent to the one given, except when J is one of the numbers σ_μ or J_ν.

$D(J)$ and critical lattices for $2 \leqslant J \leqslant 25$.

No.	Interval.	$(1-X^2)\,Y =$	$X =$	$D(J) = Y =$	Critical lattice.
1	$\sigma_0 \leqslant J \leqslant J_1$	$3-4X+2X^2$	$\dfrac{2-(J^2-J-2)^{\frac{1}{2}}}{J+2}$	$\dfrac{\{5J+2+4(J^2-J-2)^{\frac{1}{2}}\}^{\ddagger}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1+P_2$
2	$J_1 \leqslant J \leqslant \sigma_1$	$2+2X^2$	$\left(\dfrac{J-2}{J+2}\right)^{\frac{1}{2}}$	$2(J+2)^{-\frac{1}{2}}$	$P_3 = P_2$† $P_4 = -P_1$
3	$\sigma_1 \leqslant J \leqslant J_2$	$7-12X+6X^2$	$\dfrac{6-(J^2-J-6)^{\frac{1}{2}}}{J+6}$	$\dfrac{\{13J+6+12(J^2-J-6)^{\frac{1}{2}}\}^{\ddagger}}{J+6}$	$P_3 = P_1-2P_2$ $P_4 = P_1-P_2$
4	$J_2 \leqslant J \leqslant \sigma_2$	$3+4X+2X^2$	$\dfrac{-2+(J^2-J-2)^{\frac{1}{2}}}{J+2}$	$\dfrac{\{5J+2+4(J^2-J-2)^{\frac{1}{2}}\}^{\ddagger}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-P_2$
5	$\sigma_2 \leqslant J \leqslant J_3$	$18-32X+14X^2$	$-\dfrac{J-18}{J+14}$	$\dfrac{8(J-2)^{\frac{1}{2}}}{J+14}$	$P_3 = P_1-2P_2$ $P_4 = 2P_1-3P_2$
6	$J_3 \leqslant J \leqslant \sigma_3$	$6+8X+2X^2$	$\dfrac{J-6}{J+2}$	$\dfrac{4(J-2)^{\frac{1}{2}}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-2P_2$
7	$\sigma_3 \leqslant J \leqslant J_4$	$38-72X+34X^2$	$-\dfrac{J-38}{J+34}$	$\dfrac{12(J-2)^{\frac{1}{2}}}{J+34}$	$P_3 = 2P_1-3P_2$ $P_4 = 3P_1-4P_2$
8	$J_4 \leqslant J \leqslant \sigma_4$	$7+12X+6X^2$	$\dfrac{-6+(J^2-J-6)^{\frac{1}{2}}}{J+6}$	$\dfrac{\{13J+6+12(J^2-J-6)^{\frac{1}{2}}\}^{\ddagger}}{J+6}$	$P_3 = P_1+2P_2$ $P_4 = -P_1-P_2$
9	$\sigma_4 \leqslant J \leqslant J_5$	$47-84X+38X^2$	$\dfrac{42-(J^2-9J-22)^{\frac{1}{2}}}{J+38}$	$\dfrac{\{85J-298+84(J^2-9J-22)^{\frac{1}{2}}\}^{\ddagger}}{J+38}$	$P_3 = 3P_1-5P_2$ $P_4 = 2P_1-3P_2$
10	$J_5 \leqslant J \leqslant \sigma_5$	$11+12X+2X^2$	$\dfrac{-6+(J^2-9J+14)^{\frac{1}{2}}}{J+2}$	$\dfrac{\{13J-46+12(J^2-9J+14)^{\frac{1}{2}}\}^{\ddagger}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-3P_2$
11	$\sigma_5 \leqslant J \leqslant 25$	$66-128X+62X^2$	$-\dfrac{J-66}{J+62}$	$\dfrac{16(J-2)^{\frac{1}{2}}}{J+62}$	$P_3 = 3P_1-4P_2$‡ $P_4 = 4P_1-5P_2$

† Singular lattice.

‡ These values of X and Y remain true for $\sigma_5 \leqslant J \leqslant \frac{206}{7}$.

In the intervals No. 1–11 of the table, the functions $X = X(J)$ and $Y = Y(J)$ behave in the following manner:

$$\left.\begin{matrix} X \\ Y \end{matrix}\right\} \text{ is steadily } \left\{\begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix}\right\} \text{ in the intervals No. 2, 4, 6, 8, 10.}$$

$$\left.\begin{matrix} X \\ Y \end{matrix}\right\} \text{ is steadily } \left\{\begin{matrix} \text{decreasing} \\ \text{increasing} \end{matrix}\right\} \text{ in the intervals No. 1, 3, 5, 7, 9, 11.}$$

Further,

$$X = \tfrac{1}{2}, \quad Y = \frac{\sqrt{3}}{2} \quad \text{for} \quad J = \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5,$$

and

$$X = \tfrac{1}{4}, \qquad Y = \frac{\sqrt{15}}{4} \qquad \text{for} \quad J = J_1,$$

$$X = 2-\sqrt{3}, \qquad Y = \sqrt{\{4\sqrt{(3)}-6\}} \qquad \text{for} \quad J = J_2,$$

$$X = \tfrac{1}{3}, \qquad Y = \frac{2\sqrt{2}}{3} \qquad \text{for} \quad J = J_3,$$

$$X = \frac{21-4\sqrt{14}}{14}, \qquad Y = \sqrt{\left(\frac{24\sqrt{(14)}-67}{28}\right)} \qquad \text{for} \quad J = J_4,$$

$$X = \frac{4-\sqrt{7}}{3} \qquad Y = \sqrt{\left(\frac{8\sqrt{(7)}-14}{9}\right)} \qquad \text{for} \quad J = J_5.$$

The interval No. 2 is particularly interesting, since here K has only a single critical lattice, and this is singular. At the lower end $J = \frac{34}{15}$ of this interval, K has this singular lattice, and also the regular lattice

$$P_3 = P_2, \quad P_4 = -P_1+P_2,$$

and the lattice symmetrical to it in the y-axis.

The table shows that the critical lattices of K have 2, 3, 4, 5, or 6 pairs of symmetrical points on C, depending on the value of J.

The general law of the function $D(J)$ seems to be very complicated. By the table, the graph of $Y = D(J)$ is a saw-like curve for $2 \leqslant J \leqslant 25$, and possibly for all values of J. In the intervals No. 5, 6, 7, and 11, $D(J)$ takes a surprisingly simple form.

One can show that $\frac{\sqrt{3}}{2} \leqslant D(J) \leqslant \frac{\sqrt{15}}{4}$ for all values of J, and that

$$\lim_{J\to\infty} D(J) = \frac{\sqrt{3}}{2};$$

this limit equation was communicated to me by P. Erdös.

I remark finally that the problem and result of this chapter can be extended to a pair of positive definite Hermitian forms; but then the proof is preferably based on the geometrical theory of Picard's group.

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ON INFINITE SOLUBLE GROUPS (III)*

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Introduction.

The structure of finite soluble groups can be investigated by means of their nilpotent† constituents. If $\mathfrak{G}$ is a finite soluble group, it is either nilpotent or else it contains a maximal self-conjugate nilpotent subgroup, say $\mathfrak{B}_1$, viz. the join of all its self-conjugate nilpotent subgroups. Now $\mathfrak{G}/\mathfrak{B}_1$ is either nilpotent or else it contains a maximal self-conjugate nilpotent subgroup. Let $\mathfrak{B}_2$ be the corresponding self-conjugate subgroup of $\mathfrak{G}$. Continuing this process, after a finite number of steps, say k, one arrives at $\mathfrak{G}$. The series

$$1 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_k = \mathfrak{G}$$

with the recurrent rule $\mathfrak{B}_{i+1}/\mathfrak{B}_i = \mathfrak{B}_1(\mathfrak{G}/\mathfrak{B}_i)$ will be called the "*ascending nilpotent series*" of $\mathfrak{G}$.

Dually the meet $\mathfrak{C}_1$ of all self-conjugate subgroups of $\mathfrak{G}$ with nilpotent factor-groups has itself a nilpotent factor-group. If $\mathfrak{C}_1 \neq 1$, let $\mathfrak{C}_2$ be the minimal self-conjugate subgroup of $\mathfrak{C}_1$ with a nilpotent factor-group. Continuing this process, after a finite number of steps, say k', one arrives at the 1-element. The series

$$\mathfrak{G} = \mathfrak{C}_0 \supset \mathfrak{C}_1 \supset \ldots \supset \mathfrak{C}_{k'} = 1$$

* Continuation of the papers "On infinite soluble groups" (I) and (II), *Proc. London Math. Soc.* (2), 44 (1938), 53–60, 336–344. These papers will be subsequently described as I.S.G. (I) and (II) respectively. We assume the reader to be familiar with them.

† We adopt the term "nilpotent", which is used by H. Fitting in his paper "Beiträge zur Theorie der Gruppen endlicher Ordnung", *Jahresber. d. Deutsch. Math. Ver.*, 48 (1938), 77–141; *cf.* § 11, p. 98. A group $\mathfrak{G}$ is said to be *nilpotent* if its "*lower central series*" terminates in the 1-element after a finite number of steps. For finite groups this is equivalent to the definition: $\mathfrak{G}$ is the direct product of its Sylow-groups. In I.S.G. (II) we have used the term "$\mathfrak{G}$ *has a central series*" instead of "$\mathfrak{G}$ *is nilpotent*",

with the recurrent rule $\mathfrak{C}_{i+1}=\mathfrak{C}_1(\mathfrak{C}_i)$ will be called the "*descending nilpotent series*" of $\mathfrak{G}$.

The groups $\mathfrak{B}_i$ and $\mathfrak{C}_i$ are characteristic in $\mathfrak{G}$, and the $\mathfrak{C}_i$ are even "voll-invariant". It is, of course, true that $k=k'$ and $\mathfrak{C}_i \subseteq \mathfrak{B}_{k-i}$, but $\not\subseteq \mathfrak{B}_{k-i-1}$*.

In this paper we first adapt the ascending nilpotent series to the class of infinite soluble groups which we have studied, viz. the S-groups (3.1). We then show, by means of it, the importance of the *finite* factor-groups of an S-group. The main result is: An S-group $\mathfrak{G}$ is nilpotent if, and only if, all its finite factor-groups are nilpotent (3.2). This theorem is used in (3.3) to prove some characteristic properties of nilpotent S-groups.

3.1. THEOREM. *Every S-group $\mathfrak{G}$ possesses a maximal self-conjugate nilpotent subgroup $\mathfrak{B}$ which contains all self-conjugate nilpotent subgroups of $\mathfrak{G}$.*

Proof. (i) First we show that the join $\mathfrak{J}$ of two self-conjugate nilpotent subgroups $\mathfrak{R}$ and $\mathfrak{S}$ of $\mathfrak{G}$ is itself nilpotent.

Let $$\mathfrak{R}=\mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \ldots \supset \mathfrak{R}_{r+1}=1$$

and $$\mathfrak{S}=\mathfrak{S}_1 \supset \mathfrak{S}_2 \supset \ldots \supset \mathfrak{S}_{s+1}=1$$

be the lower central series of $\mathfrak{R}$ and $\mathfrak{S}$ of classes r and s respectively, and consider the series

$$\mathfrak{J}=\mathfrak{J}_1 \supseteq \mathfrak{J}_2 \supseteq \ldots, \quad \text{where} \quad \mathfrak{J}_2=(\mathfrak{J},\mathfrak{J})\dagger \quad \text{and} \quad \mathfrak{J}_{i+1}=(\mathfrak{J},\mathfrak{J}_i) \quad (i=1,2,\ldots).$$

If this series terminates in the 1-element after a finite number of steps, it is the lower central series of $\mathfrak{J}$. We shall show that $\mathfrak{J}_{r+s+1}=1$ (so that the class of the join is less than, or equal to, the sum of the classes of the components) and, moreover, that, for $i=1, 2, \ldots,$

$$(3.11) \qquad \mathfrak{J}_i \subseteq \{\mathfrak{R}_i, \mathfrak{R}_{i-1}\cap\mathfrak{S}_1, \mathfrak{R}_{i-2}\cap\mathfrak{S}_2, \ldots, \mathfrak{R}_2\cap\mathfrak{S}_{i-2}, \mathfrak{R}_1\cap\mathfrak{S}_{i-1}, \mathfrak{S}_i\}\ddagger$$

This is true for $i=1$. We apply induction and assume the formula (3.11) to be proved for a certain suffix i. Now $\mathfrak{J}_{i+1}$ is, by definition, generated by all elements of the form (I, I_i), where $I \in \mathfrak{J}$ and $I_i \in \mathfrak{J}_i$.

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* The two nilpotent series were introduced and applied to the study of finite soluble groups by Mr. P. Hall (unpublished). Some results are contained in the paper of H. Fitting, *loc. cit.*, Teil, III, §§ 11–13.

† If $\mathfrak{M}$ and $\mathfrak{N}$ are two subgroups of a group $\mathfrak{G}$, we denote by $\mathfrak{M}\cap\mathfrak{N}$ the meet, or common part, of $\mathfrak{M}$ and $\mathfrak{N}$; by $\{\mathfrak{M},\mathfrak{N}\}$ the join of, or group generated by, $\mathfrak{M}$ and $\mathfrak{N}$; and by $(\mathfrak{M},\mathfrak{N})$ the commutator subgroup of $\mathfrak{M}$ and $\mathfrak{N}$, *i.e.* the group generated by all elements $(M,N)=M^{-1}N^{-1}MN$, where $M\in\mathfrak{M}$ and $N\in\mathfrak{N}$.

‡ If the suffix exceeds the class, the group is considered to be 1.

Every element I of $\mathfrak{I}$ can be expressed in the form $R.S$, where $R \in \mathfrak{R}$ and $S \in \mathfrak{S}$; and every element I_i of $\mathfrak{I}_i$ can, by hypothesis, be expressed as a product of elements of the respective components in (3.11).

Now, by repeated application of the fundamental identities,

$$(A, BC) = (A, C).C^{-1}(A, B)C,$$

$$(AB, C) = B^{-1}(A, C)B.(B, C),$$

a commutator of arbitrary products can be reduced to a product of "pure"* commutators and of their transforms.

But for these pure commutators we have, *e.g.*,

$$(\mathfrak{R}_{i-\alpha} \cap \mathfrak{S}_\alpha, \mathfrak{R}) \subseteq \mathfrak{R}_{i-\alpha+1}, \quad \text{because} \quad (\mathfrak{R}_{i-\alpha}, \mathfrak{R}) \subseteq \mathfrak{R}_{i-\alpha+1};$$

and $$(\mathfrak{R}_{i-\alpha} \cap \mathfrak{S}_\alpha, \mathfrak{R}) \subseteq \mathfrak{S}_\alpha, \quad \text{because} \quad (\mathfrak{S}_\alpha, \mathfrak{R}) \subseteq \mathfrak{S}_\alpha;$$

therefore $$(\mathfrak{R}_{i-\alpha} \cap \mathfrak{S}_\alpha, \mathfrak{R}) \subseteq \mathfrak{R}_{i-\alpha+1} \cap \mathfrak{S}_\alpha.$$

Similarly with $\mathfrak{S}$ instead of $\mathfrak{R}$.

In each case just one of the suffixes is raised by 1. Moreover, since all groups in question are self-conjugate in $\mathfrak{G}$, they consist of complete sets of conjugate elements and, therefore, the property that a certain element is contained in one of them is not affected by the process of transforming by any element of $\mathfrak{G}$. Consequently every element of $\mathfrak{I}_{i+1}$ can be written as a product of elements belonging to the respective components of (3.11), each component with one suffix raised by 1. That is

$$\mathfrak{I}_{i+1} \subseteq \{\mathfrak{R}_{i+1}, \mathfrak{R}_i \cap \mathfrak{S}_1, \ldots, \mathfrak{R}_1 \cap \mathfrak{S}_i, \mathfrak{S}_{i+1}\},$$

which is the required result.

Finally, for $i = r+s+1$, in every component of (3.11) either the suffix of $\mathfrak{R}$ is at least $r+1$ or the suffix of $\mathfrak{S}$ at least $s+1$. So $\mathfrak{I}_{r+s+1} = 1$†.

(ii) It is now easy, by induction over the number of subgroups, to prove the following extended result:

The join $\mathfrak{I}$ of any finite number of self-conjugate nilpotent subgroups $\mathfrak{R}$, $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$, ... of $\mathfrak{G}$ is itself nilpotent. If $\mathfrak{R} = \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \ldots \supset \mathfrak{R}_{r+1} = 1$ is the lower central series of $\mathfrak{R}$, and similarly for the other subgroups, the

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* By a "pure" commutator we mean here, of course, one in which only elements of *one* component of (3.11) occur, not products of elements of different components.

† For a different proof of this part of Theorem (3.1) *cf.* Fitting, *loc. cit.*, 100–102.

classes being $r, s, t, u, \ldots$ respectively, and if $\mathfrak{I}_i$ is defined as in (i), then

$$\mathfrak{I}_i \subseteq \{\ldots, \mathfrak{R}_\alpha \cap \mathfrak{S}_\beta \cap \mathfrak{T}_\gamma \cap \mathfrak{U}_\delta \cap \ldots, \ldots\},$$

where the brackets contain all and only those components for which $\alpha+\beta+\gamma+\delta+\ldots = i$. In particular $\mathfrak{I}_{r+s+t+u+\ldots+1} = 1$, so that the class of the join is less than, or equal to, the sum of the classes of the components. We omit the formal proof.

(iii) Now $\mathfrak{G}$ satisfies the "maximal condition"*. Therefore the join $\mathfrak{B}$ of all self-conjugate nilpotent subgroups of $\mathfrak{G}$ is the join of a suitably chosen finite number of them. By (ii), this join is nilpotent. Therefore $\mathfrak{B}$ is nilpotent. This concludes the proof of Theorem (3.1). $\mathfrak{B}$ is, of course, a characteristic subgroup of $\mathfrak{G}$.

3.12. Lemma. *$\mathfrak{B}$ contains its own centraliser $\mathfrak{Z}$, i.e. the group formed by all elements of $\mathfrak{G}$ which are permutable with every element of $\mathfrak{B}$.*

Proof. If the lemma were not true, then $\mathfrak{B}$ would be contained as a proper subgroup in $\{\mathfrak{Z}, \mathfrak{B}\}$. $\{\mathfrak{Z}, \mathfrak{B}\}$ is self-conjugate in $\mathfrak{G}$, because the centraliser of a self-conjugate subgroup is a self-conjugate subgroup. Now $\{\mathfrak{Z}, \mathfrak{B}\}/\mathfrak{B}$ has Abelian subgroups which are self-conjugate in $\mathfrak{G}/\mathfrak{B}$, *e.g.* the last term of a "B-series" for $\mathfrak{G}/\mathfrak{B}$ laid through $\{\mathfrak{Z}, \mathfrak{B}\}/\mathfrak{B}$†. Let $P_1, P_2, \ldots, P_r$ be representatives in $\mathfrak{Z}$ of a set of generators for such an Abelian subgroup. Then we have the following results:

(i) $\{\mathfrak{B}, P_1, P_2, \ldots, P_r\}$ is self-conjugate in $\mathfrak{G}$.

(ii) $\{\mathfrak{B}, P_i\}$, $i = 1, 2, \ldots, r$, is nilpotent, because it is the join of the two subgroups $\mathfrak{B}$ and $\{P_i\}$ which are self-conjugate in it and nilpotent.

(iii) $\{\mathfrak{B}, P_i\}$ is self-conjugate in $\{\mathfrak{B}, P_1, P_2, \ldots, P_r\}$, because in this latter group $\mathfrak{B}$ has an Abelian factor-group.

(iv) $\{\mathfrak{B}, P_1, P_2, \ldots, P_r\}$ is the join of the subgroups $\{\mathfrak{B}, P_i\}$, $i = 1, 2, \ldots, r$.

Therefore finally $\{\mathfrak{B}, P_1, P_2, \ldots, P_r\}$ is, by (3.1), nilpotent and a self-conjugate subgroup of $\mathfrak{G}$, but is not contained in $\mathfrak{B}$. This contradicts the maximal property of $\mathfrak{B}$.

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* *Cf.* I.S.G. (I), Theorem 1.3, p. 55.

† *Cf.* I.S.G. (II), Theorem 2.312, p. 340.

3.13. THEOREM. *If $\mathfrak{H}$ is any self-conjugate subgroup of $\mathfrak{G}$ containing $\mathfrak{B}$, and if G is any element of $\mathfrak{G}$ not contained in $\mathfrak{H}$, then the automorphism induced in $\mathfrak{H}$ by G is an outer automorphism.*

Proof. If an element G of $\mathfrak{G}$ induces an inner automorphism in $\mathfrak{H}$, then there is an element $H \in \mathfrak{H}$ which gives the same inner automorphism as G does. Then $G' = G.H^{-1}$ gives the identical automorphism, *i.e.* is permutable with every element of $\mathfrak{H}$ and, in particular, of $\mathfrak{B}$. By the preceding lemma $\mathfrak{B}$ contains its centraliser. Therefore $G' \in \mathfrak{B}$ and $G = G'H \in \mathfrak{H}$. This concludes the proof*.

We may now introduce as in the case of finite soluble groups the "ascending nilpotent series" of an S-group,

$$1 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_k = \mathfrak{G},$$

where $\mathfrak{B}_1 = \mathfrak{B}$ and the $\mathfrak{B}_i$ are defined by the recurrence rule

$$\mathfrak{B}_{i+1}/\mathfrak{B}_i = \mathfrak{B}_1(\mathfrak{G}/\mathfrak{B}_i).$$

The corresponding descending series has, however, no analogue in S-groups. The factor-group of the meet of all self-conjugate subgroups of $\mathfrak{G}$ with nilpotent factor-groups need not be nilpotent itself.

This is shown, *e.g.* by the infinite dihedral group, *i.e.* the group generated by two elements P and Q with the relations $Q^{-1}PQ = P^{-1}$ and $Q^2 = 1$. In it the self-conjugate subgroups generated by P^{2^α} whose index is a power of 2 (and only they) have nilpotent factor-groups. But the meet of these groups is the 1-element, and the group itself is not nilpotent.

3.2. In the infinite dihedral group the self-conjugate subgroup generated by P^3, whose index is 6, has a factor-group which is not nilpotent. This illustrates the importance of the finite factor-groups of S-groups.

3.21. THEOREM. *Every S-group $\mathfrak{G}$ has a self-conjugate subgroup $\mathfrak{H}$, of finite index under $\mathfrak{G}$, which has no elements of finite order $\neq 1$.*

Proof. Among the self-conjugate subgroups of $\mathfrak{G}$ which are purely infinite† we select a maximal one, say $\mathfrak{H}$, *i.e.* one which cannot be embedded in another purely infinite self-conjugate subgroup of $\mathfrak{G}$. Such a group exists, because the maximal condition holds for the subgroups of $\mathfrak{G}$.

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* This proof includes, of course, the case of a finite soluble group $\mathfrak{G}$. For a different proof in this case *cf.* Fitting, *loc. cit.*, 106.

† *I.e.* which have no elements of finite order other than 1.

We proceed to show that $\mathfrak{G}/\mathfrak{H}$ is finite. We write $\mathfrak{G}/\mathfrak{H} = \mathfrak{G}'$, and similarly for any subgroup $\mathfrak{K}$ of $\mathfrak{G}$ with $\mathfrak{G} \supseteq \mathfrak{K} \supseteq \mathfrak{H}$, $\mathfrak{K}/\mathfrak{H} = \mathfrak{K}'$. Let us assume that $\mathfrak{G}'$ is infinite. Now $\mathfrak{G}'$ has no purely infinite self-conjugate subgroups, but has finite self-conjugate subgroups, *e.g.* the last term other than 1 of a "B-series" for $\mathfrak{G}$*.

Let $\mathfrak{L}'$ be a maximal self-conjugate finite subgroup of $\mathfrak{G}'$, and let $\mathfrak{M}'$ be the centraliser of $\mathfrak{L}'$ in $\mathfrak{G}'$. Then $\mathfrak{M}'$ is of finite index under $\mathfrak{G}'$. For if G is any element of $\mathfrak{G}'$, and L is any element of $\mathfrak{L}'$, then the sequence

$$G^{-1}LG,\ G^{-2}LG^2,\ \ldots$$

can only have a finite number of different terms, and therefore a certain power of G, say G^l, must be permutable with L. If m denotes the l.c.m. of the exponents l for the different elements L of $\mathfrak{L}'$, then G^m is permutable with every element of $\mathfrak{L}'$, *i.e.* G^m belongs to $\mathfrak{M}'$. Therefore $\mathfrak{G}'/\mathfrak{M}'$ has no elements of infinite order and is, in virtue of the maximal condition, a finite group.

Let now $\mathfrak{D}' = \mathfrak{L}' \cap \mathfrak{M}'$. Then $\mathfrak{D}'$ is finite and $\mathfrak{M}'/\mathfrak{D}'$ is infinite. The last term of a B-series for $\mathfrak{M}'/\mathfrak{D}'$ is a self-conjugate sub-group $\mathfrak{N}'$ such that $\mathfrak{M}' \supset \mathfrak{N}' \supset \mathfrak{D}'$, and $\mathfrak{N}'/\mathfrak{D}'$ is an elementary or a free Abelian group. Moreover $\mathfrak{N}'$ can be chosen as a characteristic subgroup of $\mathfrak{M}'$ and, therefore, a self-conjugate subgroup of $\mathfrak{G}'$. Now $\mathfrak{N}'/\mathfrak{D}'$ cannot be finite; for otherwise $\{\mathfrak{L}', \mathfrak{N}'\}$ would be finite and would contain $\mathfrak{L}'$ as a proper subgroup in contradiction to the maximal property of $\mathfrak{L}'$. So $\mathfrak{N}'/\mathfrak{D}'$ is a free Abelian group.

Now $\mathfrak{D}'$ (the central of $\mathfrak{L}'$) is contained in the central of $\mathfrak{M}'$, and therefore of $\mathfrak{N}'$. Therefore $\mathfrak{N}'$ is nilpotent and of class less than or equal to 2. If d is the l.c.m. of the orders of the elements of $\mathfrak{D}'$, and if P and Q are any elements of $\mathfrak{N}'$ and $(P, Q) = D$, then $(P, Q^d) = D^d = 1$. The d-th powers of the elements of $\mathfrak{N}'$ are therefore self-conjugate in $\mathfrak{N}'$. They generate a characteristic Abelian subgroup of $\mathfrak{N}'$ which is therefore self-conjugate in $\mathfrak{G}'$. It is obviously of infinite order, but may contain elements of $\mathfrak{D}'$. However, if we raise its elements again to the d-th power, we obtain a free Abelian self-conjugate subgroup of $\mathfrak{G}'$. This is a contradiction, and therefore $\mathfrak{G}' = \mathfrak{G}/\mathfrak{H}$ is finite.

Theorem 3.21 shows that any infinite S-group has purely infinite nilpotent self-conjugate subgroups, *e.g.* the last term other than 1 of a (characteristic) B-series for $\mathfrak{H}$. Therefore $\mathfrak{B}_1$, the first term of the ascending nilpotent series of $\mathfrak{G}$, is necessarily infinite. If now $\mathfrak{G}/\mathfrak{B}_1$ is

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* *Cf.* I.S.G. (I), Theorem 1.41, p. 57.

still infinite, then again $\mathfrak{B}_1(\mathfrak{G}/\mathfrak{B}_1) = \mathfrak{B}_i/\mathfrak{B}_1$ is infinite. Continuing this argument we prove the

3.22. COROLLARY. *In the ascending nilpotent series*

$$1 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_k = \mathfrak{G}$$

all infinite factor-groups $\mathfrak{B}_{i+1}/\mathfrak{B}_i$ occur at the beginning, and all finite ones at the end.

In the case of a nilpotent S-group $\mathfrak{G}$, Theorem 3.21 enables us to supplement Theorem 2.22* in an important point. There we proved the existence of a finite characteristic subgroup $\mathfrak{F}$ of $\mathfrak{G}$ which consists of all elements of $\mathfrak{G}$ of finite order and whose factor-group $\mathfrak{G}/\mathfrak{F} = \mathfrak{F}'$, say, is purely infinite. Now we have also a purely infinite self-conjugate subgroup $\mathfrak{H}$ of $\mathfrak{G}$ whose factor-group $\mathfrak{G}/\mathfrak{H} = \mathfrak{H}'$, say, is finite. Consider the direct product $\mathfrak{F}' \times \mathfrak{H}'$ and the mapping $G \to G\mathfrak{F} . G\mathfrak{H}$, where G is any element of $\mathfrak{G}$ and the r.h.s. is the product of two co-sets of $\mathfrak{F}$ and $\mathfrak{H}$ respectively, *i.e.* an element of $\mathfrak{F}' \times \mathfrak{H}'$. This mapping is a homomorphism, since $\mathfrak{F}$ and $\mathfrak{H}$ are self-conjugate in $\mathfrak{G}$. However the unit-element 1 of $\mathfrak{G}$ is the only element which is mapped on to the unit-element of $\mathfrak{F}' \times \mathfrak{H}'$, viz. $1\mathfrak{F} . 1\mathfrak{H}$. For if $X \to 1\mathfrak{F} . 1\mathfrak{H}$, then X must be an element of $\mathfrak{F}$ and of $\mathfrak{H}$. But $\mathfrak{F} \cap \mathfrak{H} = 1$. Hence the homomorphism is an isomorphism and we have proved

3.23. THEOREM. *Every nilpotent S-group is isomorphic with a subgroup of a direct product of a finite nilpotent and a purely infinite nilpotent group.*

In the group $\mathfrak{G} = \{P, Q\}$ generated by two elements P and Q of infinite order with the relations $(P, Q) = R$, $R^p = (P, R) = (Q, R) = 1$ (p a prime), the finite group $\mathfrak{F}$ is $\{R\}$ of order p, $\mathfrak{F}'$ is a free Abelian group of two generators. For the maximal self-conjugate purely infinite subgroup $\mathfrak{H}$ we may choose, *e.g.* $\{P^p, Q^p\}$, and $\mathfrak{H}'$ is then the non-Abelian group of order p^3 whose elements other than 1 all have order p.

3.24. THEOREM. *If an S-group $\mathfrak{G}$ is not nilpotent, then it has a self-conjugate subgroup of finite index whose factor-group is not nilpotent.*

Proof. (i) Let $1 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_k = \mathfrak{G}$ be the ascending nilpotent series of $\mathfrak{G}$. By hypothesis, $k \geqslant 2$, $\mathfrak{G}/\mathfrak{B}_{k-1}$ is nilpotent, but $\mathfrak{G}/\mathfrak{B}_{k-2}$ is not. Between $\mathfrak{B}_{k-2}$ and $\mathfrak{B}_{k-1}$ we insert the terms corresponding to a B-series

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* I.S.G. (II), 339.

for $\mathfrak{B}_{k-1}/\mathfrak{B}_{k-2}$. Let this be

$$\mathfrak{G} = \mathfrak{B}_k \supset \mathfrak{B}_{k-1} = \mathfrak{N}_0 \supset \mathfrak{N}_1 \supset \ldots \supset \mathfrak{N}_n = \mathfrak{B}_{k-2} \supset \ldots.$$

As was remarked in I.S.G. (I), p. 57, the $\mathfrak{N}_i$ can be chosen as characteristic subgroups of $\mathfrak{B}_{k-1}$ and, therefore, of $\mathfrak{G}$. There must now be an index i such that $\mathfrak{G}/\mathfrak{N}_i$ is nilpotent, but $\mathfrak{G}/\mathfrak{N}_{i+1}$ is not.

(ii) $\mathfrak{N}_i/\mathfrak{N}_{i+1}$ is an elementary or a free Abelian group. If the latter is the case, and if $P_1, P_2, \ldots, P_l$ are representatives in $\mathfrak{G}$ of generators of $\mathfrak{N}_i/\mathfrak{N}_{i+1}$, we denote by $\mathfrak{N}_{i+1}^{(p)}$ the group $\{\mathfrak{N}_{i+1}, P_1^p, P_2^p, \ldots, P_l^p\}$. $\mathfrak{N}_{i+1}^{(p)}$ is, just as $\mathfrak{N}_{i+1}$, a characteristic subgroup of $\mathfrak{G}$. It is of index p^l under $\mathfrak{N}_i$.

Now $\mathfrak{G}/\mathfrak{N}_{i+1}$ is not nilpotent. Suppose, if possible, that $\mathfrak{G}/\mathfrak{N}_{i+1}^{(p)}$ is nilpotent for every prime number p. If $\mathfrak{G}/\mathfrak{N}_{i+1}^{(p)}$ is nilpotent for a particular p, then its class is at most $c+l$, where c is the class of $\mathfrak{G}/\mathfrak{N}_i$. That is, if we define the groups $\mathfrak{H}_i$ by the recurrent rule

$$\mathfrak{H}_1 = \mathfrak{G}, \quad \mathfrak{H}_2 = (\mathfrak{G}, \mathfrak{G}), \quad \mathfrak{H}_{i+1} = (\mathfrak{H}_i, \mathfrak{G}),$$

then $\mathfrak{H}_{c+l+1}$ is contained in $\mathfrak{N}_{i+1}^{(p)}$. If this were true for every prime number p, then $\mathfrak{H}_{c+l+1}$ would also be contained in the meet of all $\mathfrak{N}_{i+1}^{(p)}$, which is evidently $\mathfrak{N}_{i+1}$. But $\mathfrak{H}_{c+l+1} \subseteq \mathfrak{N}_{i+1}$ would mean that $\mathfrak{G}/\mathfrak{N}_{i+1}$ is nilpotent. This is a contradiction.

(iii) Henceforth we may assume that $\mathfrak{N}_i/\mathfrak{N}_{i+1}$ is finite and that $\mathfrak{G}/\mathfrak{N}_{i+1}$ is infinite and not nilpotent. We write $\mathfrak{G}/\mathfrak{N}_{i+1} = \mathfrak{G}'$ and similarly for any subgroup of $\mathfrak{G}$ between $\mathfrak{G}$ and $\mathfrak{N}_{i+1}$. Let $\mathfrak{M}'$ be the centraliser of $\mathfrak{N}_i'$ in $\mathfrak{G}'$. Again, since $\mathfrak{N}_i'$ is finite, $\mathfrak{M}'$ is of finite index under $\mathfrak{G}'$ and therefore is of infinite order. By Theorem 3.21, $\mathfrak{M}'$ has a purely infinite self-conjugate subgroup $\mathfrak{K}'$ with finite factor-group $\mathfrak{M}'/\mathfrak{K}'$. $\mathfrak{K}'$ need not be self-conjugate in $\mathfrak{G}'$, but the meet $\mathfrak{D}'$ of $\mathfrak{K}'$ and all its conjugates in $\mathfrak{G}'$ is self-conjugate in $\mathfrak{G}'$. The index of $\mathfrak{D}'$ under $\mathfrak{G}'$ is finite*.

We next show that $\mathfrak{G}'/\mathfrak{D}'$ is not nilpotent. If it were, let c be the greater of the classes of $\mathfrak{G}'/\mathfrak{D}'$ and $\mathfrak{G}'/\mathfrak{N}_i'$. Then, if we put $\mathfrak{H}_1' = \mathfrak{G}'$, $\mathfrak{H}_{i+1}' = (\mathfrak{H}_i', \mathfrak{G}')$, $\mathfrak{H}_{c+1}'$ would be contained both in $\mathfrak{D}'$ and in $\mathfrak{N}_i'$. But $\mathfrak{D}'$ is purely infinite and $\mathfrak{N}_i'$ is finite. Hence $\mathfrak{H}_{c+1}' = 1$ and $\mathfrak{G}' = \mathfrak{G}/\mathfrak{N}_{i+1}$ would be nilpotent contrary to hypothesis. If $\mathfrak{D}$ corresponds in $\mathfrak{G}$ to $\mathfrak{D}'$ in $\mathfrak{G}'$, then $\mathfrak{D}$ is the required self-conjugate subgroup of $\mathfrak{G}$ whose finite factor-group $\mathfrak{G}/\mathfrak{D}$ is not nilpotent. This concludes the proof.

Another interesting inference can be drawn from Theorem 3.21. Let $\mathfrak{G}$ be an S-group and let $f(X, Y, \ldots)$ be any "word." If the equation

* Since $\mathfrak{G}'/\mathfrak{M}'$ is finite, $\mathfrak{K}'$ has only a finite number of conjugates. They are, and consequently their meet $\mathfrak{D}'$ is, of finite index under $\mathfrak{M}'$ and $\mathfrak{G}'$.

$f(X, Y, \ldots) = 1$ is not identically true in $\mathfrak{G}$, then for suitable elements $A, B, \ldots$ of $\mathfrak{G}$ we have $f(A, B, \ldots) = C \neq 1$. Now there are purely infinite self-conjugate subgroups of $\mathfrak{G}$ which do not contain C. For if $\mathfrak{H}$ is a purely infinite self-conjugate subgroup of $\mathfrak{G}$, and if $\mathfrak{N}$ is the last term other than 1 of a characteristic B-series of $\mathfrak{H}$, then either $\mathfrak{N}$ does not contain C, or if it does, then the group obtained by raising the elements of $\mathfrak{N}$ to a sufficiently high power does not contain C.

Let now $\mathfrak{M}$ be a maximal purely infinite self-conjugate subgroup of $\mathfrak{G}$ not containing C. Then $\mathfrak{G}/\mathfrak{M}$ is finite. For every purely infinite self-conjugate subgroup of $\mathfrak{G}$ between $\mathfrak{M}$ and $\mathfrak{G}$ must contain C. Assume now $\mathfrak{G}/\mathfrak{M}$ to be infinite. Then again we take the last term other than 1 of a characteristic B-series for a purely infinite self-conjugate subgroup of $\mathfrak{G}/\mathfrak{M}$ and raise the elements of this free Abelian group to a sufficiently high power. Thus we obtain a purely infinite self-conjugate subgroup of $\mathfrak{G}$ containing $\mathfrak{M}$ but not C. This is a contradiction to the maximal property of $\mathfrak{M}$. Now the equation $f(X, Y, \ldots) = 1$ is not satisfied identically in $\mathfrak{G}/\mathfrak{M}$. Hence we have proved

3.25. THEOREM. *If the equation $f(X, Y, \ldots) = 1$ (where f is any word) is not identically true in an S-group $\mathfrak{G}$, then there exists a purely infinite self-conjugate subgroup $\mathfrak{M}$ of $\mathfrak{G}$ of infinite index such that $f = 1$ is not identically true in $\mathfrak{G}/\mathfrak{M}$.*

3.3. Of the various properties of nilpotent S-groups which were proved in I.S.G. (II), 2.3*, some are characteristic in the sense that if, and only if, an S-group possesses that property, it is nilpotent.

So we are able, *e.g.*, to prove with the help of theorem (3.1) alone that Theorem 2.341 is a characteristic property of nilpotent S-groups.

3.31. THEOREM. *If in an S-group $\mathfrak{G}$ every proper subgroup $\mathfrak{K}$ is different from its normaliser, then $\mathfrak{G}$ is nilpotent.*

For the proof of (3.31) we shall use the following lemma, which is interesting in itself.

3.32. LEMMA. *Let $\mathfrak{M}$ be any maximal nilpotent subgroup of an S-group $\mathfrak{G}$, i.e. one which is not contained as a proper subgroup in any other nilpotent sub-group of $\mathfrak{G}$. Let $\mathfrak{N}$ be the normaliser of $\mathfrak{M}$ in $\mathfrak{G}$. Then $\mathfrak{N}$ is its own normaliser.*

Proof. If the lemma were not true, there would exist an element G of $\mathfrak{G}$ not contained in $\mathfrak{N}$, but permutable with $\mathfrak{N}$. Now G cannot be

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* Pp. 340 *ff.*

permutable with $\mathfrak{M}$, because it does not belong to $\mathfrak{N}$, and therefore transforms $\mathfrak{M}$ into $\mathfrak{M}' = G^{-1}\mathfrak{M}G \neq \mathfrak{M}$. $\mathfrak{M}'$ is, of course, self-conjugate in $\mathfrak{N}$ and nilpotent. But by Theorem (3.1) the group $\{\mathfrak{M}, \mathfrak{M}'\}$ is nilpotent and contains $\mathfrak{M}$ as a proper subgroup. This is a contradiction to the maximal property of $\mathfrak{M}$.

Proof of 3.31. If $\mathfrak{M}$ has the same meaning as in (3.32), its normaliser $\mathfrak{N}$ cannot be a proper subgroup of $\mathfrak{G}$ and must, therefore, coincide with $\mathfrak{G}$. Thus $\mathfrak{M}$ is self-conjugate in $\mathfrak{G}$, and, since it is nilpotent, it is contained in $\mathfrak{B}_1$. Thus every nilpotent subgroup is contained in some maximal nilpotent subgroup and therefore in $\mathfrak{B}_1$. But if G is any element of $\mathfrak{G}$, then the group $\{G\}$ is certainly nilpotent. Therefore $G \in \mathfrak{B}_1$ and $\mathfrak{B}_1 = \mathfrak{G}$.

But for finite groups much more is true. One need not know that *all* subgroups differ from their normalisers; it it sufficient to know this for the maximal subgroups. We have, more precisely, the following theorems:

If in a finite group $\mathfrak{G}$ all maximal subgroups are self-conjugate, then $\mathfrak{G}$ is nilpotent.

If in a finite group $\mathfrak{G}$ the principal subgroup $\mathfrak{D}$ has an Abelian factor-group, then $\mathfrak{G}$ is nilpotent.

If in a finite group $\mathfrak{G}$ the derived group consists only of non-generators, then $\mathfrak{G}$ is nilpotent.*

These three theorems are equivalent, because the three conditions:

(i) all maximal subgroups are self-conjugate,

(ii) the principal subgroup has an Abelian factor-group,

(iii) the derived group consists of non-generators only,

are easily seen to be equivalent†.

A short proof of the first theorem, due to Mr. P. Hall, is the following. Let $\mathfrak{S}$ be a Sylow subgroup of $\mathfrak{G}$ and let $\mathfrak{N}$ be its normaliser. Then $\mathfrak{N}$ and

* *Cf.* H. Wielandt, *Math. Zeitschrift*, 41 (1936), 281–282. For a definition of the "principal subgroup" and of a "non-generator" *cf.* I.S.G. (II), 343.

† (i) implies (ii). If a maximal subgroup is self-conjugate, then its factor-group must be a group without proper subgroups, *i.e.* cyclical of prime order. So every maximal subgroup contains the derived group and so does their meet $\mathfrak{D}$. Therefore $\mathfrak{D}$ has an Abelian factor-group.

(ii) implies (i). If $\mathfrak{D}$ has an Abelian factor-group, then every group containing $\mathfrak{D}$ and, consequently, every maximal subgroup is self-conjugate.

(ii) and (iii) imply each other, because $\mathfrak{D}$ coincides with the join $\mathfrak{F}$ of all non-generators. *Cf.* B. H. Neumann, *Journal London Math Soc.*, 12 (1937), 120–127.

every subgroup containing $\mathfrak{N}$ is its own normaliser. If $\mathfrak{N}$ were different from $\mathfrak{G}$, it would be contained in a maximal subgroup $\mathfrak{M}$. $\mathfrak{M}$ has to be its own normaliser, but is, by assumption, self-conjugate in $\mathfrak{G}$. This is a contradiction and, therefore, $\mathfrak{N} = \mathfrak{G}$ and $\mathfrak{S}$ is self-conjugate, *i.e.* $\mathfrak{G}$ is nilpotent.

We can now prove that the corresponding theorems are characteristic for nilpotent S-groups, viz. I.S.G. (II), 2.343, 2.351 and 2.352.

3.33. THEOREM. *If in an S-group $\mathfrak{G}$ all maximal subgroups are self-conjugate, then $\mathfrak{G}$ is nilpotent*.*

If in an S-group $\mathfrak{G}$ the principal subgroup $\mathfrak{D}$ has an Abelian factor-group, then $\mathfrak{G}$ is nilpotent.

If in an S-group $\mathfrak{G}$ the derived group consists of non-generators only, then $\mathfrak{G}$ is nilpotent.

Proof. Again the three theorems are equivalent and it is sufficient to prove one of them. We choose the first.

The property that every maximal subgroup of $\mathfrak{G}$ is self-conjugate in $\mathfrak{G}$ must hold in any factor-group of $\mathfrak{G}$ as well, in particular in the finite factor-groups of $\mathfrak{G}$. This means, however, that all finite factor-groups of $\mathfrak{G}$ are nilpotent and, by Theorem (3.24), that $\mathfrak{G}$ itself is nilpotent.

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* This theorem answers in the affirmative a question raised by Miss O. Taussky.

CONGRUENCES IN MULTIPLICATIVE SYSTEMS

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The set of all equivalence relations over a set Σ is a *structure* (or *lattice*) in which generalizations of various algebraic theorems such as those of Jordan-Hölder and Schreier-Zassenhaus may be established*. These theorems depend on the separation of the elements of Σ into disjoint classes of equivalent or congruential elements. In ordinary algebra such separation is effected by use of the multiplicative properties of Σ. It is the purpose of this note to determine conditions under which theorems about equivalence structures over Σ may be transferred to multiplicative properties of Σ.

Notation. Σ denotes a set closed to a binary multiplication; $a, b, c, \ldots$ denote elements of subsets $A, B, C, \ldots$ of Σ; $u, v, w, \ldots$ denote elements of Σ.

Equivalence relations will be denoted by R with appropriate suffix, and their *cross-cut* and *union* by (R_i, R_j) and $[R_i, R_j]$ respectively.

$A < B$ denotes that A is a subset of B. No assumption is made that $A, B, C, \ldots$ are closed with respect to multiplication, which will be denoted by ab and which includes such well-known products as ab, $a+b$, $b^{-1}ab$, $aba^{-1}b^{-1}$, $ab-ba$, (a, b), $[a, b]$, etc., of ordinary algebra.

1. *Equivalence determining subsets.*

Certain subsets $A, B, C, \ldots$ of Σ serve to determine equivalence relations as follows. Let $u \sim v \pmod{R_A}$ mean that a exists in A such that $u = va$.

* P. Dubreil et M-L. Dubreil-Jacotin, *Journal de Math.* (9), 18. (1939), 63–95.

This will be an equivalence relation, R_A, if (i) $u \sim u$; (ii) if $u \sim v$ then $v \sim u$; (iii) if $u \sim v$ and $v \sim w$ then $u \sim w$.

Hence R_A is an equivalence relation if, given any element w, whatever, in Σ and a_1, a_2 in A, then a exists such that

$$w = wa, \tag{1}$$

$$w = (wa_1)\, a, \tag{2}$$

$$(wa_1)\, a_2 = wa. \tag{3}$$

Definition 1. A set A satisfying these three conditions will be termed an *equivalence determining subset* or an *E-set.* Not every multiplicative set Σ or subset of Σ is an E-set. For example, of the 16 multiplicative sets of order 2 only three are E-sets, and of these two are similar.

THEOREM 1. *Every equivalence relation R determines several E-sets Σ such that $R = R_\Sigma$.*

Construct the multiplication table of Σ thus: whenever $a_i \equiv a_j \pmod{R}$ let a_i, a_j occur in the same *rows* of the table. Then if $a_i \equiv a_j \pmod{R}$, $a_i \equiv a_j \pmod{R_\Sigma^1}$ and *vice versa, i.e.* $R = R_\Sigma$.

Notation. A, B, C, ... will henceforth refer to E-subsets of Σ, which itself will be assumed to be closed to multiplication and to be an E-set.

Examples may be constructed to show that conditions (i), (ii), (iii) which are independent in the general case remain independent in the more restricted forms (1), (2) and (3).

Evidently Σ is an E-set if every element occurs in every row of its multiplication table as in the theory of groups or semi-groups.

Evidently, too, the *direct-product* of E-sets is an E-set.

The *extent* of E-sets among multiplicative sets in general is troublesome to calculate. Thus of the possible 531,441 multiplicative sets of order 4, with a fixed element as a right unit, only 15,412 are E-sets. Such sets are, however, far more numerous than the set of semi-groups, groups or associative subsets. The greater part are non-associative.

THEOREM 2. *If $A < B$, then $R_A \subseteq R_B$.*

For $u \equiv v \pmod{R_A}$ implies $u \equiv v \pmod{R_B^\gamma}$.

We cannot, however, assert that $R_A \subset R_B$, for we may have $A < B$, but $R_A = R_B$.

2. *Cross-cut.*

In general the E-subsets of Σ are not closed as regards either cross-cut or union. The following examples show that, in general, (R_A, R_B) is not equal to any R_D, D being an E-subset of Σ; that even when (B, C) is an E-subset, $R_{(B,C)} \neq (R_B, R_C)$, and that (B, C) is not necessarily an E-set.

(i)

	a	b	c	d
a	a	b	b	a
b	b	a	a	b
c	c	d	c	d
d	d	c	c	d

$A = a, b, c$; $B = a, c, d$; $C = a, b$.

$(R_A, R_B) = (R_B, R_C) = R_A = R_B = R_C : \begin{matrix} a & c \\ b & d \end{matrix}$

but $(B, C) = a$ is an E-set and

$$R_{(B,C)} : a, b, c, d \neq (R_B, R_C).$$

On the other hand $(A, B) = a, c$ is not an E-set.

(ii)

	a	b	c	d	e	f
a	a	a	a	a	c	b
b	b	b	c	c	a	a
c	c	c	c	b	b	a
d	d	d	d	d	e	e
e	e	e	e	f	d	d
f	f	f	f	e	f	f

$A = a, b, c, d$; $B = a, b, c, e, f$.

$R_A : a \begin{matrix} b & d & e \\ c & & f \end{matrix}$; $R_B = \begin{matrix} a & d & f \\ b & e & \\ c & & \end{matrix}$.

$(R_A, R_B) : a \begin{matrix} b \\ c \end{matrix} d\ e\ f$ and no E-subset D exists for which this equivalence relation is R_D.

THEOREM 3. *If (B, C) is an E-set, then $R_{(B,C)} \subseteq (R_B, R_C)$.*

For $(B, C) \leqslant B$ and $(B, C) \leqslant C$, hence $R_{(B,C)} \subseteq R_B$ and also $\subseteq R_C$, *i.e.* $\subseteq (R_B, R_C)$.

Definition 2. Let B^A denote the set of all elements b_i of B for which, whatever be w in Σ, $a_{w,b}$ exists in A such that

$$wb_i = wa_{w,b}. \tag{4}$$

As the suffix denotes, $a_{w,b}$ depends on both w and b_i. Evidently $A^\Sigma = A$ and if $B \leqslant C$ then $B^C = B$.

Notation. Σ^A will be written A_M.

A_M consists of *all* x such that $a_{w,x}$ exists satisfying $wx = wa_{w\,x}$. It is therefore *maximal* and $B^A \leqslant A_M$.

Evidently

$$B^A = (B, A_M). \tag{5}$$

THEOREM 4. *If $B < C$, then $B^A \leqslant C^A$ and $A^B \leqslant A^C$.*

Since $B < C$, $(B, A_M) \leqslant (C, A_M)$, *i.e.* $B^A \leqslant C^A$. The second part follows from definition 2.

THEOREM 5. $(B, C)^A = (B^A, C^A)$.

For $(B, C)^A = ((B, C), A_M) = ((B, A_M), (C, A_M)) = (B^A, C^A)$.

THEOREM 6. $D = (B, C) \leqslant B^C$.

Let w be any element in Σ, d any element in D. Then $wd = wb = wc$, hence d is in B^C and also in C^B.

COROLLARY 1. $(B, C) \leqslant (B^C, C^B)$. *But $B^C \leqslant B$ and $C^B \leqslant C$, hence* $(B^C, C^B) \leqslant (B, C)$. *Hence*

COROLLARY 2. $(B, C) = (B^C, C^B)$.

COROLLARY 3. $(B, C) \leqslant (B_M, C_M)$.

Generalization of the Cancellation Law.

The above theorems are true without restriction, but in order that B^A shall be an E-set conditions are necessary.

Postulate $P:1$. If, for any element u, $ua = ub$, then d exists in $D = (A, B)$ such that $ua = ub = ud$. Here, of course, A and B are assumed to be E-sets and D non-void.

This postulate will be assumed henceforth where required.

A stronger form is

$P:2$. If $ua = ub$, then a and b are in D.

$P:3$. If $ua = ub$, then $a = b$. (The ordinary left-hand cancellation law.)

THEOREM 7. *B^A is an E-set.*

For, if w is any element in Σ, a_w, b_w exist such that $w = wa_w = wb_w$. Hence, by $P:1$, d_w exists in (A, B) and therefore, by Theorem 6, in B^A such that $w = wd_w$, *i.e.* (1) is satisfied.

Next, let b_1, b_2 be any elements in B^A and let w be any element in Σ, then, Definition 2, a_1, a_2 exist in A such that $(wb_1)b_2 = (wa_1)a_2 = (wb_1)a_2$

and, since A and B are E-sets, a_3 and b_3 exist such that

$$(wb_1)\,b_2 = wb_3 = (wa_1)\,a_2 = wa_3.$$

Hence, by $P:1$, d_3 exists in (A, B) and therefore in B^A, such that $(wb_1)\,b_2 = wd_3$, *i.e.* (3) is satisfied.

Finally, let b be any element in B^A; then a exists such that $wb = wa$ and b', a' exist such that $w = (wb)\,b' = (wa)\,a' = (wb)\,a'$. Hence, by P:1, d exists in (A, B) and therefore in B^A, such that $w = (wb)\,d$, *i.e.* (2) is satisfied. Therefore B^A is an E-set.

COROLLARY 1. *A_M is an E-set.*

COROLLARY 2. *$(B, C)^A$ is an E-set.*

THEOREM 8. *$D = (B, C)$ is an E-set.*

For any w in Σ, c_w, b_w exist such that $w = wc_w = wb_w$. Hence, by $P:1$, d_w exists in (B, C) such that $w = wd_w$, *i.e.* (1) is satisfied.

Next, let d be any element in D then $wd = wb = wc$ and b', c' exist such that $w = (wd)\,b' = (wd)\,c'$. Therefore, by P:1, d' exists in D such that $w = (wd)\,d'$, *i.e.* (2) is satisfied.

Finally, let d_1, d_2 be any elements in D, and let w be any element in Σ; then $(wd_1)\,d_2 = (wc_1)\,c_2 = wc_3$ and also $= (wb_1)\,b_2 = wb_3$. Hence, by $P:1$, d_3 exists in D such that $(wd_1)\,d_2 = wd_3$, *i.e.* (3) is satisfied and $D = (B, C)$ is an E-set.

It will be noted that D has been supposed to be non-void.

COROLLARY 1. *(A^B, A^C) is an E-set.*

COROLLARY 2. *(A_M, B_M) is an E-set.*

THEOREM 9. *If $A < B$, then $R_A = R_{B^A} \subseteq R_B$.*

Since $A < B$, $A \leqslant B^A$, *i.e.* $R_A \subseteq R_{B^A}$. But, if $u \equiv v \pmod{R_{B^A}}$, then $u = vb_i$, where b_i is in B^A, so that a_i exists such that $u = vb_i = va_i$, *i.e.* $u \equiv v \pmod{R_A}$. Hence $R_{B^A} \subseteq R_A$, and therefore $R_A = R_{B^A}$.

COROLLARY. $R_A = R_{A_M}$.

THEOREM 10. *If $R_C \subseteq R_A$, then $C = C^A \leqslant A_M$.*

Let w be any element in Σ, and let c be any element in C; then $u \equiv w \pmod{R_c}$ implies $u \equiv w \pmod{R_A}$, *i.e.* if $u = wc$, then a exists such that $u = wc = wa$. Hence c is in C^A and therefore also in A_M, *i.e.* $C = C^A \leqslant A_M$.

COROLLARY 1. *Of all E-sets for which* $R_C = R_A$, A_M *is* maximal, *i.e.* $C \leqslant A_M$.

COROLLARY 2. *If* $R_B = R_C$, *then* $B = B^C$ *and* $C = C^B$.

THEOREM 11. $R_{(B,C)} = (R_B, R_C)$.

For $(B, C) \leqslant B$, therefore $R_{(B,C)} \subseteq R_B$. Similarly $R_{(B,C)} \subseteq R_C$, *i.e.* $R_{(B,C)} \subseteq (R_B, R_C)$.

Next, let $u \equiv v \pmod{(R_B, R_C)}$; then $u \equiv v \pmod{R_B}$ and also $u \equiv v \pmod{R_C}$. Hence b and c exist such that $u = vb = vc$ and therefore, $P:1$, d exists in $D = (B, C)$, such that $u = vd$, *i.e.* $u = v \pmod{R_{(B,C)}}$. Hence $(R_B, R_C) \subseteq R_{(B,C)}$, *i.e.*

$$R_{(B,C)} = (R_B, R_C).$$

Without some restriction such as P:1, all that can be asserted, even when (B, C) and $[B, C]$ are E-sets, is that

$$R_{(B,C)} \subseteq (R_B, R_C) \subseteq [R_B, R_C] \subseteq R_{[B,C]}. \tag{6}$$

THEOREM 12. $C^{(A,B)} = (C^A, C^B)$.

Since $(A, B) \leqslant A$ and also $(A, B) \leqslant B$, we have $C^{(A,B)} \leqslant (C^A, C^B)$. Next, let z be any element in (C^A, C^B), then a and b exist such that $wz = wa = wb$, and therefore, by $P:1$, d exists in (A, B) such that $wz = wd$. Hence z is in $C^{(A,B)}$, *i.e.* $(C^A, C^B) \leqslant C^{(A,B)}$, *i.e.*

$$C^{(A,B)} = (C^A, C^B).$$

COROLLARY. $(A, B)_M = \Sigma^{(A,B)} = (\Sigma^A, \Sigma^B) = (A_M, B_M)$. (7)

THEOREM 13. *If* $P^Q = P$ *and* $Q^A = Q$, *then* $P^A = P$.

Since $P^Q = P$ for every element p in P and any w in Σ, q exists in Q such that $wp = wq$ and, since $Q^A = Q$, a exists in A such that $wq = wa$, *i.e.* $wp = wa$. Therefore $P \leqslant P^A$, *i.e.* $P = P^A$.

3. *Minimal sets.*

If $A^P = A^Q = A$, then, by Theorem 12, $(A^P, A^Q) = A^{(P,Q)} = A$. Hence (P, Q) is also a member of the same set and there exists therefore a *minimal* set A_m for which $A^{A_m} = A$ and moreover A_m is unique.

Evidently if $A^P = A$, then $A_m \leqslant P$, otherwise $(A_m, P) < A_m$ contrary to hypothesis.

THEOREM 14. $A_m \leqslant A$.

For $A^A = A$, hence $A_m \leqslant A$.

THEOREM 15. *If* $B < C$, *then* $B_m \leqslant C_m$.

Since $B < C$ for any b in B and any w in Σ, c_m exists in C_m such that $wb = wc_m$. Therefore $B^{C_m} = B$, *i.e.* $B_m \leqslant C_m$.

COROLLARY. $B_m \leqslant \Sigma_m$.

THEOREM 16. $A_m = (A, \Sigma_m)$.

Let the elements of Σ_m be $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_s, \sigma_{s+1}, \ldots, \sigma_t$, where $a_1, a_2, \ldots, a_s$ are in A and $a_1, a_2, \ldots, a_m$ in A_m. Also all the elements of A_m are in Σ_m and therefore are the elements $a_1, a_2, \ldots, a_m$ themselves. Hence $a_{m+1}, \ldots, a_s$ may be removed from Σ_m, for if $wx = wa_i$, $m < i \leqslant s$, then a_j exists, $1 \leqslant j \leqslant m$ such that $wx = wa_j$. Therefore, unless $m = s$, Σ_m is not minimal, contrary to hypothesis. Hence $A_m = (A, \Sigma_m)$.

THEOREM 17. $(A, B)_m = (A_m, B_m)$.

For
$$(A_m, B_m) = ((A, \Sigma_m), (B, \Sigma_m))$$
$$= ((A, B), \Sigma_m) = (A, B)_m.$$

THEOREM 18. $B^A = B^{(A,B)} = B^{(A,B)_m} = B^{A_m}$.

Let b_i be any element in B^A and let $wb_i = wa$; then, by $P:1$, d exists in (A, B) such that $wb_i = wd$. Therefore $B^A \leqslant B^{(A,B)}$. On the other hand, since $(A, B) \leqslant A$, $B^{(A,B)} \leqslant B^A$, *i.e.* $B^A = B^{(A,B)}$.

Again, $wb_i = wa = wa_m$ where a_m is in A_m, *i.e.* $B^A \leqslant B^{A_m}$, but, since $A_m \leqslant A$, $B^{A_m} \leqslant B^A$, *i.e.* $B^A = B^{A_m}$.

COROLLARY 1. *If* $B^A = B$, *then* $B_m \leqslant A_m$.

COROLLARY 2. *If* $B^A = B$ *and* $A^B = A$, *then* $A_m = B_m$.

COROLLARY 3. *If* $B^A = B$, $C^A = C$, *then*
$$(B, C)^A = (B^A, C^A) = (B, C).$$

Hence $(B, C)_m \leqslant (A, B, C)_m$ and $(B, C)_m \leqslant A_m$.

COROLLARY 4. $(A^B, B^A) = (A, B)$.

See also Corollary 2, Theorem 6.

THEOREM 19. *If* $R_A = R_B$, *then* $A_m = B_m$ *and* $A_M = B_M$.

For then $A^B = A$ and $B^A = B$, hence $A_m = B_m$. Next, $R_A = R_{A_M}$, $R_B = R_{B_M}$, hence $A_M \leqslant B_M \leqslant A_m$, *i.e.* $A_M = B_M$.

COROLLARY. $A_m \leqslant B \leqslant A_M$, *consequently all E-sets B for which* $R_B = R_A$ *lie in the* quotient A_M/A_m *of O. Ore*.*

THEOREM 20. *If* $A_m = B_m$, *then* $A^B = A$, $B^A = B$, $R_A = R_B$.

For given any a, a_m exists such that $wa = wa_m$, *i.e.* $wa = wb_m$, since $A_m = B_m$. Hence $A^B = A$. Similarly $B^A = B$ and $R_A = R_B$.

COROLLARY. *If* $A_m = B_m$, *then* $A_M = B_M$.

THEOREM 21. $(A_m)_m = A_m$; $(A_M)_M = A_M$; $(A_m)_M = A_M$; $(A_M)_m = A_m$.

Proof obvious.

COROLLARY. *If* $A_M = B_M$, *then* $A_m = B_m$.

These theorems show that the correspondence $R_A \to A_M/A_m$ is (1.1) and is a *cross-cut isomorphism*. This isomorphism will be discussed in §4 after properties relating to union have been established. When the ordinary left-hand cancellation laws hold, $A_M = A_m = A$ and the quotients reduce to single sets. As far as cross-cut properties are concerned, no assumption is needed about the closure of A, B, C, ... with respect to multiplication.

4. *Union.*

Further conditions must be imposed if the E-subsets of Σ are to be closed with respect to union. One of these is that the E-subsets $A, B, C, \ldots$ must be closed with respect to multiplication, for since $[A, B]$ denotes the set of all possible products of the elements of A and B taken in any order and with any arrangement, and, since we require that $[A, A] = A$, it follows that A must be closed to multiplication. From now on this will be assumed where necessary.

The other conditions which suffice for our purpose will be taken to be generalizations of the associative laws:

$P:5$. $w(ab) = (\ldots(((wa)b_2)a_3)b_4\ldots)a_{2s+1}$, where the product on the right is found by successive multiplications on the right of (wa).

* *Annals of Math.*, 36 (1935), 406–437.

$P:6$. $(wa)b = w[\ldots((ab_2)a_3)b_4\ldots)a_{2s+1}]$, where the product in [ ] is is obtained from (ab_2) by successive right-hand multiplication.

Henceforth $P:1$, $P:5$, and $P:6$ will be assumed as required.

THEOREM 23. *Every element of* $[A, B]$ *may be expressed in one of the forms* $(\ldots((a_1b_2)a_3)b_4\ldots)a_{2s+1}\ldots$, *or* $(\ldots((b_1a_2)b_3)a_4\ldots)b_{2r+1}\ldots$, *in which the products are right-hand successive multiples of* a_1b_2 *and* b_1a_2 *respectively.*

a, b are as prescribed.

$a_1a_2 = a_3$, a_1b_2, $b_1b_2 = b_3$, b_1a_2 are as prescribed.

In the following the sequences will be assumed to be $(ab)a')b'\ldots$, since those in which the b's precede the a's are obtained by interchanging a with b.

The products of length 3, viz.:

$$(a_1a_2)b_4 = a_3b_4; \quad (a_1b_2)a_3; \quad a_1(b_2a_3) = ((a_1b_2)a_4)b_5\ldots;$$

$$a_1(a_2b_3) = (a_1a_2)b_4)a_5)b_6\ldots = (a_3b_4)a_5)b_6\ldots;$$

$$(a_1b_2)b_3 = a_1b_4$$

are all as prescribed.

The products of length 4, viz.:

$$(a_1b_2)(a_3b_4) = (a_1b_2)a_3)b_5)a_6)b_7\ldots;$$

$$a_1[(b_2a_3)b_4] = \{a_1(b_2a_3)\}b_3)a_4)b_5\ldots$$

$$= \{(a_1b_2)a_5)b_6)a_7\ldots\}b_3)a_4)b_5\ldots$$

$$= (a_1b_2)a_5)b_6)a_7\ldots;$$

$$a_1[b_2(a_3b_4)] = a_1[(b_2a_3)b_6)a_7)b_8\ldots)a_{2s+1}]$$

$$= a_1[m(a_{2s+1})]$$

$$= (a_1m)a_r)b_{r+1})a_{r+2}\ldots;$$

but $$a_1m = a_1[\ldots(b_2a_3)b_6)a_7)\ldots b_{2s}]$$

$$= [a_1[(b_2a_3)b_6)a_7\ldots a_{2s-1}]]b_t)a_{t+1}\ldots$$

and so on $$= (a_1(b_2a_3))b_4')a_5'\ldots$$

$$= (a_1b_2)a_3')b_4')a_5'\ldots;$$

$$\{a_1(b_2a_3)\}b_4 = [(a_1b_2)a_5)b_6)a_7\ldots]b_4$$

are all of the prescribed form.

In products in which consecutive a's or b's appear they may be absorbed by means of (2), leaving one a followed by one b.

Assume, then, that m_1 and m_2 are of the prescribed form; then

$$\begin{aligned} m_1 m_2 &= m_1[(\dots(a_1 b_2) a_3) b_4) a_5 \dots) b_{2s}) a_{2s+1}] \\ &= ([m_1(\dots(a_1 b_2) a_3) b_4) a_5 \dots) b_{2s}] a_r) b_{r+1}) a_{r+2} \dots \\ &= (\dots [m_1(\dots(a_1 b_2) a_3) b_4) a_5 \dots] b_t) a_{t+1} \dots] a_r) b_{r+1}) \dots \end{aligned}$$

and so on until by absorbing m_1 and decreasing the length of the product multiplying it we get

$$\begin{aligned} &= (\dots [m_1\{(a_1 b_2) a_p) b_{p+1}) a_{p+2} \dots] b_t) a_{t+1} \dots \\ &= (\dots (([(m_1 a_1) b_2] a_{z+1} \dots) a_x) b_{x+1} \dots, \end{aligned}$$

which is of the prescribed form.

Hence, by induction, the theorem is true generally.

Note that $P:6$ has not been used.

THEOREM 24. *If $[A, B]$ is an E-set and A and B are closed to multiplication, then $R_{[A,B]} = [R_A, R_B]$.*

For $A \leqslant [A, B]$ and $B \leqslant [A, B]$. Hence $[R_A, R_B] \subseteq R_{[A,B]}$. Next, let $u \equiv w \pmod{R_{[A,B]}}$, *i.e.* $u = wm$, where m is in $[A, B]$ and is therefore of the form $m = (a_1 b_2) a_3) b_4 \dots$; *i.e.*

$$u = w[(\dots (a_1 b_2) a_3) b_4) \dots].$$

A repetition of the arguments used in the proof of Theorem 23 gives

$$u = (wa_1) b_2') a_3') \dots;$$

hence $w_1, w_2, \dots, w_{2s-1}$ exist such that

$$u = w_1 \pmod{R_B}; \quad w_1 \equiv w_2 \pmod{R_A}; \quad \dots; \quad w_{2s-1} \equiv w \pmod{R_A},$$

i.e. $u \equiv w \pmod{[R_A, R_B]}$.

Hence $$R_{[A,B]} \subseteq [R_A, R_B],$$

i.e. $$R_{[A,B]} = [R_A, R_B].$$

COROLLARY 1. $[R_A, R_B] = [R_{A_M}, R_{B_M}] = R_{[A_M, B_M]}$.

Hence, Theorem 10, $[A_M, B_M]_M = [A, B]_M$.

COROLLARY 2. $[A_m, B_m]_M = [A, B]_M$.

COROLLARY 3. $[A, B]_m = [A_m, B_m]_m$.

THEOREM 25. $[A_m, B_m] = [A, B]_m$.

For $A_m \leqslant \Sigma_m$; $B_m \leqslant \Sigma_m$; hence $[A_m, B_m] \leqslant \Sigma_m$. Hence

$$[A_m, B_m]_m = ([A_m, B_m], \Sigma_m) = [A_m, B_m],$$

i.e.
$$[A_m, B_m]_m = [A, B]_m = [A_m, B_m].$$

THEOREM 26. $[A, B]$ *is an E-set.*

First, $wa_w = w$ and a_w is in $[A, B]$, so that (1) is satisfied.
Next, let m be any element in $[A, B]$; then, by Theorem 23,

$$wm = w[((a_1 b_2) a_3) b_4 \ldots]$$

and, by a repetition of the argument used in the proof of Theorem 23,

$$wm = (((wa_1) b_2') a_3') b_4' \ldots) a'_{2s+1}.$$

Hence
$$(wm)\bar{a}_{2s+1} = ((wa_1) b_2') a_3' \ldots) b'_{2s},$$

i.e.
$$((wm)\bar{a}_{2s+1})\bar{b}_{2s} = ((wa_1) b_2') a_3' \ldots) a'_{2s-1},$$

and so

$$((wm)\bar{a}_{2s+1})\bar{b}_{2s})\bar{a}_{2s+1}) \ldots)\bar{a}_3)\bar{b}_2 = w.$$

The length of the product which is headed by (wm) may be reduced by means of $P:6$. Thus let

$$\begin{aligned} w &= ((((wm) a_1) b_2) a_3) b_4 \\ &= (((wm) a_1) b_2)[(a_3') b_4') a_5') b_6' \ldots] \\ &= ((wm) a_1) b_2) m_2 \\ &= ((wm) a_1)[(b_2' m_2') b_3') m_4' \ldots] \\ &= ((wm) a_1) m_3, \quad \text{where } m_3 \text{ is in } [A, B] \\ &= (wm)[(a_1 m_3'') a_3'') m_4'' \ldots] \\ &= (wm)\bar{m}, \quad \text{where } \bar{m} \text{ is in } [A, B]. \end{aligned}$$

Obviously the method applies generally. Hence (2) is satisfied.

Finally, let m_1, m_2 be any elements in $[A, B]$; then

$$(wm_1)m_2 = w[(m_1m_2')m_3']m_4' \ldots]$$

$$= wm_3,$$

since $[A, B]$ is closed to multiplication. Hence (3) is satisfied, *i.e.* $[A, B]$ is an E-set.

THEOREM 27. *Every element in* $[A_M, B_M]$ *is of one of the forms* a_M, b_M, $a_M\delta$, $b_M\delta$, *where* δ *is in* $[A_m, B_m]$.

By Theorem 23, every element in $[A_M, B_M]$ is of the form

$$(\ldots(((a_Mb_1)a_2)b_3)a_4\ldots)a_{2s},$$

but, by definition, § 3, $a_mb_1 = a_mb_1'$, where b_1' is in B_m, and

$$(a_mb_1')a_2 = (a_mb_1')a_2',$$

where a_2' is in A_m, and so on. By previous arguments this is equal to $a_m\delta$, where δ is in $[A_m, B_m]$.

In order to complete the investigation we postulate somewhat stronger forms of the associative laws, viz.:

$P:7$. $w(ab) = (wa)b'$, where b' is in B.

$P:8$. $(wa)b = w(ab')$, where b' is in B.

THEOREM 28. $[A_M, B_M]_M = [A_M, B_M]$.

Let z be any element in $[A_M, B_M]_M$; then, for any w, m exists in $[A_M, B_M]$ such that $wz = wm$.

Take for the m the various forms of Theorem 27.

(i) $wz = wa_M$; then z is in $(A_M)_M = A_M$ and therefore also in $[A_M, B_M]$.

(ii) $wz = w(a_M\delta)$, where δ is in $[A_m, B_m]$,

$wz = (wa_M)\delta'$, where δ' is in $[A_m, B_m]$.

Since $[A_m, B_m]$ is an E-set, $\bar{\delta}$ exists in it such that $(wz)\bar{\delta} = wa_M$, *i.e.* $wa_M = w(z\delta'')$, $P:8$, where δ'' is in $[A_m, B_m]$.

Hence $z\delta''$ is in $(A_M)_M = A_M$. Let $z\delta'' = a_M'$; then δ''' exists in $[A_m, B_m]$ such that $z = (z\delta'')\delta''' = a_M'\delta'''$, *i.e.* z is in $[A_M, B_M]$.

Hence $[A_M, B_M]_M \leqslant [A_M, B_M]$; but $[A_M, B_M] \leqslant [A_M, B_M]_M$, and so $[A_M, B_M] = [A_M, B_M]_M$.

COROLLARY. $[A, B]_M = [A_M, B_M]_M = [A_M, B_M]$.

The *principal theorem* of this paper may now be proved, viz.:

THEOREM 29. *The correspondence $R_A \to A_M/A_m$ between the equivalence relations, R_A, arising from closed E-subsets of Σ and the quotients, A_M/A_m, of such E-subsets is a structure isomorphism.*

Let $R_A \to A_M/A_m$, *i.e.* the quotient in the sense of O. Ore*, consisting of all E-subsets, C, for which $R_C = R_A$, and let $R_B \to B_M/B_m$. Then (A, B) and $[A, B]$ are closed E-subsets of Σ and

$$R_{[A, B]} \to [A, B]_M/[A, B]_m,$$

$$R_{(A, B)} \to (A, B)_M/(A, B)_m.$$

But*

$$(A_M, B_M)/(A_m, B_m) = (A_M/A_m, B_M/B_m),$$

$$[A_M, B_M]/[A_m, B_m] = [A_M/A_m, B_M/B_m],$$

while $$R_{[A, B]} = [R_A, R_B]; \quad R_{(A, B)} = (R_A, R_B).$$

Therefore $$(R_A, R_B) \to (A_M/A_m, B_M/B_m)$$

and $$[R_A, R_B] \to [A_M/A_m, B_M/B_m].$$

Hence the correspondence is a structure isomorphism.

It is interesting to note that O. Ore's *ad hoc* definition of the cross-cut and union of two quotients is necessary in order to establish the isomorphism, also that $P:7$ and $P:8$ are postulated by Murdoch and Ore in their work on semi-groups†.

Remarks. In the work of P. Dubreil-Jacotin (*loc. cit.*) commutative subsets are replaced by *associated sets*. In these, R_i and R_j are associated if, whenever $u \equiv v \pmod{R_i}$, $v \equiv w \pmod{R_j}$ then s exists such that

$$u \equiv s \pmod{R_j}, \quad s \equiv w \pmod{R_i},$$

* *Annals of Math.*, 36 (1935), 406–437.

† *Annals of Math.*, 37 (1936), 265–292.

and *vice versa*. This is equivalent to

$P:9$. Whatever be w in Σ and a and b in A and B respectively, then a_1, b_1 exist such that

$$(wa)\,b = (wb_1)\,a_1. \tag{8}$$

Evidently, if A and B are commutative, *i.e.* $AB = BA$, then R_A and R_B are associated. The converse is not true.

Definition. A and B will be termed *linked* if $P:9$ holds.

THEOREM 30. *If A and B are linked, then every element of $[A, B]$ is of one of the forms a; b; ab; ba.*

By Theorem 23 we have

$$m = [\{(a_1 b_2)\,a_3\}\,b_4]\,a_5 \ldots,$$

and, if P:9 holds, $(a_1 b_2)\,a_3 = (a_1 a_4)\,b_5 = a_6 b_5$ and $(a_6 b_5)\,b_4 = a_6 b_7$, and so on. The proof may be completed by induction.

THEOREM 31. *If A and B are commutative, and if $A \leqslant C$, then*

$$(C, [A, B]) = [A, (B, C)].$$

If $A \leqslant C$, then without restriction

$$(C, [A, B]) \geqslant [A, (B, C)].$$

Next, $[A, B]$ consists entirely of elements of the forms a, b, ba.

Let w be in $(C, [A, B])$ and take the various possible forms in succession.

(i) $w = c = a$, *i.e.* w is in $[A, (B, C)]$.

(ii) $w = c = b$, *i.e.* w is in (B, C) and therefore in $[A, (B, C)]$.

(iii) $w = c = ba$, hence $\bar{a}$ exists such that $c\bar{a} = b$, *i.e.* since $A \leqslant C$, b is in C and therefore in (B, C). Hence $c = ba$ is in $[A, (B, C)]$.

Hence in all cases w is in $[A, (B, C)]$, *i.e.* $(C, [A, B]) \leqslant [A, (B, C)]$. Therefore $(C, [A, B]) = [A, (B, C)]$.

COROLLARY 1. *The sub E-sets of Σ commutative in pairs and satisfying the prescribed conditions form a Dedekind (modular) structure.*

The normal sub-groups of a group form an example of this result.

COROLLARY 2. *If A and B are commutative, $AB = [A, B]$.*

The condition that A and B are commutative may be replaced by a weaker condition, viz.:

THEOREM 32. *If C is such that the equation $c_1 x = c_2$ is always solvable in C, and if A and B are linked, then, if $A \leqslant C$, $(C, [A, B]) = [A, (B, C)]$.*

For the elements of $[A, B]$ are now of the form a; b; ab; ba. The new form ab gives, following the proof of Theorem 31, $w = c = ab$, and since $A \leqslant C$, $\bar{c}$ exists in C such that $w = c = ab = a\bar{c}$, hence, by P : 1, d exists in (B, C) such that $w = ad$, *i.e.* w is in $[A, (B, C)]$, and the structure is modular.

Another concept used by Dubreil-Jacobin is: R_A and R_B are *semi-consecutive* if for any specified w_i of Σ either $w_i B \leqslant w_i A$ or $w_i A \leqslant w_i B$. If A and B are E-sets, it does not follow that they are semi-consecutive.

On the other hand the following example shows that R_A and R_B may be semi-consecutive and yet (R_A, R_B) may correspond to no E-set.

	a	b	c	d	e	f	g	h
a	a	b	c	d	a	b	a	b
b	a	b	c	d	a	b	a	b
c	a	b	c	d	c	d	c	d
d	a	b	c	d	c	d	c	d
e	e	f	e	f	e	f	g	h
f	e	f	e	f	e	f	g	h
g	g	h	g	h	e	f	g	h
h	g	h	g	h	e	f	g	h

$A = a, b, c, d$;

$B = e, f, g, h$;

R_A: $a\ e\ g$ / $b\ f\ h$ / c / d;

R_B: $a\ c\ e$ / $b\ d\ f$ / g / h;

R_A and R_B are semi-consecutive and

$$(R_A, R_B): \begin{matrix} a & c & e & g \\ b & d & f & h \end{matrix},$$

but this corresponds to no E-set.

THEOREM 34. *A^B is semi-consecutive to B.*

For, by Definition 2, for all w, $wA^B \leqslant wB$, *i.e.* A^B is semi-consecutive to B.

COROLLARY 1. *A_M, A_m are semi-consecutive to A and also to A_M.*

COROLLARY 2. *A^B is semi-consecutive to (A, B).*

Under certain conditions the group of automorphisms of Σ and the groupoid of operators on Σ may be shown to be isomorphic with quotients of equivalence relations, R_A/R_B, and much of the theory of the groups of inertia can be generalized.

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ON THE DISTRIBUTION OF TIDES OVER A CHANNEL

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1. *Introduction.*

The part of the channel considered in this paper lies on a flat portion of the earth's surface, and is bounded by two parallel vertical planes near the coasts. The depth may vary across the channel, but all transverse sections are equal. Over this uniform part the tides are taken to be free from frictional forces, but no restrictions are placed either on those parts of the channel which lie outside the part considered or on the tides over those parts. In either direction the channel may close or it may open into a basin of any shape, and in such a basin frictional forces may act on the water. Dissipation of energy may also take place between the two lateral bounding planes and the corresponding coasts.

The mathematical problem considered is that of the expression of the elements of the tides over the part of the channel considered, in terms of the elevations and of the longitudinal currents over one transverse section lying within this part. Along the two lateral bounding planes the transverse current is assumed to have the same phase as the elevation, and the amplitude of the transverse current is assumed to be proportional to that of the elevation. The constant of proportionality is assumed to be uniform along each bounding plane, but it may be different on the two sides of the channel. In this way it is possible to allow for dissipation of energy near the coasts, as in a recent paper on the effects of coastal friction on the tides (Proudman, 1940).

The solution of the mathematical problem is given by an expansion theorem. When the rotation of the earth and the dissipation are neglected,

the expansions reduce to those of Sturm-Liouville type, and this particular case has been used in an earlier paper on the curvature of cotidal lines across a channel (Proudman, 1928). The leading ideas in the proof of the general theorem of the present paper are taken from a paper by A. C. Dixon (1904) on a class of expansions in oscillating functions, but the expansions of the present paper are not particular cases of those considered by Dixon.

It is proposed to use the results of the paper to investigate the distribution of the tides over the Atlantic Ocean. In the differential equations for the tides of an ocean the coordinate of longitude enters as simply as a Cartesian coordinate for a flat sea, so that the earth's curvature along a parallel of latitude does not invalidate the theorem. The coordinate of latitude does enter in a more complicated way than a Cartesian coordinate, but this difficulty may be avoided by restricting the part of the ocean considered to a narrow zonal band, and then neglecting the variation of latitude over this band. A separate application of the theorem will therefore be made to a part of the ocean lying between two near parallels of latitude, and between two meridians, one meridian lying near each of the opposite coasts of the ocean. It is these meridians which correspond to the lateral bounding planes specified above. The variations of depth along a parallel of latitude may be taken into account. The theorem will allow the tidal elements along one of the parallels of latitude to be expressed in terms of those along the other. Separate applications can be made to contiguous zonal bands, and for different zonal bands the bounding meridians, the distributions of depth, and the coefficients of coastal dissipation, will be different. For this application an important result is that the expansion for the tidal elevation is valid on the lateral bounding planes.

2. *Fundamental equations.*

Denote by:

g the acceleration of gravity;

ω the vertical component of the earth's angular velocity;

b the breadth of the channel between the two bounding planes;

bx, by horizontal Cartesian coordinates so that the lateral bounding planes are given by $x = 0$ and $x = 1$;

h sometimes written as $h(x)$, the mean depth of the water at any place;

$\bar{h}$ a convenient value of h, as, for example, the mean value;

h_0, h_1 the effective depths of the coastal shelves, in terms of which the dissipation of energy near the coasts may be expressed;

t the time;

ζ the elevation of the free surface of the water at any place and time;

$\bar{\zeta}$ the equilibrium-value of ζ corresponding to the external tide-generating forces;

u, v the components of tidal current in the directions respectively of increasing x and y;

σ the speed, *i.e.* 2π/period, of the harmonic constituent of the tides considered.

It is convenient to write

$$f = \frac{\sigma}{2\omega}, \quad \beta = (f^2-1)\,\frac{4\omega^2 b^2}{g\bar{h}}, \quad c = -(f^2-1)\,\frac{2i\omega b}{fg},$$

$$k_0 = (f^2-1)\,\frac{2\omega b}{h(0)}\left(\frac{h_0}{g}\right)^{\frac{1}{2}}, \quad k_1 = (f^2-1)\,\frac{2\omega b}{h(1)}\left(\frac{h_1}{g}\right)^{\frac{1}{2}}.$$

It will be supposed that the depth h does not vanish, even for $x = 0$ and $x = 1$. The constant f may be taken as positive, but the special case in which $f = 1$ will be excluded from consideration. Accents will sometimes be used to denote differentiation with respect to x. Complex harmonic motion with time-factor $e^{i\sigma t}$ will be used throughout.

The equation of continuity may be written as

$$\frac{\partial}{\partial x}(hu) + h\frac{\partial v}{\partial y} + i\sigma b\zeta = 0 \qquad (2.1)$$

and the equations of motion as

$$\left.\begin{aligned} ifu-v &= -\frac{g}{2\omega b}\frac{\partial}{\partial x}(\zeta-\bar{\zeta}), \\ ifv+u &= -\frac{g}{2\omega b}\frac{\partial}{\partial y}(\zeta-\bar{\zeta}). \end{aligned}\right\} \tag{2.2}$$

The boundary conditions may be written as

$$\left.\begin{aligned} -hu &= (gh_0)^{\frac{1}{2}}\zeta \quad (x=0), \\ hu &= (gh_1)^{\frac{1}{2}}\zeta \quad (x=1), \end{aligned}\right\} \tag{2.3}$$

these equations representing losses of energy equal to those carried by simple progressive waves in canals of depths h_0 and h_1 respectively.

On solving the equations (2.2) algebraically, we have

$$\left.\begin{aligned} cu &= \left(\frac{\partial}{\partial x}-\frac{i}{f}\frac{\partial}{\partial y}\right)(\zeta-\bar{\zeta}), \\ cv &= \left(\frac{\partial}{\partial y}+\frac{i}{f}\frac{\partial}{\partial x}\right)(\zeta-\bar{\zeta}), \end{aligned}\right\} \tag{2.4}$$

and then, on substituting into (2.1),

$$\left\{\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\right)+h\frac{\partial^2}{\partial y^2}-\frac{i}{f}h'\frac{\partial}{\partial y}\right\}(\zeta-\bar{\zeta})+\beta\bar{h}\zeta=0. \tag{2.5}$$

The boundary conditions (2.3) may, in view of (2.4), be written as

$$\left.\begin{aligned} \left(\frac{\partial}{\partial x}-\frac{i}{f}\frac{\partial}{\partial y}\right)(\zeta-\bar{\zeta}) &= \frac{ik_0}{f}\zeta \quad (x=0), \\ \left(\frac{\partial}{\partial x}-\frac{i}{f}\frac{\partial}{\partial y}\right)(\zeta-\bar{\zeta}) &= -\frac{ik_1}{f}\zeta \quad (x=1). \end{aligned}\right\} \tag{2.6}$$

It will be assumed that

$$\bar{\zeta}=\bar{\zeta}(x)\,e^{\lambda y+i\sigma t}, \tag{2.7}$$

where $\bar{\zeta}(x)$ is a function of x only, and λ is a complex constant; this form covers the tide-producing potential of the moon and sun.

3. *Elemental solutions.*

In the differential equation (2.5) take

$$\bar{\zeta}=0, \quad \zeta=Z(\lambda, x)\,e^{\lambda y+i\sigma t},$$

where λ is a complex constant, so that

$$\frac{d}{dx}\{hZ'(\lambda, x)\}+\left(\lambda^2 h-\frac{i}{f}\lambda h'+\beta\bar{h}\right)Z(\lambda, x)=0. \qquad (3.1)$$

Complete the determination of $Z(\lambda, x)$ by the conditions

$$Z(\lambda, 0)=1, \quad Z'(\lambda, 0)=\frac{i}{f}(\lambda+k_0),$$

and write $$V(\lambda, x)=\lambda Z(\lambda, x)+\frac{i}{f}Z'(\lambda, x).$$

Also, let $R(\lambda, x)$ denote another solution of (3.1) with the conditions

$$R(\lambda, 1)=1, \quad R'(\lambda, 1)=-\frac{i}{f}(\lambda-k_1),$$

and write $$W(\lambda, x)=\lambda R(\lambda, x)+\frac{i}{f}R'(\lambda, x).$$

Write

$$D(\lambda)=h(x)\{Z(\lambda, x)\,R'(\lambda, x)-Z'(\lambda, x)\,R(\lambda, x)\}, \qquad (3.2)$$

this quantity being independent of x. On taking its values at $x=0$ and $x=1$, we have respectively

$$D(\lambda)=h(0)\left\{R'(\lambda, 0)-\frac{i}{f}(\lambda+k_0)\,R(\lambda, 0)\right\}$$

$$=-h(1)\left\{Z'(\lambda, 1)-\frac{i}{f}(\lambda-k_1)\,Z(\lambda, 1)\right\}. \qquad (3.3)$$

When the depth is uniform, the functions $Z(\lambda, x)$, $R(\lambda, x)$, $V(\lambda, x)$, $W(\lambda, x)$ and $D(\lambda)$ can all be expressed in terms of circular functions, and the zeros of $D(\lambda)$ are all simple. We shall assume that the depth h is such as always to make these functions analytic and the zeros of $D(\lambda)$, say (λ_s), simple. Each of $Z(\lambda_s, x)$ and $R(\lambda_s, x)$ will satisfy both boundary conditions (2.6), and in fact one of these functions is a multiple of the other. On writing, for shortness,

$$Z_s(x)=Z(\lambda_s, x), \quad V_s(x)=V(\lambda_s, x),$$

$$R_s(x)=R(\lambda_s, x), \quad W_s(x)=W(\lambda_s, x),$$

we have

$$\frac{Z_s(x)}{R_s(x)} = \frac{V_s(x)}{W_s(x)} = \frac{1}{R_s(0)} = Z_s(1). \tag{3.4}$$

Let μ denote a second value of λ; then from (3.1) we deduce

$$\int_0^1 \frac{d}{dx}\{hZ'(\lambda, x)\} Z(\mu, x)\, dx + \int_0^1 \left(\lambda^2 h - \frac{i}{f}\lambda h' + \beta\bar{h}\right) Z(\lambda, x) Z(\mu, x)\, dx = 0, \tag{3.5}$$

and on subtracting from (3.5) the corresponding equation with λ and μ reversed, we have, after some reduction,

$$(\lambda - \mu)\int_0^1 h(x)\{Z(\lambda, x) V(\mu, x) + Z(\mu, x) V(\lambda, x)\}\, dx = Z(\mu, 1) D(\lambda) - Z(\lambda, 1) D(\mu). \tag{3.6}$$

By making $\mu \to \lambda$, (3.6) yields

$$2\int_0^1 h(x) Z(\lambda, x) V(\lambda, x)\, dx = \{Z(\lambda, 1)\}^2 \frac{d}{d\lambda}\left\{\frac{D(\lambda)}{Z(\lambda, 1)}\right\}. \tag{3.61}$$

From (3.6) we also deduce that

$$\int_0^1 h(x)\{Z_r(x) V_s(x) + Z_s(x) V_r(x)\}\, dx = 0 \quad (\lambda_r \neq \lambda_s), \tag{3.62}$$

and, from (3.61),

$$2\int_0^1 h(x) Z_s(x) V_s(x)\, dx = Z_s(1) D_s, \tag{3.63}$$

where

$$D_s = \frac{d}{d\lambda} D(\lambda) \quad (\lambda = \lambda_s).$$

From the assumption that all the zeros of $D(\lambda)$ are simple, it follows that D_s can never vanish.

In view of (3.4), the equations (3.62) and (3.63) may be written in the respective forms

$$\int_0^1 h(x)\{Z_r(x) W_s(x) + R_s(x) V_r(x)\}\, dx = 0 \quad (\lambda_r \neq \lambda_s) \tag{3.64}$$

and

$$\int_0^1 h(x)\{Z_s(x) W_s(x) + R_s(x) V_s(x)\}\, dx = D_s. \tag{3.65}$$

If we write

$$U_s(x) = Z_s'(x) - \frac{i}{f}\lambda Z_s(x)$$

a solution of the equations of § 2 for free tides is given by

$$\left.\begin{aligned} \zeta &= Z_s(x)\, e^{\lambda_s y + i\sigma t}, \\ cu &= U_s(x)\, e^{\lambda_s y + i\sigma t}, \\ cv &= V_s(x)\, e^{\lambda_s y + i\sigma t}. \end{aligned}\right\} \tag{3.7}$$

When there is no dissipation, so that $k_0 = k_1 = 0$, the solution (3.7) was studied in an earlier paper on the tides in a channel (Proudman, 1925), special attention being given to a channel of parabolic section. When there is no dissipation and the depth is uniform, the solution was given by Poincaré (1910).

4. *Particular integral.*

If we retain $\bar{\zeta}$ as given by (2.7) and write

$$\zeta - \bar{\zeta} = \zeta(x)\, e^{\lambda y + i\sigma t}, \tag{4.1}$$

the differential equation (2.5) becomes

$$\frac{d}{dx}\{h\zeta'(x)\} + \left(\lambda^2 h - \frac{i}{f}\lambda h' + \beta\bar{h}\right)\zeta(x) = -\beta\bar{h}\,\bar{\zeta}(x), \tag{4.2}$$

while the boundary conditions (2.6) become

$$\left.\begin{aligned} \zeta'(0) - \frac{i}{f}(\lambda + k_0)\,\zeta(0) &= \frac{ik_0}{f}\,\bar{\zeta}(0), \\ \zeta'(1) - \frac{i}{f}(\lambda - k_1)\,\zeta(1) &= -\frac{ik_1}{f}\,\bar{\zeta}(1). \end{aligned}\right\} \tag{4.3}$$

The method of variation of parameters then gives, as the solution of (4.2) and (4.3),

$$\begin{aligned} \zeta(x) = \frac{R(\lambda, x)}{D(\lambda)}\left\{\frac{i}{f}k_0 h(0)\,\bar{\zeta}(0) - \beta\bar{h}\int_0^x \bar{\zeta}(\xi)\, Z(\lambda, \xi)\, d\xi\right\} \\ + \frac{Z(\lambda, x)}{D(\lambda)}\left\{\frac{i}{f}k_1 h(1)\,\bar{\zeta}(1) - \beta\bar{h}\int_x^1 \bar{\zeta}(\xi)\, R(\lambda, \xi)\, d\xi\right\}. \end{aligned} \tag{4.4}$$

On writing

$$\left.\begin{aligned} cu(x) &= \zeta'(x) - \frac{i}{f}\lambda\zeta(x), \\ cv(x) &= \lambda\zeta(x) + \frac{i}{f}\zeta'(x), \end{aligned}\right\} \tag{4.5}$$

a particular integral of the equations of § 2 for forced tides is given by

$$\left.\begin{aligned} \zeta &= \{\bar{\zeta}(x)+\zeta(x)\}\, e^{\lambda y+i\sigma t}, \\ u &= \quad u(x)\ e^{\lambda y+i\sigma t}, \\ v &= \quad v(x)\ e^{\lambda y+i\sigma t}. \end{aligned}\right\} \tag{4.6}$$

5. *General solution.*

Consider the general sum of the elemental solutions (3.7) and the particular integral (4.6), viz.:

$$\left.\begin{aligned} \zeta &= \{\bar{\zeta}(x)+\zeta(x)\}\, e^{\lambda y+i\sigma t} + \sum_s A_s Z_s(x)\, e^{\lambda_s y+i\sigma t}, \\ cu &= \quad cu(x)\ e^{\lambda y+i\sigma t} + \sum_s A_s U_s(x)\, e^{\lambda_s y+i\sigma t}, \\ cv &= \quad cv(x)\ e^{\lambda y+i\sigma t} + \sum_s A_s V_s(x)\, e^{\lambda_s y+i\sigma t}, \end{aligned}\right\} \tag{5.1}$$

the coefficients A_s being constants and $\sum_s$ denoting summation over all the roots of the equation $D(\lambda)=0$. The part of the channel over which (5.1) applies will, of course, be that over which the series in (5.1) converge and satisfy the differential equations, and the greater the extent of this part the greater will be the restrictions on the coefficients A_s.

On the section $y=0$, let

$$\zeta - \{\bar{\zeta}(x)+\zeta(x)\}\, e^{i\sigma t} = Z(x)\, e^{i\sigma t},$$

$$cv - cv(x)\, e^{i\sigma t} = V(x)\, e^{i\sigma t},$$

$Z(x)$ and $V(x)$ being any independent functions such as could occur in the actual tides. From (5.1) we have

$$\sum_r A_r Z_r(x) = Z(x), \quad \sum_r A_r V_r(x) = V(x). \tag{5.2}$$

On multiplying these two equations by $h(x)\,W_s(x)$ and $h(x)\,R_s(x)$ respectively, adding, integrating, and using (3.64) and (3.65), we have

$$D_s A_s = \int_0^1 h(x)\{Z(x)\,W_s(x)+V(x)\,R_s(x)\}\,dx. \tag{5.3}$$

The remainder of the paper will be devoted to the proof that the values of A_s given by (5.3) make valid the two expansions (5.2), the first for all values of x including $x=0$ and $x=1$, and the second for all values of x excluding $x=0$ and $x=1$. This is the expansion theorem.

The expansions (5.2) will be established for arbitrary distributions of $Z(x)$ and $V(x)$, and for these the region of convergence of the series in (5.1) will, in general, reduce to the section $y=0$ itself. But the distributions of ζ and v on $y=0$ are not arbitrary; they are, in fact, restricted by the conditions of convergence of the series in (5.1) over the region to which these formulae apply.

6. *Equivalent expansions.*

Let $F(x)$ denote any function defined in the range $0 \leqslant x \leqslant 1$, which may be differentiated in this range and is such that

$$|F(x)| < P, \quad |F'(x)| < Q \quad (0 \leqslant x \leqslant 1),$$

where P and Q are constants. On writing

$$Z(x) = 0, \quad V(x) = F(x)$$

in (5.2) and (5.3), we have

$$\sum_s \frac{Z_s(x)}{D_s} \int_0^1 h(\xi)\,F(\xi)\,R_s(\xi)\,d\xi = 0 \tag{6.1}$$

and

$$\sum_s \frac{V_s(x)}{D_s} \int_0^1 h(\xi)\,F(\xi)\,R_s(\xi)\,d\xi = F(x), \tag{6.2}$$

while, on writing

$$Z(x) = F(x), \quad V(x) = 0,$$

in (5.2) and (5.3), we have

$$\sum_s \frac{Z_s(x)}{D_s} \int_0^1 h(\xi)\, F(\xi)\, W_s(\xi)\, d\xi = F(x) \tag{6.3}$$

and

$$\sum_s \frac{V_s(x)}{D_s} \int_0^1 h(\xi)\, F(\xi)\, W_s(\xi)\, d\xi = 0. \tag{6.4}$$

It will be sufficient to establish the validity of the expansions (6.1) and (6.3) for all values of x in the range (0, 1), and the expansions (6.2) and (6.4) for all these values except $x = 0$ and $x = 1$.

In view of (3.4), the expansion (6.1) may be written in the form

$$\sum_s \left\{ \frac{R_s(x)}{D_s} \int_0^x h(\xi)\, F(\xi)\, Z_s(\xi)\, d\xi + \frac{Z_s(x)}{D_s} \int_x^1 h(\xi)\, F(\xi)\, R_s(\xi)\, d\xi \right\} = 0 \tag{6.5}$$

which is specially convenient for the argument which follows. The expansions (6.2), (6.3), and (6.4) may be written in similar forms.

7. *Subsidiary functions.*

For shortness, write

$$Z = \frac{1}{D(\lambda)} \int_0^x h(\xi)\, F(\xi)\, Z(\lambda, \xi)\, d\xi, \qquad R = \frac{1}{D(\lambda)} \int_x^1 h(\xi)\, F(\xi)\, R(\lambda, \xi)\, d\xi,$$

$$Z' = \frac{1}{D(\lambda)} \int_0^x h(\xi)\, F(\xi)\, Z'(\lambda, \xi)\, d\xi, \quad R' = \frac{1}{D(\lambda)} \int_x^1 h(\xi)\, F(\xi)\, R'(\lambda, \xi)\, d\xi,$$

$$V = \frac{1}{D(\lambda)} \int_0^x h(\xi)\, F(\xi)\, V(\lambda, \xi)\, d\xi, \quad W = \frac{1}{D(\lambda)} \int_x^1 h(\xi)\, F(\xi)\, W(\lambda, \xi)\, d\xi,$$

and take

$$I_1 = ZR(\lambda, x) + RZ(\lambda, x), \qquad I_2 = ZW(\lambda, x) + RV(\lambda, x),$$

$$I_3 = VR(\lambda, x) + WZ(\lambda, x), \qquad I_4 = VW(\lambda, x) + WV(\lambda, x),$$

$$\lambda J_2 = ZR'(\lambda, x) + RZ'(\lambda, x), \qquad \lambda J_3 = Z'\, R(\lambda, x) + R'\, Z(\lambda, x),$$

$$\lambda^2 J_4 = Z'\, R'(\lambda, x) + R'\, Z'(\lambda, x).$$

Then from the definitions of $V(\lambda, x)$ and $W(\lambda, x)$ in § 3 we have

$$\left.\begin{aligned} I_2 &= \lambda\left(I_1+\frac{i}{f}J_2\right), \quad I_3 = \lambda\left(I_1+\frac{i}{f}J_3\right), \\ I_4 &= \lambda^2\left\{I_1+\frac{i}{f}(J_2+J_3)-\frac{1}{f^2}J_4\right\}. \end{aligned}\right\} \tag{7.1}$$

The process of integration by parts may be applied to Z' and R' directly, and to Z and R after $Z(\lambda, \xi)$ and $R(\lambda, \xi)$ have been replaced by their equivalents according to the differential equation (3.1). In this way we obtain, on writing

$$G(x) = \frac{\lambda^2 h(x) F(x)}{\lambda^2 h(x) - if^{-1}\lambda h'(x) + \beta\bar{h}}$$

and using (3.2),

$$\left.\begin{aligned} \lambda I_1 &= \frac{1}{\lambda}G(x)+K_3-\frac{i}{f}L_3, \quad \lambda J_2 = K_4-\frac{i}{f}L_4, \\ \lambda J_3 &= -K_1+L_1, \qquad\qquad \lambda J_4 = \frac{1}{\lambda}F(x)-K_2+L_2, \end{aligned}\right\} \tag{7.2}$$

where K_1 and K_2 only differ from I_1 and J_2 respectively in having

$$\frac{1}{h(\xi)}\frac{d}{d\xi}\{h(\xi)F(\xi)\}$$

in place of $F(\xi)$, while K_3 and K_4 only differ from J_3 and J_4 respectively in having $G'(\xi)$ in place of $F(\xi)$, and

$$L_1 = \frac{1}{D(\lambda)}\{h(1)F(1)Z(\lambda, x)-h(0)F(0)R(\lambda, x)\},$$

$$L_2 = \frac{1}{\lambda D(\lambda)}\{h(1)F(1)Z'(\lambda, x)-h(0)F(0)R'(\lambda, x)\},$$

$$L_3 = \frac{1}{D(\lambda)}\left\{\left(1-\frac{k_1}{\lambda}\right)h(1)G(1)Z(\lambda, x)-\left(1+\frac{k_0}{\lambda}\right)h(0)G(0)R(\lambda, x)\right\},$$

$$L_4 = \frac{1}{\lambda D(\lambda)}\left\{\left(1-\frac{k_1}{\lambda}\right)h(1)G(1)Z'(\lambda, x)-\left(1+\frac{k_0}{\lambda}\right)h(0)G(0)R'(\lambda, x)\right\}.$$

From (7.1) and (7.2) we deduce

$$\left.\begin{aligned} I_2 &= \frac{1}{\lambda} G(x)+K_3+\frac{i}{f} K_4-\frac{i}{f} L_3+\frac{1}{f^2} L_4, \\ I_3 &= \frac{1}{\lambda} G(x)+K_3-\frac{i}{f} K_1-\frac{i}{f} (L_3-L_1), \\ I_4 &= G(x)-\frac{1}{f^2} F(x)+\frac{i}{f} \lambda(K_4-K_1)+\lambda\left(K_3+\frac{1}{f^2} K_2\right) \\ &\qquad -\frac{i}{f} \lambda(L_3-L_1)+\frac{1}{f^2} \lambda(L_4-L_2). \end{aligned}\right\} \quad (7.3)$$

8. *Contour integrals and asymptotic forms.*

For any particular value of x, $Z(\lambda, x)$ and $R(\lambda, x)$ are integral functions of λ, and so are $V(\lambda, x)$ and $W(\lambda, x)$. Hence the poles of I_1, I_2, I_3 and I_4 in the λ-plane are given by the zeros of $D(\lambda)$, viz. (λ_s).

Consider now the four contour-integrals

$$\frac{1}{2\pi i} \int I_n d\lambda \quad (n = 1, 2, 3, 4) \qquad (8.1)$$

taken round a closed contour in the λ-plane which does not pass through any of the values λ_s. By Cauchy's theorem it follows that the integrals (8.1) are equal respectively to (6.5) and the other three similar expansions, except that $\sum_s$ must be restricted to those values of s for which the points λ_s lie within the contour. It will be shown that, as this contour tends to infinity in all directions, the integrals (8.1) tend respectively to 0, $F(x)$, $F(x)$, 0 for $n = 1, 2, 3, 4$ provided that for $n = 2$ and 4 we exclude the end values, $x = 0$ and $x = 1$.

For large values of $|\lambda|$, we have

$$Z(\lambda, x) = q\left\{\frac{h(0)}{h(x)}\right\}^p e^{i\lambda x}+p\left\{\frac{h(0)}{h(x)}\right\}^q e^{-i\lambda x},$$

$$R(\lambda, x) = q\left\{\frac{h(1)}{h(x)}\right\}^p e^{-i\lambda(1-x)}+p\left\{\frac{h(1)}{h(x)}\right\}^q e^{i\lambda(1-x)},$$

$$D(\lambda) = -2ipqh(1)\lambda\left[\left\{\frac{h(0)}{h(1)}\right\}^p e^{i\lambda}-\left\{\frac{h(0)}{h(1)}\right\}^q e^{-i\lambda}\right],$$

where

$$p = \tfrac{1}{2}\left(1-\frac{1}{f}\right), \quad q = \tfrac{1}{2}\left(1+\frac{1}{f}\right).$$

We deduce that

$$e^{2i\lambda_s} = \left\{\frac{h(0)}{h(1)}\right\}^{1/f}, \quad D_s = 4pq\,h(1)\left\{\frac{h(0)}{h(1)}\right\}^{\frac{1}{2}},$$

so that $D_s \neq 0$.

Also, for large values of $|\lambda|$,

$$\lambda h(x)\{G(x)-F(x)\} = \frac{i}{f}\,h'(x)\,F(x),$$

so that

$$\lambda(L_3-L_1) = \frac{1}{D(\lambda)}\left\{\left[\frac{i}{f}\,h'(1)-k_1 h(1)\right]F(1)\,Z(\lambda, x)\right.$$
$$\left.-\left[\frac{i}{f}\,h'(0)+k_0 h(0)\right]F(0)\,R(\lambda, x)\right\},$$

and $\lambda(L_4-L_2)$ only differs from this in having $Z'(\lambda, x)/\lambda$ and $R'(\lambda, x)/\lambda$ respectively in place of $Z(\lambda, x)$ and $R(\lambda, x)$.

Write $\lambda = m+in$, where m and n are real, and take M and N large positive numbers such that

$$N^2 = M, \quad \sin M = 1.$$

For each integral (8.1) take a contour made up of four parts, the two arcs of the circle passing through $m = \pm M$, $n = \pm N$, which lie between the two straight lines $m = \pm M$, and the parts of these two straight lines which are cut off by the circle.

Now if H denotes the maximum value of $h(\xi)/h(x)$ for all values of x and ξ in the range (0, 1), we deduce

$$|Z(\lambda, x)| < 2q\,H^q e^{|n|x}$$

and

$$|R(\lambda, x)| < 2q\,H^q e^{|n|(1-x)}.$$

When $m = M$, we deduce that

$$D(\lambda) = 2pqh(1)\lambda\left[\left\{\frac{h(0)}{h(1)}\right\}^p e^{-n}+\left\{\frac{h(0)}{h(1)}\right\}^q e^{n}\right],$$

and then that, on all parts of the contour,

$$|D(\lambda)| > |pq|\hbar(1)H^{-q}e^{|n|}|\lambda|.$$

Hence, on all parts of the contour,

$$|I_1| < \frac{4q}{|p|}\frac{PH^{3q+1}}{|\lambda|}\left\{\int_0^x e^{|n|(\xi-x)}d\xi + \int_x^1 e^{|n|(x-\xi)}d\xi\right\}$$

and

$$|L_1| < \frac{2}{|p|}\frac{PH^{2q+1}}{|\lambda|}\{e^{|n|(x-1)} + e^{-|n|x}\}.$$

On the straight parts of the contour

$$\int_0^x e^{|n|(\xi-x)}d\xi + \int_x^1 e^{|n|(x-\xi)}d\xi < 1$$

and

$$e^{|n|(x-1)} + e^{-|n|x} < 2,$$

while

$$\int\left|\frac{d\lambda}{\lambda}\right| < \frac{4N}{M} = \frac{4}{N}.$$

On the circular arcs, where $|n| \geqslant N$,

$$\int_0^x e^{|n|(\xi-x)}d\xi + \int_x^1 e^{|n|(x-\xi)}d\xi < \frac{2}{N},$$

and

$$e^{|n|(x-1)} + e^{-|n|x} < 2e^{-Na},$$

if, in the second of these, $0 < a \leqslant x \leqslant 1-a < 1$, so that the end-points are excluded from the range of x; also, for the circular arcs,

$$\int\left|\frac{d\lambda}{\lambda}\right| < 2\pi.$$

Hence, by taking N sufficiently large, we can make the three integrals

$$\int I_1 d\lambda, \quad \int \frac{1}{\lambda} L_1 d\lambda, \quad \int L_1 d\lambda$$

as near to zero as we please, provided, that in the third, x is not at either end of its range.

In the same way and for the same contour it may be shown that the integrals of

$$K_1, \quad K_2, \quad K_3, \quad K_4, \quad L_3 - L_1, \quad L_4 - L_2$$

tend to vanish, for $0 \leqslant x \leqslant 1$, while the integrals of

$$L_2, \quad L_3, \quad L_4, \quad \lambda(L_3 - L_1), \quad \lambda(L_4 - L_2)$$

tend to vanish, for $0 < a \leqslant x \leqslant 1-a < 1$.

On neglecting these vanishing integrals, we deduce from (7.2) and the definitions immediately following (7.2) that

$$\lambda K_1 = \frac{1}{\lambda} \frac{1}{h(x)} \frac{d}{dx} \{h(x) F(x)\},$$

$$\lambda K_2 = \lambda K_3 = 0,$$

$$\lambda K_4 = \frac{1}{\lambda} F'(x),$$

for $0 < a \leqslant x \leqslant 1-a < 1$. Then, by using (7.3), we have

$$I_1 = 0, \quad I_3 = \frac{1}{\lambda} F(x),$$

for $0 \leqslant x \leqslant 1$, and

$$I_2 = \frac{1}{\lambda} F(x),$$

$$I_4 = \left\{1 + \frac{i}{f} \frac{1}{\lambda} \frac{h'(x)}{h(x)}\right\} F(x) - \frac{1}{f^2} F(x) + \frac{i}{f} \frac{1}{\lambda} \left\{F'(x) - \frac{1}{h(x)} \frac{d}{dx} [h(x) F(x)]\right\}$$

$$= \left(1 - \frac{1}{f^2}\right) F(x),$$

for $0 < a \leqslant x \leqslant 1-a < 1$. It follows that

$$\frac{1}{2\pi i} \int I_1 \, d\lambda = 0, \quad \frac{1}{2\pi i} \int I_3 \, d\lambda = F(x)$$

for $0 \leqslant x \leqslant 1$, and

$$\frac{1}{2\pi i} \int I_2 \, d\lambda = F(x), \quad \frac{1}{2\pi i} \int I_4 \, d\lambda = 0$$

for $0 < a \leqslant x \leqslant 1-a < 1$.

This completes the proof of the validity of the expansions (6.1)–(6.4) for the ranges of x stated in § 6.

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A MEAN VALUE THEOREM INVOLVING CESÀRO MEANS

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1. The following theorem is due essentially to M. Riesz*.

THEOREM. *If* $0 < \lambda \leqslant 1$, $a < b < \beta$ *and if* $f(t)$ *is absolutely integrable in* (a, b), *then*†

$$\left| \int_a^b (\beta - t)^{\lambda-1} f(t)\, dt \right| \leqslant \operatorname*{ess.\overline{bound}}_{a<x<b} \left| \int_a^x (x-t)^{\lambda-1} f(t)\, dt \right|. \tag{1.1}$$

By a method of proof similar to that of (1.1) or by a change of variables, it can be shown that

$$\left| \int_a^b (t-\alpha)^{\lambda-1} f(t)\, dt \right| \leqslant \operatorname*{ess.\overline{bound}}_{a<x<b} \left| \int_x^b (t-x)^{\lambda-1} f(t)\, dt \right|,$$

where $a < \alpha < b$.

In this note we obtain analogous results involving C_λ-means‡ and at

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* Cf. M. Riesz, **5**, 115–117; G. H. Hardy and M. Riesz, **4**, 28; S. Verblunsky, **7**, 173–174. See also L. S. Bosanquet, **2**, for an analogue in which integrals are replaced by sums.

† Ess.$\overline{\text{bound}}$ is used to denote the essential upper bound. In the case when $0 < \lambda < 1$, $\int_a^x (x-t)^{\lambda-1} f(t)\, dt$ may fail to exist for values of x forming a sub-set of (a, b) of measure zero; these values may, however, be neglected as the essential upper bound is considered.

‡ $C_\lambda(f, a, b)$, the λ-th Cesàro mean of $f(t)$ in (a, b), is defined by the equation

$$C_\lambda(f, a, b) = \int_a^b |b-t|^{\lambda-1} f(t)\, dt \Big/ \int_a^b |b-t|^{\lambda-1}\, dt,$$

the integrals involved being C_s-P integrals, where $s = \max(\lambda - 1, 0)$.

the same time extend* (1.1) to the case when the integrals involved are C_0-P or Denjoy-Perron integrals.

It will be shown that if $0 < \lambda \leqslant 1$, $a < b < \beta$ and if $f(t)$ is integrable in the C_0-P sense in (a, b), then†

$$(1.2)\quad \operatorname*{ess.bound}_{a<x<b} C_\lambda(f, a, x) \leqslant \frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt}{\int_a^b (\beta-t)^{\lambda-1}\,dt} \leqslant \operatorname*{ess.\overline{bound}}_{a<x<b} C_\lambda(f, a, x),$$

and also

$$(1.3)\quad \operatorname*{ess.bound}_{a<x<b} C_\lambda(f, b, x) \leqslant \frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt}{\int_a^b (\beta-t)^{\lambda-1}\,dt} \leqslant \operatorname*{ess.\overline{bound}}_{a<x<b} C_\lambda(f, b, x).$$

Corresponding results can of course be obtained involving

$$\int_a^b (t-\alpha)^{\lambda-1} f(t)\,dt,$$

where $\alpha < a < b$.

It is of interest to note that, although (1.2) remains true when $C_\lambda(f, a, x)$ is replaced by $C_\lambda(f, b, x)$, $\int_a^x (x-t)^{\lambda-1} f(t)\,dt$ cannot be replaced by $\int_x^b (t-x)^{\lambda-1} f(t)\,dt$ in (1.1); for, if we take $f(t) \equiv (b-t)^{-\lambda}$, where $0 < \lambda < 1$, it can be shown without difficulty that, for values of β sufficiently near to b,

$$\left| \int_a^b (\beta-t)^{\lambda-1} f(t)\,dt \right| > \operatorname*{ess.\overline{bound}}_{a<x<b} \left| \int_x^b (t-x)^{\lambda-1} f(t)\,dt \right|.$$

It is easy to show that (1.2) and (1.3) are false for $\lambda > 1$. In fact, more than this can be proved: it is not possible to find a constant K_1

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* (1.1) has been extended by Verblunsky to the case of the general Denjoy integral, but his work has not been published. *Cf.* J. C. Burkill, **3**, 224. The proof given here is also valid for the general Denjoy integral.

† (1.1) cannot be replaced by the corresponding inequalities

$$\operatorname*{ess.bound}_{a<x<b} \int_a^x (x-t)^{\lambda-1} f(t)\,dt \leqslant \int_a^b (\beta-t)^{\lambda-1} f(t)\,dt \leqslant \operatorname*{ess.\overline{bound}}_{a<x<b} \int_a^x (x-t)^{\lambda-1} f(t)\,dt,$$

as is shown by taking $f(t) \equiv (t-a)^{-\lambda}$.

such that

$$\left| \int_a^b (\beta-t)^{\lambda-1} f(t)\, dt \right| \leqslant K_1 \omega_1 \int_a^b (\beta-t)^{\lambda-1}\, dt,$$

where $\omega_1 = \overline{\underset{a<x<b}{\text{bound}}} |C_\lambda(f, a, x)|$; nor is it possible to find a constant K_2 such that

$$\left| \int_a^b (\beta-t)^{\lambda-1} f(t)\, dt \right| \leqslant K_2 \omega_2 \int_a^b (\beta-t)^{\lambda-1}\, dt,$$

where $\omega_2 = \overline{\underset{a<x<b}{\text{bound}}} |C_\lambda(f, b, x)|$. For, if we give β some fixed value greater than b and take $f(t) \equiv (t-a)^m$, where $m > -1$, it can be shown that

$$\int_a^b (\beta-t)^{\lambda-1} f(t)\, dt \Big/ \left\{ \omega_1 \int_a^b (\beta-t)^{\lambda-1}\, dt \right\}$$

exceeds $(\beta-b)^{\lambda-1}\Gamma(\lambda+m+1)/\{(\beta-a)^{\lambda-1}\Gamma(\lambda+1)\Gamma(m+2)\}$, and therefore tends to infinity as m tends to infinity; while, if we take $f(t) = (b-t)^m$, then

$$\int_a^b (\beta-t)^{\lambda-1} f(t)\, dt \Big/ \left\{ \omega_2 \int_a^b (\beta-t)^{\lambda-1}\, dt \right\}$$

tends to infinity as m tends to infinity.

We shall, however, show that there is a constant K, depending only on λ (being independent of a, b, β and the particular function considered), such that

$$\left| \int_a^b (\beta-t)^{\lambda-1} f(t)\, dt \right| \leqslant K\omega \int_a^b (\beta-t)^{\lambda-1}\, dt, \tag{1.4}$$

where $\omega = \max(\omega_1, \omega_2)$ and the integrals involved are $C_{\lambda-1}$-P integrals.

It can be proved without difficulty that the constant K does depend on λ and tends to infinity as λ tends to infinity. For instance, if we define $f(t) = 1$ for $1 \leqslant t \leqslant 2$, $f(t) = 0$ otherwise, and take $a = 0$, $b = 3$, $\lambda \geqslant 2$, then it can be shown that $\omega = (2^\lambda - 1)/3^\lambda$, and hence

$$\frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\, dt}{\omega \int_a^b (\beta-t)^{\lambda-1}\, dt} = \frac{(\beta-1)^\lambda - (\beta-2)^\lambda}{\beta^\lambda - (\beta-3)^\lambda} \frac{3^\lambda}{2^\lambda - 1} > \tfrac{1}{2}(\tfrac{3}{2})^{\lambda-1}$$

for sufficiently large values of β.

It follows that $K > \frac{1}{2}(\frac{3}{2})^{\lambda-1}$, so that K tends to infinity as λ tends to infinity.

It is not possible to extend (1.1) in a similar way to the case $\lambda > 1$. If we take* $f(t) \equiv 1$ and write

$$\omega' = \max\left\{\overline{\underset{a<x<b}{\text{bound}}} \left|\int_a^x (x-t)^{\lambda-1} f(t)\,dt\right|,\ \overline{\underset{a<x<b}{\text{bound}}} \left|\int_x^b (t-x)^{\lambda-1} f(t)\,dt\right|\right\},$$

then $\frac{1}{\omega'}\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt$ tends to infinity as β tends to infinity, and there is therefore no constant K (independent of β) such that

$$\left|\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt\right| \leqslant K\omega'.$$

2. THEOREM I. *If $0 < \lambda \leqslant 1$, $a < b < \beta$ and if $f(t)$ is integrable in the C_0-P sense in (a, b), then*

$$(2.1)\quad \underset{a<x<b}{\text{ess.bound}}\, C_\lambda(f, a, x) \leqslant \frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt}{\int_a^b (\beta-t)^{\lambda-1}\,dt} \leqslant \overline{\underset{a<x<b}{\text{ess.bound}}}\, C_\lambda(f, a, x),$$

and†

$$(2.2)\quad \underset{a<x<b}{\text{ess.bound}}\, C_\lambda(f, b, x) \leqslant \frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt}{\int_a^b (\beta-t)^{\lambda-1}\,dt} \leqslant \overline{\underset{a<x<b}{\text{ess.bound}}}\, C_\lambda(f, b, x).$$

If $\lambda = 1$, the results (2.1) and (2.2) follow at once from the continuity of $\int_a^x f(t)\,dt$ and $\int_x^b f(t)\,dt$ at the points b, a respectively. We therefore suppose that $0 < \lambda < 1$ and use a method of proof similar to that due to M. Riesz‡.

Define

$$\phi(t) = \frac{1}{\Gamma(\lambda)\,\Gamma(1-\lambda)} \int_t^b (u-t)^{-\lambda} (\beta-u)^{\lambda-1}\,du \quad (a \leqslant t \leqslant b).$$

Then

$$(2.3)\qquad (\beta-t)^{\lambda-1} = -\int_t^b (u-t)^{\lambda-1} \phi'(u)\,du \qquad (a \leqslant t < b),$$

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* *Cf.* L. S. Bosanquet, **2**, 146.

† See below for a result which is better than (2.2).

‡ M. Riesz, **5**, 115–117. *Cf.* also S. Verblunsky, **7**, 173–174.

and hence

$$\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt = -\int_a^b f(t)\,dt \int_t^b (u-t)^{\lambda-1}\phi'(u)\,du.$$

It will be proved below that it is legitimate to invert the order of integration, so that

$$(2.4) \qquad \int_a^b (\beta-t)^{\lambda-1} f(t)\,dt = -\int_a^b \phi'(u)\,du \int_a^u (u-t)^{\lambda-1} f(t)\,dt$$

$$= -\frac{1}{\lambda}\int_a^b \phi'(u)(u-a)^{\lambda} C_\lambda(f, a, u)\,du.$$

Now* $\phi'(u) \leqslant 0$ for $a \leqslant u < b$ and, by the result (2.4) with $f(t) \equiv 1$,

$$\int_a^b (\beta-t)^{\lambda-1}\,dt = -\frac{1}{\lambda}\int_a^b \phi'(u)(u-a)^{\lambda}\,du.$$

It therefore follows that (2.1) is satisfied.

In order to obtain (2.2) in the case when $0 < \lambda < 1$, we define

$$\psi(t) = \frac{1}{\Gamma(\lambda)\,\Gamma(1-\lambda)}\int_a^t (t-u)^{-\lambda}(\beta-u)^{\lambda-1}\,du \quad (a \leqslant t \leqslant b).$$

Then

$$\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt = \int_a^b f(t)\,dt \int_a^t (t-u)^{\lambda-1}\psi'(u)\,du,$$

and (2.2) can be obtained by a method similar to that used above.

The following corollary shows that if $C_\lambda(f, a, b)$ exists, then (2.1) also holds when $\beta = b$.

COROLLARY. *If* $0 < \lambda \leqslant 1$ *and if* $C_\lambda(f, a, b)$ *exists, then*

$$\operatorname{ess.}\underline{\operatorname{bound}}_{a<x<b} C_\lambda(f, a, x) \leqslant C_\lambda(f, a, b) \leqslant \operatorname{ess.}\overline{\operatorname{bound}}_{a<x<b} C_\lambda(f, a, x).$$

Take ϵ such that $0 < \epsilon < b-a$. It can be deduced from Theorem I that

$$\operatorname{ess.}\underline{\operatorname{bound}}_{a<x<b} C_\lambda(f, a, x) \leqslant \frac{\int_a^{b-\epsilon}(b-t)^{\lambda-1} f(t)\,dt}{\int_a^{b-\epsilon}(b-t)^{\lambda-1}\,dt} \leqslant \operatorname{ess.}\overline{\operatorname{bound}}_{a<x<b} C_\lambda(f, a, x),$$

and the result stated is obtained by letting ϵ tend to zero.

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* *Cf.* S. Verblunsky, *loc. cit.*

Note added* 30 *August*, 1945. The following result, which is better† than (2.2), can be obtained by integration by parts.

THEOREM A. *If* $0<\lambda\leqslant 1$, $a<b<\beta$, *and if* $f(t)$ *is integrable in the* C_0-P *sense in* (a, b), *then*

$$(2.5)\quad \underline{\text{bound}}_{a<x<b} C_1(f, b, x) \leqslant \frac{\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt}{\int_a^b (\beta-t)^{\lambda-1}\,dt} \leqslant \overline{\text{bound}}_{a<x<b} C_1(f, b, x).$$

Further, if the bounds of $C_1(f, b, x)$ *are finite, then* $(b-t)^{\lambda-1}f(t)$ *is integrable in the* C_0-P *sense in* (a, b), *and* (2.5) *also holds when* $\beta=b$.

As in Theorem I, it is sufficient to consider the case when $0<\lambda<1$. Let $\theta(x)=\int_x^b f(t)\,dt$ for $a\leqslant x\leqslant b$. Then, whenever $\beta>b$,

$$(2.6)\quad \int_a^b (\beta-t)^{\lambda-1}f(t)\,dt=(\beta-a)^{\lambda-1}\theta(a)+(1-\lambda)\int_a^b (\beta-t)^{\lambda-2}\theta(t)\,dt.$$

Since $\theta(x)=(b-x)\,C_1(f, b, x)$, and $\theta(x)$ is continuous for $x=a$, it can be deduced without difficulty that (2.5) is satisfied.

Now suppose that the bounds of $C_1(f, b, x)$ are finite, and take ϵ such that $0<\epsilon<b-a$. Then

$$\int_a^{b-\epsilon} (b-t)^{\lambda-1}f(t)\,dt=-\Big[(b-t)^{\lambda-1}\theta(t)\Big]_a^{b-\epsilon}+(1-\lambda)\int_a^{b-\epsilon}(b-t)^{\lambda-2}\theta(t)\,dt,$$

and hence, letting $\epsilon\to+0$, $(b-t)^{\lambda-1}f(t)$ is integrable in the C_0-P sense in (a, b), and (2.6) holds when $\beta=b$. It follows that (2.5) also holds when $\beta=b$.

3. The result (1.1) can be deduced from (2.4) in the usual way when $0<\lambda<1$. Since $\phi'(u)\leqslant 0$ for $a\leqslant u<b$, while

$$\int_a^b \phi'(u)\,du=\lim_{\epsilon\to+0}\int_a^{b-\epsilon}\phi'(u)\,du=-\phi(a),$$

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* This note replaces a corollary in which the existence of C_0-$P\int_a^b (b-t)^{\lambda-1}f(t)\,dt$ was established under less general conditions.

† Since $C_1(f, b, x)$ is continuous for $a<x<b$, (2.5) is equivalent to (2.2) in the case $\lambda=1$. If $0<\lambda<1$, it follows from the consistency theorem for fractional integrals (cf. L. S. Bosanquet, **1**, Theorem IV; the proof given here for the general Denjoy integral can be adapted to apply to the C_0-P integral) that the bounds of $C_1(f, b, x)$ are contained between the essential bounds of $C_\lambda(f, b, x)$ for $a<x<b$.

it follows from (2.4) that

$$\phi(a)\,\text{ess.}\underline{\text{bound}}_{a<x<b}\int_a^x (x-t)^{\lambda-1}f(t)\,dt \leqslant \int_a^b (\beta-t)^{\lambda-1}f(t)\,dt$$

$$\leqslant \phi(a)\,\text{ess.}\overline{\text{bound}}_{a<x<b}\int_a^x (x-t)^{\lambda-1}f(t)\,dt.$$

Since $0<\phi(a)<1$, it therefore follows that (1.1) remains true for functions integrable in the C_0-P sense in (a, b).

4. We now show that, with the notation of Theorem I,

$$(4.1)\qquad \int_a^b f(t)\,dt\int_t^b (u-t)^{\lambda-1}\phi'(u)\,du = \int_a^b \phi'(u)\,du\int_a^u (u-t)^{\lambda-1}f(t)\,dt,$$

a result which was assumed in the proof of Theorem I.

We first prove two lemmas*.

LEMMA I. *If* $a<c<b$, *then*

$$(4.2)\qquad \int_a^c f(t)\,dt\int_t^c (u-t)^{\lambda-1}\phi'(u)\,du = \int_a^c \phi'(u)\,du\int_a^u (u-t)^{\lambda-1}f(t)\,dt.$$

It can easily be shown that $\phi''(u)$ is continuous for $a\leqslant u\leqslant c$, and hence, writing $F(t)=\int_a^t f(u)\,du$, it follows that

$$\int_a^c f(t)\,dt\int_t^c (u-t)^{\lambda-1}\phi'(u)\,du$$

$$= -\int_a^c F(t)\,\frac{d}{dt}\left\{\int_t^c (u-t)^{\lambda-1}\phi'(u)\,du\right\}dt$$

$$= -\int_a^c F(t)\left\{\int_t^c (u-t)^{\lambda-1}\phi''(u)\,du-(c-t)^{\lambda-1}\phi'(c)\right\}dt$$

$$= \phi'(c)\int_a^c (c-t)^{\lambda-1}F(t)\,dt-\int_a^c \phi''(u)\,du\int_a^u (u-t)^{\lambda-1}F(t)\,dt$$

$$= \int_a^c \phi'(u)\left\{\frac{d}{du}\int_a^u (u-t)^{\lambda-1}F(t)\,dt\right\}du$$

$$= \int_a^c \phi'(u)\,du\int_a^u (u-t)^{\lambda-1}f(t)\,dt,$$

* For similar lemmas see L. S. Bosanquet, **1**, 136–139.

the inversion in the order of integration following from Fubini's theorem and the last step from the consistency theorem for fractional integrals.

LEMMA II. *If* $a \leqslant c < d < b$, *then*

$$(4.3) \qquad \int_c^d f(t)\,dt \int_d^b (u-t)^{\lambda-1}\phi'(u)\,du = \int_d^b \phi'(u)\,du \int_c^d (u-t)^{\lambda-1} f(t)\,dt.$$

Take ϵ such that $c < d-\epsilon < d$. Then

$$\int_c^{d-\epsilon} f(t)\,dt \int_d^b (u-t)^{\lambda-1}\phi'(u)\,du$$

$$= \left[F(t)\int_d^b (u-t)^{\lambda-1}\phi'(u)\,du\right]_{t=c}^{t=d-\epsilon} - (1-\lambda)\int_c^{d-\epsilon} F(t)\,dt \int_d^b (u-t)^{\lambda-2}\phi'(u)\,du$$

$$= \int_d^b \phi'(u)\left\{\left[(u-t)^{\lambda-1}F(t)\right]_{t=c}^{t=d-\epsilon} - (1-\lambda)\int_c^{d-\epsilon}(u-t)^{\lambda-2}F(t)\,dt\right\} du$$

$$= \int_d^b \phi'(u)\,du \int_c^{d-\epsilon} (u-t)^{\lambda-1} f(t)\,dt.$$

In order to prove that (4.3) is satisfied, it is therefore sufficient to show that

$$(4.4) \qquad \lim_{\epsilon\to+0} \int_d^b \phi'(u)\,du \int_{d-\epsilon}^d (u-t)^{\lambda-1} f(t)\,dt = 0.$$

Whenever $d < u \leqslant b$, it follows from the second mean value theorem that

$$\int_{d-\epsilon}^d (u-t)^{\lambda-1} f(t)\,dt = (u-d)^{\lambda-1}\int_{\xi_u}^d f(t)\,dt,$$

where $d-\epsilon < \xi_u < d$. Hence, since $\phi'(u) \leqslant 0$ for $d \leqslant u < b$,

$$\left|\int_d^b \phi'(u)\,du \int_{d-\epsilon}^d (u-t)^{\lambda-1} f(t)\,dt\right| \leqslant \overline{\operatorname{bound}}_{d-\epsilon<\xi<d}\left|\int_\xi^d f(t)\,dt\right| \times \int_d^b (u-d)^{\lambda-1}\{-\phi'(u)\}\,du$$

$$= (\beta-d)^{\lambda-1}\,\overline{\operatorname{bound}}_{d-\epsilon<\xi<d}\left|\int_\xi^d f(t)\,dt\right|,$$

by (2.3), so that (4.4) is satisfied.

We now consider (4.1). Since

$$\int_a^b f(t)\,dt \int_t^b (u-t)^{\lambda-1}\phi'(u)\,du$$

$$= \lim_{\epsilon\to+0}\left\{\int_a^{b-\epsilon} f(t)\,dt \int_t^{b-\epsilon}(u-t)^{\lambda-1}\phi'(u)\,du + \int_a^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\phi'(u)\,du\right\}$$

$$= \lim_{\epsilon\to+0}\left\{\int_a^{b-\epsilon}\phi'(u)\,du \int_a^{u}(u-t)^{\lambda-1}f(t)\,dt + \int_a^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\phi'(u)\,du\right\},$$

by Lemma I, in order to prove that (4.1) is satisfied it is sufficient to show that

$$\lim_{\epsilon\to+0}\int_a^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du = 0. \tag{4.5}$$

Now suppose that $a<c<b$ and take ϵ such that $c<b-\epsilon<b$. Then $\int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du$ is positive and increases as t increases for $a\leqslant t\leqslant c$ and hence, by the second mean value theorem,

$$\left|\int_a^c f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du\right|$$

$$= \left|\int_\xi^c f(t)\,dt \int_{b-\epsilon}^{b}(u-c)^{\lambda-1}\{-\phi'(u)\}\,du\right| \quad (a<\xi<c)$$

$$\leqslant \int_{b-\epsilon}^{b}(u-c)^{\lambda-1}\{-\phi'(u)\}\,du \times \overline{\text{bound}}_{a<x<c}\left|\int_x^c f(t)\,dt\right|.$$

It follows that

$$\lim_{\epsilon\to+0}\int_a^c f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du = 0$$

whenever $a<c<b$, and hence that (4.5) may be replaced by

$$\lim_{c\to b-0}\overline{\lim_{\epsilon\to+0}}\left|\int_c^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du\right| = 0. \tag{4.6}$$

By Lemma II,

$$\int_c^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b}(u-t)^{\lambda-1}\{-\phi'(u)\}\,du = \int_{b-\epsilon}^{b}\{-\phi'(u)\}\,du \int_c^{b-\epsilon}(u-t)^{\lambda-1}f(t)\,dt$$

$$= \int_{b-\epsilon}^{b}\{-\phi'(u)\}(u-b+\epsilon)^{\lambda-1}\,du \int_{\xi_u}^{b-\epsilon} f(t)\,dt,$$

by the second mean value theorem, where $c < \xi_u < b-\epsilon$. Hence

$$\left| \int_c^{b-\epsilon} f(t)\,dt \int_{b-\epsilon}^{b} (u-t)^{\lambda-1} \{-\phi'(u)\}\,du \right|$$

$$\leqslant 2 \int_{b-\epsilon}^{b} (u-b+\epsilon)^{\lambda-1} \{-\phi'(u)\}\,du \times \overline{\operatorname{bound}}_{c\leqslant\xi\leqslant b} \left| \int_\xi^b f(t)\,dt \right|$$

$$= 2(\beta-b+\epsilon)^{\lambda-1} \overline{\operatorname{bound}}_{c\leqslant\xi\leqslant b} \left| \int_\xi^b f(t)\,dt \right|,$$

by (2.3), so that (4.6) is satisfied, and this is the result required.

5. In order to extend Theorem I to the case $\lambda > 1$, we first prove two lemmas.

LEMMA III. *If $0 < \lambda < \mu$ and if $f(t)$ is integrable in the C_s-P sense in (a, b), where $s = \max(\lambda-1, 0)$, then*

$$(5.1) \qquad \overline{\operatorname{bound}}_{a<x<b} |C_\mu(f, a, x)-f(a)| \leqslant \overline{\operatorname{bound}}_{a<x<b} |C_\lambda(f, a, x)-f(a)|,$$

$$(5.2) \qquad \overline{\operatorname{bound}}_{a<x<b} |C_\mu(f, b, x)-f(b)| \leqslant \overline{\operatorname{bound}}_{a<x<b} |C_\lambda(f, b, x)-f(b)|.$$

The method of proof is similar to that of a result due to Burkill. It can be shown* that

$$\int_a^x (x-t)^{\mu-1} \{f(t)-f(a)\}\,dt$$

$$= \frac{\Gamma(\mu)}{\Gamma(\lambda)\,\Gamma(\mu-\lambda)} \int_a^x (x-t)^{\mu-\lambda-1}\,dt \int_a^t (t-u)^{\lambda-1} \{f(u)-f(a)\}\,du,$$

subject to the existence† of the inner integral on the right for $a \leqslant t \leqslant x$. The result (5.1) can be deduced without difficulty and (5.2) can be proved in a similar manner.

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* Obtained by integrating by parts until the integrals involved are C_0-P integrals, and then using the consistency theorem for fractional integrals. Cf. J. C. Burkill, **3**, 221.

† The integral exists for all t in (a, b) if $\lambda \geqslant 1$.

LEMMA IV. *If $f(t)$ is integrable in the C_s-P sense in (a, b), where $s = \max(\lambda-1, 0)$, $\lambda > 0$, and if*

$$\omega = \max\left\{\overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(f, a, x)-f(a)|,\ \overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(f, b, x)-f(b)|\right\},$$

then there is a constant H, depending only on λ (being independent of a, b and the particular function $f(t)$ considered), such that

$$|f(b)-f(a)| \leqslant H\omega.$$

This lemma has been established in a previous paper* in the case in which λ is an integer. Writing $\mu = [\lambda]+1$ and using Lemma III, it can easily be seen that it also holds for non-integral values of λ.

We now prove the main theorem for the case $\lambda > 1$.

THEOREM II. *If $\lambda > 1$, $a < b < \beta$ and if $f(t)$ is integrable in the $C_{\lambda-1}$-P sense in (a, b), then there is a constant K, depending only on λ (being independent of a, b, β and the particular function $f(t)$ considered), such that*

$$\left|\int_a^b (\beta-t)^{\lambda-1} f(t)\,dt\right| \leqslant K\omega \int_a^b (\beta-t)^{\lambda-1}\,dt, \tag{5.3}$$

where†

$$\omega = \max\left\{\overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(f, a, x)|,\ \overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(f, b, x)|\right\}. \tag{5.4}$$

Let $\lambda = n+p$, where $0 < p \leqslant 1$ and n is a positive integer, and define

$$F_0(x) = f(x),$$

$$F_r(x) = (C_{\lambda-r}\text{-}P)\int_a^x F_{r-1}(t)\,dt \quad (r = 1, 2, 3, \ldots, n).$$

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* W. L. C. Sargent, **6**, 216.

† There is no restriction in considering upper bounds instead of essential upper bounds in the definition of ω; it is pointed out below that, when $\lambda > 1$, the upper bounds of $|C_\lambda(f, a, x)|$ and $|C_\lambda(f, b, x)|$ for $a < x < b$ coincide with the essential upper bounds.

If we integrate by parts r times, we find that*

$$(5.5) \qquad C_\lambda(f, a, x) = \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)}{(x-a)^r} C_{\lambda-r}(F_r, a, x) \qquad (r = 1, 2, 3, \dots, n),$$

so that, by (5.4),

$$(5.6) \qquad \overline{\operatorname{bound}}_{a<x<b} |C_{\lambda-r}(F_r, a, x)| \leqslant \frac{(b-a)^r \omega}{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)} \qquad (r = 1, 2, 3, \dots, n).$$

We proceed to show that, for $r = 1, 2, 3, \dots, n$,

$$(5.7) \qquad \overline{\operatorname{bound}}_{a<x<b} |C_{\lambda-r}(F_r, b, x) - F_r(b)| \leqslant \frac{A_r(b-a)^r \omega}{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)},$$

where A_r is a constant depending only on λ and r. Since $F_r(a) = 0$, it then follows from (5.6) and Lemma IV that

$$(5.8) \qquad |F_r(b)| \leqslant \frac{B_r(b-a)^r \omega}{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)},$$

where B_r is a constant depending only on λ and r.

Since

$$F_r(x) - F_r(b) = (C_{\lambda-r}\text{-}P) \int_b^x F_{r-1}(t)\, dt \qquad (r = 1, 2, 3, \dots, n),$$

a single integration by parts shows that

$$(5.9) \qquad C_{\lambda-r+1}(F_{r-1}, b, x) = \frac{\lambda-r+1}{x-b} \{C_{\lambda-r}(F_r, b, x) - F_r(b)\} \qquad (r = 1, 2, 3, \dots, n),$$

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* By (5.5), with $r = n$,

$$C_\lambda(f, a, x) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)\, C_p(F_n, a, x)/(x-a)^n.$$

Now $F_n(x)$, being an indefinite C_p-P integral, is C_p-continuous for $a \leqslant x \leqslant b$. Thus $\lim_{h \to +0} C_p(F_n, x, x-h) = F_n(x)$, and hence, by the consistency theorem for fractional integrals, $\lim_{h \to +0} C_1(F_n, x, x-h) = F_n(x)$ for $a < x \leqslant b$. It follows from Theorem A and the corollary to Theorem I that $C_p(F_n, a, x)$ exists for $a < x \leqslant b$, and

$$\operatorname{ess.}\underline{\operatorname{bound}}_{a<t<x} C_p(F_n, a, t) \leqslant C_p(F_n, a, x) \leqslant \operatorname{ess.}\overline{\operatorname{bound}}_{a<t<x} C_p(F_n, a, t).$$

It can be deduced without difficulty that the upper bound of $|C_\lambda(f, a, x)|$ for $a < x < b$ coincides with the essential upper bound, whilst the upper bound of $|C_\lambda(f, b, x)|$ can be considered in a similar way.

and it follows from (5.4) and (5.9) that (5.7) is satisfied for $r=1$. Further, if (5.7) and (5.8) hold for some positive integer r less than n, then

$$\overline{\operatorname{bound}}_{a<x<b} |C_{\lambda-r}(F_r, b, x)| \leqslant \frac{(A_r+B_r)(b-a)^r\omega}{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)},$$

and it follows from (5.9) that (5.7) also holds when r is replaced by $r+1$.

Thus (5.7) and (5.8) hold for $r=1, 2, 3, \dots, n$ and hence, writing $B=\max_{1\leqslant r\leqslant n}(B_r/{}^nC_r)$, we have

$$\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)|F_r(b)| \leqslant {}^nC_r B(b-a)^r\omega \quad (r=1, 2, 3, \dots, n), \tag{5.10}$$

where B is a constant depending only on λ.

Further, in view of (5.6), with $r=n$, and Theorem I,

$$\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)\int_a^b(\beta-t)^{p-1}F_n(t)\,dt \leqslant (b-a)^n\omega\int_a^b(\beta-t)^{p-1}dt. \tag{5.11}$$

If we integrate by parts n times, we find that

$$\lambda\int_a^b(\beta-t)^{\lambda-1}f(t)\,dt=\sum_{r=1}^{n}\lambda(\lambda-1)(\lambda-2)\dots(\lambda-r+1)(\beta-b)^{\lambda-r}F_r(b) +\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n)\int_a^b(\beta-t)^{p-1}F_n(t)\,dt,$$

so that, by (5.10) and (5.11),

$$\begin{aligned}
&\left|\lambda\int_a^b(\beta-t)^{\lambda-1}f(t)\,dt\right| \\
&\leqslant B\omega(\beta-b)^p\sum_{r=1}^{n}{}^nC_r(\beta-b)^{n-r}(b-a)^r+\omega(b-a)^n p\int_a^b(\beta-t)^{p-1}dt \\
&= B\omega(\beta-b)^p\{(\beta-a)^n-(\beta-b)^n\}+\omega(b-a)^n\{(\beta-a)^p-(\beta-b)^p\} \\
&< B\omega\{(\beta-a)^{n+p}-(\beta-b)^{n+p}\}+\omega(b-a)^{n+p} \\
&< (B+1)\,\omega\{(\beta-a)^{n+p}-(\beta-b)^{n+p}\} \\
&= \lambda K\omega\int_a^b(\beta-t)^{\lambda-1}dt,
\end{aligned}$$

where $K=B+1$. The theorem is therefore established.

Finally, I should like to say how much I am indebted to Dr. Bosanquet for his valuable criticisms and suggestions.

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ON METRIC SPACES BASED ON A VECTOR DENSITY

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Introduction.

Underhill (1)* and Landsberg (2) introduced invariants associated with the simple integral $\int F(x, y, x', y')\,dt$ which are generalizations of certain notions occurring in the theory of surfaces. The possibility of basing a differential geometry on such notions was first pointed out by Finsler in his thesis. This geometry, since known as Finsler geometry, has attained a final form, including a definition of parallel transport which preserves lengths and angles, in a tract by Cartan (3).

The invariants of Underhill were generalized in a series of papers by Koschmieder (5, 6, 7) to multiple integrals. In particular he deduced a covariant tensor of the second order which could be used as the fundamental metric tensor in a differential geometry. He also obtained an invariant U_0* in terms of which the expression for the second variation of a multiple integral is invariant for both coordinate and parameter transformations.

Independently of Koschmieder the possibility of basing a geometry upon the notion of area rather than length was investigated by Cartan (4), who constructed a geometry associated with the multiple integral

$$\iint F(x, y, p, q)\,dx\,dy$$

in which the integral represents the area of a surface. This geometry has been studied in relation to the analytical work of Koschmieder by Berwald (8), who succeeded in expressing the fundamental invariant of Koschmieder in terms of the curvature and torsion of the Cartan space.

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* These numbers refer to the list of papers at the end.

The geometries mentioned, associated with the names of Finsler and Cartan, differ essentially from Riemannian geometry in that their fundamental quantities (metric and connection coefficients) are not only functions of position, but involve an element of support, which in Finsler geometry is a line element, and in Cartan geometry is a hypersurface element. These two cases have been envisaged from a single point of view by Schouten and Haantjes (9), who took the element of support to be a vector density of arbitrary weight p. Finsler geometry then comes in as a special case in which the element of support is a contravariant vector density of weight $p = 0$, while Cartan geometry has an element of support which is a covariant vector density of weight $p = -1$. The conditions for the existence of such a geometry were studied in all their generality by Schouten and Haantjes, and the metric and connection coefficients deduced from the "fundamental function" for all permissible values of p.

In the geometries of Finsler and Cartan there are two vectors which are of fundamental importance. They are (i) the unit vector $\vec{l}$ associated with the element of support, and (ii) the vector $\vec{A}$, which appears in the forms:

$$(a)\ A_i = \tfrac{1}{2}L\frac{\partial \log g}{\partial x'^i}\ \text{(Finsler)}, \quad \text{and} \quad (b)\ A^i = -L\frac{\partial(1/\sqrt{g})}{\partial u_i}\ \text{(Cartan)}.$$

The object of this paper is to go a little further in the general case, in order to show the significance of the two vectors just mentioned.

1. *Metric coefficients.*

We denote by u_i a covariant vector density of weight $-p$, and by u^i a contravariant vector density of weight $+p$. To each of these we associate a scalar $L(x, u)$ which is positive homogeneous of the first degree in the u. This scalar is the "fundamental function" of the geometry. We shall in the sequel use the notation of Berwald (8), and we shall indicate by (a) any symbols or equations referring to the covariant case, and by (b) those referring to the contravariant case.

We first deduce the metric coefficients from the fundamental function. Since

$$(1.1) \qquad (a)\ a^{ij} = \frac{\partial^2(\frac{1}{2}L^2)}{\partial u_i \partial u_j}, \qquad (b)\ a_{ij} = \frac{\partial^2(\frac{1}{2}L^2)}{\partial u^i \partial u^j}.$$

transform like tensor densities of weights $2p$ and $-2p$ respectively, we can put

$$(1.2) \qquad (a)\ a^{ij} = g^p g^{ij}, \qquad (b)\ a_{ij} = g^{-p} g_{ij},$$

where g in each case denotes the determinant of the g_{ij} with *low* indices. Defining a in *each* case as $|a_{ij}|$, so that for the covariant case $|a^{ij}| = 1/a$, we have, respectively,

$$(a) \quad \frac{1}{a} = g^{np-1}, \qquad (b) \quad a = g^{-(np-1)},$$

so that

$$(1.3) \qquad (a) \quad g = a^{-1/(np-1)}, \qquad (b) \quad g = a^{-1/(np-1)}.$$

From these relations we have, since we are going to use (1.2) in the form

$$(a) \quad g^{ij} = g^{-p} a^{ij}, \qquad (b) \quad g_{ij} = g^p a_{ij},$$

the following definitions for the metric coefficients:

$$(1.4) \quad (a) \quad g^{ij} = a^{p/(np-1)} \frac{\partial^2(\frac{1}{2}L^2)}{\partial u_i \partial u_j}, \qquad (b) \quad g_{ij} = a^{-p/(np-1)} \frac{\partial^2(\frac{1}{2}L^2)}{\partial u^i \partial u^j}.$$

2. *The unit vector* $\vec{l}$ *and the vector* $\vec{A}$.

From (1.1) and (1.2), on carrying out the differentiation, we get

$$(2.1) \quad (a) \quad g^p g^{ij} = L \frac{\partial^2 L}{\partial u_i \partial u_j} + \frac{\partial L}{\partial u_i} \frac{\partial L}{\partial u_j}, \qquad (b) \quad g^{-p} g_{ij} = L \frac{\partial^2 L}{\partial u^i \partial u^j} + \frac{\partial L}{\partial u^i} \frac{\partial L}{\partial u^j},$$

and hence, using the homogeneity property of the L in the u,

$$(2.2) \qquad (a) \quad g^p g^{ij} u_j = L \frac{\partial L}{\partial u_i}, \qquad (b) \quad g^{-p} g_{ij} u^j = L \frac{\partial L}{\partial u^i}.$$

$$(2.3) \qquad (a) \quad g^p g^{ij} u_i u_j = L^2, \qquad (b) \quad g^{-p} g_{ij} u^i u^j = L^2,$$

i.e.

$$(a) \quad g^{ij} \left(\frac{\sqrt{g^p}}{L} u_i\right) \left(\frac{\sqrt{g^p}}{L} u_j\right) = 1, \qquad (b) \quad g_{ij} \left(\frac{u^i}{L\sqrt{g^p}}\right) \left(\frac{u^j}{L\sqrt{g^p}}\right) = 1,$$

so that

$$(2.4) \qquad (a) \quad l_i = \frac{\sqrt{g^p}}{L} u_i, \qquad (b) \quad l^i = \frac{1}{L\sqrt{g^p}} u^i,$$

and from (2.2) and (2.3), since $g^{ij} l_i l_j = l^i l_i = 1$, we have

$$(a) \quad \sqrt{g^p}\, g^{ij} \left(\frac{\sqrt{g^p}}{L} u_j\right) = \frac{\partial L}{\partial u_i}, \qquad (b) \quad \frac{1}{\sqrt{g^p}} g_{ij} \frac{u^j}{L\sqrt{g^p}} = \frac{\partial L}{\partial u^i};$$

hence

$$(a) \quad l^i = g^{ij} l_j = \frac{1}{\sqrt{g^p}} \frac{\partial L}{\partial u_i}, \qquad (b) \quad l_i = g_{ij} l^j = \sqrt{g^p} \frac{\partial L}{\partial u^i},$$

so that the components of the unit vector associated with the element of support are

$$(2.5)\qquad (a)\quad \begin{cases} l_i = \dfrac{\sqrt{g^p}}{L}\, u_i, \\ l^i = \dfrac{1}{\sqrt{g_p}}\dfrac{\partial L}{\partial u_i}, \end{cases} \qquad (b)\quad \begin{cases} l^i = \dfrac{1}{L\sqrt{g^p}}\, u^i, \\ l_i = \sqrt{g^p}\dfrac{\partial L}{\partial u^i}. \end{cases}$$

Differentiating (2.2) and using (2.1), we have

$$(2.6)\quad (a)\quad \frac{\partial g^p}{\partial u_k}\, g^{ij} u_j + g^p \frac{\partial g^{ij}}{\partial u_k}\, u_j = 0, \qquad (b)\quad \frac{\partial (g^{-p})}{\partial u^k}\, g_{ij} u^j + g^{-p}\frac{\partial g_{ij}}{\partial u^k}\, u^j = 0.$$

On modifying we get

$$(2.7)\qquad \begin{cases} (a) \quad p \cdot \tfrac{1}{2}\dfrac{\partial \log g}{\partial u_k}\, g^{ij} u_j + \tfrac{1}{2}\dfrac{\partial g^{ij}}{\partial u_k}\, u_j = 0, \\ (b) \quad -p \cdot \tfrac{1}{2}\dfrac{\partial \log g}{\partial u^k}\, g_{ij} u^j + \tfrac{1}{2}\dfrac{\partial g_{ij}}{\partial u^k}\, u^j = 0. \end{cases}$$

If we define, for any $f(x, u)$ which is homogeneous in the u, the operator

$$(a)\quad f\|^i = \frac{L}{\sqrt{g^p}}\frac{\partial f}{\partial u_i}, \qquad (b)\quad f\|_i = L\sqrt{g^p}\frac{\partial f}{\partial u^i},$$

which produces another function homogeneous in the u, we have, if the function happens to be of degree zero in the u,

$$(a)\quad f\|^0 = 0, \qquad (b)\quad f\|_0 = 0.$$

Hence, if, following what is done in Finsler and Cartan geometries, we define the tensors

$$(2.8)\qquad (a)\quad A^{ikh} = -\tfrac{1}{2} g^{ik}\|^h, \qquad (b)\quad A_{ikh} = +\tfrac{1}{2} g_{ik}\|_h,$$

and the vectors

$$(2.9)\qquad (a)\quad A^i = \tfrac{1}{2}(\log g)\|^i, \qquad (b)\quad A_i = \tfrac{1}{2}(\log g)\|_i,$$

the equations (2.7) become

$$(a)\quad p \cdot \tfrac{1}{2}(\log g)\|^k \cdot l_i + \tfrac{1}{2} g^{ij}\|^k l_j = 0, \qquad (b)\quad -p \cdot \tfrac{1}{2}(\log g)\|_k \cdot l_i + \tfrac{1}{2} g_{ij}\|_k l^j = 0,$$

giving therefore, since contraction with l is denoted by the letter o,

$$(2.10)\quad (a)\quad A^{iok} = A^{oik} = p l^i A^k, \qquad (b)\quad A_{iok} = A_{oik} = p l_i A_k,$$

so that

$$(a)\quad A^i = \frac{1}{p}\, A^{ooi}, \qquad (b)\quad A_i = \frac{1}{p}\, A_{ooi}.$$

From (2.8) and (2.9) it follows that

(2.11) (a) $A^{ik\circ}=0$, (b) $A_{ik\circ}=0$,

(2.12) (a) $A^{\circ}=0$, (b) $A_{\bullet}=0$.

Since for the determinant $g=|g_{ij}|$

$$\frac{\partial \log g}{\partial u}=\frac{1}{g}\frac{\partial g}{\partial u}=g^{jk}\frac{\partial g_{jk}}{\partial u},$$

we have

(a) $g^{jk}.\frac{1}{2}g_{jk}\|^{i}=\frac{1}{2}(\log g)\|^{i}$, (b) $g^{jk}.\frac{1}{2}g_{jk}\|_{i}=\frac{1}{2}(\log g)\|_{i}$

i.e.

(a) $A_{m}{}^{mi}=A^{i}$, (b) $A=A^{m}{}_{mi}$,

so that the vector $\vec{A}$ is equal to one of the contractions of the torsion tensor.

We may easily prove, following the method employed by Berwald, that simple relations exist between the two contracted tensors which can be deduced from the torsion tensor, *i.e.* between

(a) $A_{m}{}^{mi}$ $(=A^{i})$ and $A^{im}{}_{m}$, (b) $A^{m}{}_{mi}$ $(=A_{i})$ and $A_{im}{}^{m}$.

The relations are

(2.13) (a) $A^{m}{}_{im}=A^{i}\{1-(n-1)p\}$, (b) $A_{im}{}^{m}=A_{i}\{1-(n-1)p\}$,

giving for (a) $p=1$ (Cartan) and (b) $p=0$ (Finsler) the known results

(2.14) (a) $A^{m}{}_{im}=(2-n)A^{i}$, (b) $A_{im}{}^{m}=A_{i}=A^{m}{}_{mi}$.

3. *The absolute differential of the unit vector.*

From the definition (2.5) we get immediately

(3.1) (a) $L\|^{i}=Ll^{i}$, (b) $L\|_{i}=Ll_{i}$.

The following results will also be needed:

(3.2) (a) $(\sqrt{g^{p}})\|^{i}=p\sqrt{g^{p}}.A^{i}$, (b) $(\sqrt{g^{p}})\|_{i}=p\sqrt{g^{p}}.A_{i}$,

(3.3) (a) $\left(\frac{1}{\sqrt{g^{p}}}\right)\Big\|^{i}=-\frac{p}{\sqrt{g^{p}}}A^{i}$, (b) $\left(\frac{1}{\sqrt{g^{p}}}\right)\Big\|_{i}=-\frac{p}{\sqrt{g^{p}}}A_{i}$.

Using these we have

(3.4) (a) $l^{i}\|^{h}=g^{ij}-l^{i}(l^{h}+pA^{h})$, (b) $l^{i}\|_{h}=\delta_{h}{}^{i}-l^{i}(l_{h}+pA_{h})$,

(3.5) (a) $l_{i}\|^{h}=\delta_{i}{}^{h}-l_{i}(l^{h}-pA^{h})$, (b) $l_{i}\|_{h}=g_{ij}-l_{i}(l_{h}-pA_{h})$.

Passing now to the consideration of the absolute differential of a vector, we write first, for any vector X^i,

$$(a) \quad DX^i = dX^i + X^k(\Gamma^i_{kh}\,dx^h + C^{ih}_k\,du_h),$$

$$(b) \quad DX^i = dX^i + X^k(\Gamma^i_{kh}\,dx^h + C^i_{kh}\,du^h),$$

and on modifying and using (2.10), we have for the unit vector

$$(3.6) \quad \begin{cases} (a) \quad Dl^i = \varpi^i = dl^i + \Gamma^i_{oh}\,dx^h + pl^iA^h\,dl_h, \\ (b) \quad Dl^i = \varpi^i = dl^i + \Gamma^i_{oh}\,dx^h + pl^iA_h\,dl^h, \end{cases}$$

$$(3.7) \quad \begin{cases} (a) \quad \varpi_i = dl_i - \Gamma_{ioh}\,dx^h - pl_iA^h\,dl_h. \\ (b) \quad \varpi_i = dl_i - \Gamma_{ioh}\,dx^h - pl_iA_h\,dl^h. \end{cases}$$

The vanishing of A^{iko} enables us to write

$$(3.8) \quad (a) \quad A^{kh}_i\,dl_h = A^{kh}_i(\varpi_h + \Gamma_{hor}\,dx^r), \qquad (b) \quad A^i_{kh}\,dl^h = A^i_{kh}(\varpi^h - \Gamma^h_{or}\,dx^r),$$

so that, if we put

$$(3.9) \quad (a) \quad \overset{*}{\Gamma}{}^i_{kh} = \Gamma^i_{kh} + A_k{}^{ir}\,\Gamma_{roh}, \qquad (b) \quad \overset{*}{\Gamma}{}^i_{kh} = \Gamma^i_{kh} - A^i_{kr}\,\Gamma^r_{oh},$$

the absolute differential of any vector may be written

$$(3.10) \quad \begin{cases} (a) \quad DX^i = dX^i + \overset{*}{\Gamma}{}^i_{kh}\,X^k\,dx^h + A^{ih}_k\,X^k\,\varpi_h, \\ (b) \quad DX^i = dX^i + \overset{*}{\Gamma}{}^i_{kh}\,X^k\,dx^h + A^i_{kh}\,X^k\,\varpi^h. \end{cases}$$

Applying this to the unit vector, we obtain finally

$$(3.11) \quad \varpi^i = (\delta_h{}^i + pl^iA_h)\left(dl^h + \overset{*}{\Gamma}{}^h_{or}\,dx^r\right),$$

$$(3.12) \quad \varpi_i = (\delta_i{}^h - pl_iA^h)\left(dl_h - \overset{*}{\Gamma}_{hor}\,dx^r\right).$$

4. *Angular metric.*

Following Cartan for Finsler spaces and for spaces based on the notion of area, we may, under the condition $dx^i = 0$, define the angular metric at a point as given by $d\phi^2 = \varpi_i\varpi^i$. Under the condition $dx^i = 0$, the equations (3.11) and (3.12) become respectively

$$(4.1) \quad \varpi^i = dl^i + pl^iA_h\,dl^h$$

and

$$(4.2) \quad \varpi_i = dl_i - pl_iA^h\,dl_h.$$

These expressions can be further simplified on using the definition (2.5), so that

$$(4.3)\qquad \begin{cases} (a)\quad dl^i = \dfrac{1}{\sqrt{g^p}}\, d\left(\dfrac{\partial L}{\partial u_i}\right) + \dfrac{\partial L}{\partial u_i}\, d\left(\dfrac{1}{\sqrt{g^p}}\right), \\ (b)\quad dl^i = \dfrac{1}{\sqrt{g^p}}\, d\left(\dfrac{u^i}{L}\right) + \dfrac{u^i}{L}\, d\left(\dfrac{1}{\sqrt{g^p}}\right), \end{cases}$$

$$(4.4)\qquad \begin{cases} (a)\quad dl_i = \sqrt{g^p}\, d\left(\dfrac{u_i}{L}\right) + \dfrac{u_i}{L}\, d(\sqrt{g^p}), \\ (b)\quad dl_i = \sqrt{g^p}\, d\left(\dfrac{\partial L}{\partial u^i}\right) + \dfrac{\partial L}{\partial u^i}\, d(\sqrt{g^p}). \end{cases}$$

And now. since

$$(4.5)\quad (a)\quad d\left(\frac{1}{\sqrt{g^p}}\right) = \left(\frac{1}{\sqrt{g^p}}\right)\Big\|^h dl_h, \qquad (b)\quad d\left(\frac{1}{\sqrt{g^p}}\right) = \left(\frac{1}{\sqrt{g^p}}\right)\Big\|_h dl^h,$$

with the corresponding relation for $\sqrt{g^p}$, and by (3.2) and (3.3), the ϖ^i and ϖ_i reduce simply to

$$(4.6)\qquad (a)\quad \varpi^i = \frac{1}{\sqrt{g^p}}\, d\left(\frac{\partial L}{\partial u_i}\right), \qquad (b)\quad \varpi^i = \frac{1}{\sqrt{g^p}}\, d\left(\frac{u^i}{L}\right),$$

$$(4.7)\qquad (a)\quad \varpi_i = \sqrt{g^p}\, d\left(\frac{u_i}{L}\right), \qquad (b)\quad \varpi_i = \sqrt{g^p}\, d\left(\frac{\partial L}{\partial u^i}\right),$$

so that

$$(4.8)\quad (a)\quad d\phi^2 = d\left(\frac{\partial L}{\partial u_i}\right) d\left(\frac{u_i}{L}\right) \qquad (b)\quad d\phi^2 = d\left(\frac{\partial L}{\partial u^i}\right) d\left(\frac{u^i}{L}\right)$$

$$= \frac{\partial^2 L}{\partial u_i \partial u_j}\, du_j \left(\frac{du_i}{L} - u_i \frac{dL}{L^2}\right), \qquad = \frac{\partial^2 L}{\partial u^i \partial u^j}\, du^j \left(\frac{du^i}{L} - u^i \frac{dL}{L^2}\right),$$

and from the homogeneity of the L function in the u, we shall have finally

$$(4.9)\quad (a)\quad d\phi^2 = \frac{1}{L}\frac{\partial^2 L}{\partial u_i \partial u_j}\, du_i du_j, \qquad (b)\quad d\phi^2 = \frac{1}{L}\frac{\partial^2 L}{\partial u^i \partial u^j}\, du^i du^j.$$

5. *Covariant derivation.*

Following Cartan, we may consider the elementary vector increment undergone by any vector X^i when the unit vector l^i is transported by parallelism from the point x^i to the near point $x^i + dx^i$. In that case the

absolute differential can be written in the form $X^i|_h dx^h$, where

$$(5.1)\qquad \begin{cases} (a) \quad X^i|_h = \partial_h X^i + X^i\|^m \overset{*}{\Gamma}_{moh} + \overset{*}{\Gamma}^i_{mh} X^m, \\ (b) \quad X^i|_h = \partial_h X^i - X^i\|_m \overset{*}{\Gamma}^m_{oh} + \overset{*}{\Gamma}^i_{mh} X^m \end{cases}$$

are the "covariant derivatives" of X^i in the two cases.

In particular for the unit vector, on using (3.4) and (3.5), we have

$$(5.2)\qquad \begin{cases} (a) \quad l^i|_h = \partial_h l^i - p l^i A^m \overset{*}{\Gamma}_{moh} - l^i \overset{*}{\Gamma}_{ooh} + \overset{*}{\Gamma}^{io}_{\ h} + \overset{*}{\Gamma}^{oi}_{\ h} = 0, \\ (b) \quad l^i|_h = \partial_h l^i + p l^i A_m \overset{*}{\Gamma}^m_{oh} + l^i \overset{*}{\Gamma}_{ooh} = 0, \end{cases}$$

$$(5.3)\qquad \begin{cases} (a) \quad l_i|_h = \partial_h l_i + p l_i A^m \overset{*}{\Gamma}_{moh} - l_i \overset{*}{\Gamma}_{ooh} = 0, \\ (b) \quad l_i|_h = \partial_h l_i - p l_i A_m \overset{*}{\Gamma}^m_{oh} + l_i \overset{*}{\Gamma}_{ooh} - \overset{*}{\Gamma}_{oih} - \overset{*}{\Gamma}_{ioh} = 0. \end{cases}$$

Since they will be needed we write down the expressions for the covariant derivatives of the densities $\sqrt{g^p}$ and $1/\sqrt{g^p}$

$$(5.4)\qquad \begin{cases} (a) \quad (\sqrt{g^p})|_h = \partial_h(\sqrt{g^p}) + p\sqrt{g^p}\left(A^m \overset{*}{\Gamma}_{moh} - \overset{*}{\Gamma}^m_{mh}\right) = 0, \\ (b) \quad (\sqrt{g^p})|_h = \partial_h(\sqrt{g^p}) - p\sqrt{g^p}\left(A_m \overset{*}{\Gamma}^m_{oh} + \overset{*}{\Gamma}^m_{mh}\right) = 0, \\ (a) \quad \left(\frac{1}{\sqrt{g^p}}\right)\Big|_h = \partial_h\left(\frac{1}{\sqrt{g^p}}\right) - \frac{p}{\sqrt{g^p}}\left(A^m \overset{*}{\Gamma}_{moh} - \overset{*}{\Gamma}^m_{mh}\right) = 0, \\ (b) \quad \left(\frac{1}{\sqrt{g^p}}\right)\Big|_h = \partial_h\left(\frac{1}{\sqrt{g^p}}\right) + \frac{p}{\sqrt{g^p}}\left(A_m \overset{*}{\Gamma}^m_{oh} + \overset{*}{\Gamma}^m_{mh}\right) = 0. \end{cases}$$

From the definition of the unit vector, we have

$$(5.5)\qquad \begin{cases} (a) \quad \frac{\partial l_i}{\partial x^h} = u_i\left[\frac{1}{L}\frac{\partial(\sqrt{g^p})}{\partial x^h} - \sqrt{g^p}\,\frac{1}{L^2}\frac{\partial L}{\partial x^h}\right], \\ (b) \quad \frac{\partial l^i}{\partial x^h} = w^i\left[\frac{1}{L}\frac{\partial(1/\sqrt{g^p})}{\partial x^h} - \frac{1}{\sqrt{g^p}}\,\frac{1}{L^2}\frac{\partial L}{\partial x^h}\right]. \end{cases}$$

On comparing the expressions on the right-hand sides of these with corresponding expressions deduced from (5.2), we have, on using (5.4) and rearranging,

$$(5.6)\quad (a)\quad \frac{\partial L}{\partial x^h} + L\left(\overset{*}{\Gamma}_{ooh} - p\overset{*}{\Gamma}^m_{mh}\right) = 0, \qquad (b)\quad \frac{\partial L}{\partial x^h} - L\left(\overset{*}{\Gamma}_{00h} - p\overset{*}{\Gamma}^m_{mh}\right) = 0.$$

6. *Some alternation formulae.*

If we write $X^i\|^{hk}$ for $X^i\|^h\|^k$, then

$$(6.1)\quad \begin{cases} (a) & X^i\|^{hk}-X^i\|^{kh}=X^i\|^h(l^k-pA^k)-X^i\|^k(l^h-pA^h), \\ (b) & X^i\|_{hk}-X^i\|_{kh}=X^i\|_h(l_k+pA_k)-X^i\|_k(l_h+pA_h). \end{cases}$$

Similarly, if we write $X^i|_{hk}$ for $X^i|_h|_k$, we obtain, on using (5.4) and (5.6),

$$(6.2)\quad X^i|_{hk}-X^i|_{kh}=\begin{pmatrix} i & n \\ m \end{pmatrix} \overset{*}{R}_n{}^m{}_{hk},$$

where

$$(a)\quad \begin{pmatrix} i & n \\ m \end{pmatrix}=\delta_m{}^i X^n+X^i\|^n l_m, \qquad (b)\quad \begin{pmatrix} i & n \\ m \end{pmatrix}=\delta_m{}^i X^n-X^i\|_m l^n,$$

and where

$$(6.3)\quad \begin{cases} (a) & \overset{*}{R}_m{}^i{}_{hk}=\partial_k \overset{*}{\Gamma}{}^i_{mh}+\overset{*}{\Gamma}{}^i_{mh}\|^n \Gamma_{nok}+\overset{*}{\Gamma}{}^i_{nk}\overset{*}{\Gamma}{}^n_{mh}-h|k, \\ (b) & \overset{*}{R}_m{}^i{}_{hk}=\partial_k \overset{*}{\Gamma}{}^i_{mh}-\overset{*}{\Gamma}{}^i_{mh}\|_n \overset{*}{\Gamma}{}^n_{ok}+\overset{*}{\Gamma}{}^i_{nk}\overset{*}{\Gamma}{}^n_{mh}-h|k, \end{cases}$$

with

$$(6.4)\quad \begin{cases} (a) & \overset{*}{R}_{mohk}=\partial_k \overset{*}{\Gamma}_{moh}+\overset{*}{\Gamma}_{moh}\|^n \overset{*}{\Gamma}_{nok}-h|k, \\ (b) & \overset{*}{R}_o{}^m{}_{hk}=\partial_k \overset{*}{\Gamma}{}^m_{oh}-\overset{*}{\Gamma}{}^m_{oh}\|_n \overset{*}{\Gamma}{}^n_{ok}-h|k, \end{cases}$$

the symbol $h|k$ denoting the terms which can be obtained from those written down by interchanging h and k.

Finally, we have

$$(6.5)\quad \begin{cases} (a) & X^i|_h\|^k-X^i\|^k|_h=\begin{pmatrix} i & n \\ m \end{pmatrix} \overset{*}{\Gamma}{}^m_{nh}\|^k, \\ (b) & X^i|_h\|_k-X^i\|_k|_h=\begin{pmatrix} i & n \\ m \end{pmatrix} \overset{*}{\Gamma}{}^m_{nh}\|_k. \end{cases}$$

7. *Lie derivation.*

We next introduce an operator which, when the tensors are functions of position only, has been used in deformation problems by Schouten (**10**, p. 148), and in the theory of invariant integrals by Weitzenböck (**11**, p. 374) and by Hodge (**12**, p. 274).

Let us consider the infinitesimal transformation

$$(7.1)\quad \bar{x}^i=x^i+\xi(x)\,dt$$

and let us denote by a bar the transformed quantities, so that, to the first order, we have

$$(7.2)\quad \begin{cases} (a)\quad \bar{u}_i(\bar{x}) = \left|\dfrac{\partial x}{\partial \bar{x}}\right|^{-p} u_m \dfrac{\partial x^m}{\partial \bar{x}^i} = u_i - (u_m \partial_i \xi^m - p u_i \partial_m \xi^m)\, dt, \\ (b)\quad \bar{u}^i(\bar{x}) = \left|\dfrac{\partial x}{\partial \bar{x}}\right|^{p} u^m \dfrac{\partial \bar{x}^i}{\partial x^m} = u_i + (u^m \partial_m \xi^i - p u^i \partial_m \xi^m)\, dt, \end{cases}$$

so that, if we consider a contravariant vector field $X^i(x, u)$, using the notation of Schouten (*loc. cit.*), we write

$$(a)\quad \overset{n}{d} X^i = \frac{\partial X^i}{\partial x^j}\,\delta x^j + \frac{\partial X^i}{\partial u_j}\,\delta u_j, \qquad (b)\quad \overset{n}{d} X^i = \frac{\partial X^i}{\partial x^j}\,\delta x^j + \frac{\partial X^i}{\partial u^j}\,\delta u^j$$

with δx^j and δu defined by (7.1) and (7.2) respectively, *i.e.* where $\delta u = \bar{u}(\bar{x}) - u(x) = \overset{m}{d} u$. The explicit expressions for $\overset{m}{d} u$ are therefore

$$(a)\quad \overset{m}{d} u_j = -(u_m \partial_j \xi^m - p u_j \partial_m \xi^m)\, dt,$$

$$(b)\quad \overset{m}{d} u^j = (u^m \partial_m \xi^j - p u^j \partial_m \xi^m)\, dt,$$

and, on taking account of the homogeneity of degree zero of the $X^i(x, u)$, we can write

$$(7.3)\quad \begin{cases} (a)\quad \overset{n}{d} X^i = \left[\xi^j \partial_j X^i - \dfrac{\partial X^i}{\partial u_j}\, u_m \partial_j \xi^m\right] dt = [\xi^j \partial_j X^i - X^i \|^j l_m \partial_j \xi^m]\, dt, \\ (b)\quad \overset{n}{d} X^i = \left[\xi^j \partial_j X^i + \dfrac{\partial X^i}{\partial u^j}\, u^m \partial_m \xi^j\right] dt = [\xi^j \partial_j X^i + X^i \|_j l^m \partial_m \xi^j]\, dt, \end{cases}$$

and for the two cases

$$\overset{m}{d} X^i = X^k \partial_k \xi^i\, dt,$$

so that, writing $\underset{L}{D} = \dfrac{\overset{n}{d} - \overset{m}{d}}{dt}$, we have

$$(7.4)\quad \begin{cases} (a)\quad \underset{L}{D} X^i = \xi^k X^i|_k - X^k \xi^i|_k X^i \|^m l_k \xi^k|_m, \\ (b)\quad \underset{L}{D} X^i = \xi^k X^i|_k - X^k \xi^i|_k + X^i \|_k \xi^k|_0, \end{cases}$$

$$(7.5)\quad \begin{cases} (a)\quad \underset{L}{D} X_i = \xi^k X_i|_k + X_k \xi^k|_i - X_i \|^k l_m \xi^m|_k, \\ (b)\quad \underset{L}{D} X_i = \xi^k X_i|_k + X_k \xi^k|_i + X_i \|_k \xi^k|_0. \end{cases}$$

The extension to tensors in general will be sufficiently obvious from these two cases. Let us consider in particular the metric tensor, for which we have

$$(7.6)\qquad \begin{cases} (a) & \underset{L}{D} g_{ij} = \xi^m g_{ij}|_m + g_{mj}\xi^m|_i + g_{im}\xi^m|_j - g_{ij}\|^m l_n \xi^n|_m, \\ (b) & \underset{L}{D} g_{ij} = \xi^m g_{ij}|_m + g_{mj}\xi^m|_i + g_{im}\xi^m|_j + g_{ij}\|_m \xi^m|_0. \end{cases}$$

Since we have a Euclidean connection with $g_{ij}|_m = 0$, these may be modified to the simpler expressions

$$(7.7)\qquad \begin{cases} (a) & \underset{L}{D} g_{ij} = \xi_i|_j + \xi_j|_i - 2A^m_{ij} l_n \xi^n|_m = 2[\xi_{(i}|_{j)} - A^m_{ij} l_n \xi^n|_m], \\ (b) & \underset{L}{D} g_{ij} = \xi_i|_j + \xi_j|_i + 2A_{ijm}\xi^m|_0 = 2[\xi_{(i}|_{j)} + A_{ijm}\xi^m|_0]. \end{cases}$$

"Motions" in these spaces would then be determined by examining the conditions of integrability of the equations $\underset{L}{D} g_{ij} = 0$, which has been done for Riemannian spaces by Killing (**13**, p. 167), the equations just written being generalizations of the so-called "equations of Killing". The equations $\underset{L}{D} g_{ij} = 0$ have been studied for Finsler spaces by Knebelman (**14**). We are not concerned with the problem of motions in this paper, but we shall need the above expressions for the Lie derivative of the metric coefficients. We shall also need the Lie derivative of the connection parameters $\overset{*}{\Gamma}{}^i_{jk}$, which, as has been proved for Riemannian spaces, is a tensor. For these

$$(7.8)\qquad \begin{cases} (a) & \overset{n}{d}\overset{*}{\Gamma}{}^i_{jk} = (\xi^m \partial_m \overset{*}{\Gamma}{}^i_{jk} - \overset{*}{\Gamma}{}^i_{jk}\|^m l_n \partial_m \xi^n)\, dt, \\ (b) & \overset{n}{d}\overset{*}{\Gamma}{}^i_{jk} = (\xi^m \partial_m \overset{*}{\Gamma}{}^i_{jk} + \overset{*}{\Gamma}{}^i_{jk}\|_m l^n \partial_n \xi^m)\, dt, \end{cases}$$

$$(7.9)\qquad \overset{m}{d}\overset{*}{\Gamma}{}^i_{jk} = -\left(\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \overset{*}{\Gamma}{}^i_{jm}\partial_k \xi^m + \overset{*}{\Gamma}{}^i_{mk}\partial_j \xi^m - \overset{*}{\Gamma}{}^m_{jk}\partial_m \xi^i\right) dt,$$

and we readily obtain

$$(7.10)\qquad \begin{cases} (a) & \underset{L}{D}\overset{*}{\Gamma}{}^i_{jk} = \xi^i|_{jk} + \xi^m \overset{*}{R}{}^i_{jkm} - \overset{*}{\Gamma}{}^i_{jk}\|^n l_m \xi^m|_n, \\ (b) & \underset{L}{D}\overset{*}{\Gamma}{}^i_{jk} = \xi^i|_{jk} + \xi^m \overset{*}{R}{}^i_{jkm} + \overset{*}{\Gamma}{}^i_{jk}\|_m \xi^m|_0. \end{cases}$$

By using (6.2), (6.5), (7.4) and (7.10) we now easily deduce, for the interchange of the orders of covariant derivation and Lie derivation, the

following

$$\underset{L}{D}(X^i|_h)-(\underset{L}{D}X^i)|_h=\binom{i\ n}{m}\underset{L}{D}\overset{*}{\Gamma}{}^m_{nh}. \tag{7.11}$$

8. *Theory of hypersurfaces.*

We shall consider the points of the hypersurface as given parametrically by

$$x^i=x^i(v^1, v^2, \ldots, v^{n-1}). \tag{8.1}$$

For a geometry whose element of support is a covariant vector density, the natural procedure is to regard the hypersurface as the locus of its elements of support, which is the point of view taken by Cartan and by Berwald. No other possibility presents itself in this case. For the contravariant case, however, two cases have been considered for the particular value $p=0$, and the two are still possible in the general case. The contravariant vector density can either be taken as tangential to the surface, or normal to it. In the latter case, which has been treated in detail for $p=0$ by Wegener (15), the theory is identical with that of Berwald for Cartan geometry. We shall briefly give the essentials of the theory for arbitrary p, which will apply to the covariant case, as well as for the contravariant case when the element of support is normal to the hypersurface. The theory of subordination depends upon the two equations

$$l_i x_\rho{}^i=0 \quad \text{and} \quad l_i\varpi^i=0, \tag{8.2}$$

so that we can write $\varpi^i=x_\rho{}^i\varpi^\rho$ or $\varpi_i=x_i{}^\rho\varpi_\rho$.

For a displacement along the hypersurface we write, following the authors mentioned,

$$\varpi^\rho=-a^\rho{}_\sigma dv^\sigma \quad \text{or} \quad \varpi_\rho=-a_{\rho\sigma}dv^\sigma. \tag{8.3}$$

For any vector $X^i(x, u)$, from

$$DX^i=X^i|_h dx^h+(X^i\|^h+A^{ih}_k X^k)\varpi_h, \tag{8.4}$$

on using (8.3) and the fact that $dx^h=x_\rho{}^h dv^\rho$ we obtain

$$DX^i=D_\sigma X^i dv^\sigma, \tag{8.5}$$

where D_σ is to be the generalization of the D-symbol of van der Waerden

and Bortolotti (**10**, § 9) to this geometry. Explicitly

$$(8.6)\quad \begin{cases} (a)\quad D_\sigma X^i = \partial_\sigma X^i + X^i\|^m \left(\overset{*}{\Gamma}_{mo\hbar} x_\sigma{}^\hbar - x_m{}^\rho a_{\rho\sigma}\right) \\ \qquad\qquad + X^m \left(\overset{*}{\Gamma}{}^i_{m\hbar} x_\sigma{}^\hbar - A^{i\hbar}_m x_\hbar{}^\rho a_{\rho\sigma}\right), \\ (b)\quad D_\sigma X^i = \partial_\sigma X^i - X^i\|_m \left(\overset{*}{\Gamma}{}^m_{o\hbar} x_\sigma{}^\hbar + x_\rho{}^m a^\rho_{\cdot\sigma}\right) \\ \qquad\qquad + X^m \left(\overset{*}{\Gamma}{}^i_{m\hbar} x_\sigma{}^\hbar - A^i_{m\hbar} x_\rho{}^\hbar a^\rho_{\cdot\sigma}\right). \end{cases}$$

We shall use the abbreviations

$$(8.7)\quad (a)\quad \Gamma^i_{m\sigma} = \overset{*}{\Gamma}{}^i_{m\hbar} x_\sigma{}^\hbar - A^{i\hbar}_m x_\hbar{}^\rho a_{\rho\sigma}, \qquad (b)\quad \Gamma^i_{m\sigma} = \overset{*}{\Gamma}{}^i_{m\hbar} x_\sigma{}^\hbar - A^i_{m\hbar} x_\rho{}^\hbar a^\rho_{\cdot\sigma}.$$

If we now apply this D_σ-operator to the tangent vector $x^i_{(\rho)}$, where (ρ) indicates that the index is merely ordinal, we can easily verify that the $a_{\rho\sigma}$ as defined in (8.3) is also given by

$$(8.8)\qquad a_{\rho\sigma} = l_i D_\sigma x^i_{(\rho)} = l_i \left(\frac{\partial^2 x^i}{\partial v^\rho \partial v^\sigma} + \Gamma^i_{m\sigma} x_\rho{}^m\right),$$

and if we define the connection parameters in the hypersurface by

$$(8.9)\qquad \gamma^\tau_{\rho\sigma} = x_i{}^\tau D_\sigma x^i_{(\rho)},$$

we can write

$$(8.10)\qquad D_\sigma x_\rho{}^i = \frac{\partial^2 x^i}{\partial v^\rho \partial v^\sigma} + \Gamma^i_{m\sigma} x_\rho{}^m - \gamma^\tau_{\rho\sigma} x_\tau{}^i = l^i a_{\rho\sigma}.$$

Similarly $D_\sigma l^i = -(l^i\|^\hbar + A^{io\hbar}) x_\hbar{}^\rho a_{\rho\sigma}$, the right-hand side of which, on using (2.10) and (3.4), becomes $-x_\rho{}^i a^\rho_{\cdot\sigma}$, and the left-hand side is $\partial_\sigma l^i + \Gamma^i_{o\sigma}$. The fundamental equations of hypersurface-theory are therefore

$$(8.11)\qquad D_\sigma x_\rho{}^i = l^i a_{\rho\sigma} \quad \text{and} \quad D_\sigma l^i = -x_\rho{}^i a^\rho_{\cdot\sigma},$$

which for Cartan geometry are equations (25.6) and (25.7) in Berwald (**8**), and for Finsler geometry are equations (13) in Wegener (**15**).

The second fundamental form of a hypersurface in Riemannian geometry is introduced by Bianchi (**16**, p. 450) as the variation of the first fundamental form in passing from the given hypersurface to a near one obtained by cutting off on the normal at every point a constant infinitesimal segment. Let us see if the quadratic form with the coefficients $a_{\rho\sigma}$ just obtained will admit of such a geometrical interpretation.

Following Schouten (**10**, p. 151), we write for the variation of the coefficients of the first fundamental form $\overset{a}{\delta} g_{\rho\sigma} = \left(\underset{L}{\delta} g_{ij}\right) x_\rho^i x_\sigma^j$. On using (7.6) and (8.6) and noting that ξ^i is a function of position only, the term $(\xi_i|_j + \xi_j|_i) x_\rho^i x_\sigma^j$, on quoting

$$\xi^i|_j x_\sigma^j = D_\sigma \xi^i + A_m^{ih} \xi^m x_h^\rho a_{\rho\sigma}, \tag{8.12}$$

can be modified to

$$2g_{ij} x^i_{(\rho} D_{\sigma)} \xi^j + 2A^h_{im} x_h^\mu \xi^m x^i_{(\rho} a_{\sigma)\mu},$$

and if $\xi^i = \epsilon l^i$ the second of these terms vanishes, and we are left, by (8.11), with the single term $-2\epsilon a_{\rho\sigma}$. There will then remain for consideration the terms

$$(a) \quad -2A^m_{\rho\sigma} l_n \xi^n|_m, \qquad (b) \quad 2A_{\rho\sigma m} \xi^m|_o.$$

For the case (a), the index m is a hypersurface index, in view of $A_{\rho\sigma o} = 0$, so that the term becomes $-2A^\tau_{\rho\sigma} l_n (\xi^n|_m x_\tau^m)$, and using (8.16) for the particular value $\xi^i = \epsilon l^i$, we shall have $-2\epsilon p A^\tau_{\rho\sigma} A^\pi a_{\tau\pi}$, which vanishes either when $p = 0$, or when $A^\rho = 0$.

In case (b) for the same particular value of ξ^i, we shall have

$$2\epsilon A_{\rho\sigma m} l^n \left(\frac{\partial l^m}{\partial x^n} + \overset{*}{\Gamma}{}^m_{on}\right),$$

and if dy represents the element of arc in the congruence of curves transversal to the hypersurface

$$\frac{dl^i}{dy} + \overset{*}{\Gamma}{}^i_{oo} = (\delta_j{}^i - p l^i A_j) \frac{\varpi^j}{dy},$$

and since $A_{\rho\sigma o} = 0$, it will finally become $2\epsilon A_{\rho\sigma i} \dfrac{\varpi^i}{dy}$, involving the first curvature of the curve of the congruence transversal to the hypersurface.

Let us consider in the same way the geometry of the hypersurface when the element of support is a contravariant vector density which is tangential to the hypersurface, so that $l^i = x_\rho{}^i l^\rho$. The ϖ^i will then have a component $\overline{\varpi}^i$ in the hypersurface and a component $\hat{\varpi}^i$ normal to it. If the unit normal vector to the hypersurface is denoted by n^i, we shall have, on using the expression (3.17) for ϖ^i and the fact that $n^\rho = n_o = 0$, the explicit expression for $\hat{\varpi}^i$ in the form

$$\hat{\varpi}^i = n^i \left[n_h \left(\frac{\partial^2 x^h}{\partial v^\rho \partial v^\sigma} l^\rho + \overset{*}{\Gamma}{}^h_{o\sigma} \right) \right] dv^\sigma = H_\sigma{}^i dv^\sigma. \tag{8.13}$$

Hence for any displacement along the hypersurface we can write (8.4) in the form

$$DX^i = D_\sigma X^i dv^\sigma + (X^i\|_h + A^i_{kh} X^k)\,\overline{\omega}^h$$

with

$$D_\sigma X^i = X^i|_h x_\sigma{}^h + (X^i\|_h + A^i_{kh} X)\,H_\sigma{}^h, \tag{8.14}$$

where $H_\sigma{}^h$ is so defined, as the generalization of the D-symbol (**10**, § 9) in that case.

The coefficient of X^k in the right-hand side of (8.14) is

$$\Gamma^i_{k\sigma} = \overset{*}{\Gamma}{}^i_{hk} x_\sigma{}^h + A^i_{kh} H_\sigma{}^h.$$

For the tangent vector $x_\rho{}^i$ not involving the element of support

$$D_\sigma x^i_{(\rho)} = \frac{\partial^2 x^i}{\partial v^\rho \partial v^\sigma} + \Gamma^i_{k\sigma} x_\rho{}^k.$$

The normal and tangential components of these vectors are, on denoting by n^i the unit normal vector

$$H^{\cdot\cdot i}_{\rho\sigma} = n^i n_h D_\sigma x^h_{(\rho)} \quad \text{and} \quad \gamma^\tau_{\rho\sigma} = x_i{}^\tau D_\sigma x^i_{(\rho)},$$

so that we can also write

$$H^{\cdot\cdot i}_{\rho\sigma} = D_\sigma x_\rho{}^i = \frac{\partial^2 x^i}{\partial v^\rho \partial v^\sigma} + \Gamma^i_{k\sigma} x_\rho{}^k - \gamma^\tau_{\rho\sigma} x_\tau{}^i = n^i h_{\rho\sigma}.$$

The tensor $H^{\cdot\cdot i}_{\rho\sigma}$ corresponds to the first tensor of Eulerian curvature in Riemannian geometry. To prove this let us consider the vector $X^i = x_\rho{}^i X^\rho$, which is tangential to the hypersurface. It can be transported from the element $(x+dx,\ u+du)$ to the element $(x,\ u)$ either (i) as a vector of the general space, or (ii) as a vector of the hypersurface. The vectorial difference between the two results will be $DX^i - x_\rho{}^i DX^\rho$, which we shall now express in terms of the $H^{\cdot\cdot i}_{\rho\sigma}$.

On quoting

$$D_\sigma X^i = D_\sigma (x_\rho{}^i X^\rho) = X^\rho H^{\cdot\cdot i}_{\rho\sigma} + x^{\cdot i} D_\sigma X^\rho$$

and

$$DX^\rho = D_\sigma X^\rho dv^\sigma + (X^\rho\|_\sigma + A^\rho_{\tau\sigma} X^\tau)\,\overline{\omega}^\sigma.$$

we shall obtain, on simplification,

$$DX^i - x_\rho{}^i DX^\rho = X^\rho H^{\cdot\cdot i}_{\rho\sigma} dv^\sigma + n^i (n_h A^h_{\rho\sigma})\, X^\rho \overline{\omega}^\sigma.$$

If the X^i is taken to be the unit vector l^i, we have

$$\varpi^i - x_\rho{}^i \bar{\varpi}^\rho = H_{00}^{\cdot\cdot i} dv^\sigma, \tag{8.15}$$

so that, on remarking that the right-hand side is $\hat{\varpi}^i$ and comparing with (8.13), we have $H_{0\sigma}^{\cdot\cdot i} = H_\sigma{}^i$. If, further, the dv^σ is in the direction of the element of support, and if we divide by the element of length ds, so that $dv^\sigma/ds = l^\sigma$, then, from (8.15), we have

$$\frac{\varpi^i}{ds} - x_\rho{}^i \frac{\bar{\varpi}^\rho}{ds} = H_{00}^{\cdot\cdot i};$$

the right-hand side is therefore the normal curvature vector, the normal curvature being

$$n_i \left\{ x_{00}^i + \overset{*}{\Gamma}{}_{00}^i \right\}.$$

On applying the D_σ-operator to the relation $x_j{}^\rho n^j = 0$, we have

$$x_j{}^\rho D_\sigma n^j + n^j D_\sigma x_j{}^\rho = 0.$$

On contracting with $x_\rho{}^i$ and remarking that, since $D_\sigma n^i$ is perpendicular to n^i, we have

$$D_\sigma n^i = x_\rho{}^i x_j{}^\rho D_\sigma n^j = -n^j x_\rho{}^i D_\sigma x_j{}^\rho.$$

Using again

$$D_\sigma x_j{}^\rho = g^{\rho\tau} g_{jk} D_\sigma x_\tau{}^k = g^{\rho\tau} g_{jk} n^k h_{\tau\sigma},$$

we have $D_\sigma n^i = -x_\rho{}^i g^{\rho\tau} h_{\tau\sigma}$, so that the fundamental formulae of hypersurface theory become

$$D_\sigma x_\rho{}^i = n^i h_{\rho\sigma} \quad \text{and} \quad D_\sigma n^i = -x_\rho{}^i h^\rho_{\cdot\sigma}. \tag{8.16}$$

Relations between the curvature tensors of the general space and the hypersurface are then obtained by further derivation and alternation of the sets of equations (8.11) and (8.16). These would constitute generalizations of the equations of Gauss, Codazzi, Kuhne and Ricci. The coefficients of the second fundamental form $h_{\rho\sigma}$ in this case do not coincide with the expression for the variation of the coefficients of the first fundamental form. In fact a calculation similar to that carried out after (8.12) for a displacement of amount ϵn^i along the normal to the hypersurface gives

$$\overset{a}{\delta} g_{\rho\sigma} = -2\epsilon [h_{(\rho\sigma)} + g_{ik} x_{(\rho}^i H_{\sigma)}^n A_{mn}^k n^m - A_{\rho\sigma\cdot} n^j |_\bullet].$$

9. *Transformation by dilatation.*

This is a topic treated by Cartan for a geometry based on the notion of area. Let us consider the existence of such transformations in case (*a*), and in case (*b*) when the element of support is normal to the hypersurface.

Let us cut off on the normal to the hypersurface at the element (x, u) a segment of constant length ϵ, so that the coordinates of the point thus obtained are $\bar{x}^i = x^i + \delta x^i$, with $\delta x^i = \epsilon l^i$. Let dx^i be a displacement on the hypersurface itself, so that $l_i dx^i = 0$. We want to consider whether, if we pass a plane parallel to the tangent plane of the hypersurface at x^i through the point $\bar{x}^i$, the planes so obtained can be tangent to one and the same hypersurface. In symbols we have to examine whether the equation $\delta l_i . dx^i + l_i \delta dx^i = 0$ can be satisfied in these geometries, where δl_i is to be determined from $\varpi_i(\delta) = 0$, so that

$$\delta l_i = \overset{*}{\Gamma}_{ior} \delta x^r = \epsilon \overset{*}{\Gamma}_{ior} l^r \quad \text{and} \quad \delta dx^i = d\delta x^i = \epsilon dl^i.$$

Taking account of these, the condition becomes

$$l_i dl^i + \overset{*}{\Gamma}_{ior} l^r dx^i = 0.$$

When we use the symmetry of $\overset{*}{\Gamma}_{ior}$ in i and r and the relation $l_i dl^i = l^i dl_i$, the condition becomes $l^i(dl_i - \overset{*}{\Gamma}_{ior} dx^r) = 0$; on using (3.16) and the fact that $\varpi_0 = 0$, we get the condition in the final form $pA^i \varpi_i = 0$. Such a transformation by dilatation exists therefore for $p = 0$, so that the condition always holds for Finsler spaces. For Cartan spaces the condition, in the form $A^i \varpi_i = 0$, differs from the equation given by Cartan (4, p. 30) only by the factor $\sqrt{g}/L$. The condition is satisfied for all values of p provided $A^i = 0$.

10. *Theory of curves.*

It has been shown that, for Finsler and Cartan spaces a theory of curves is possible with appropriate Frenet formulae (3, 4). For a Finsler space this theory is possible by regarding the curve as the locus of its line elements of support, whereas for Cartan space every point of the curve must have associated with it the normal hypersurface element. The extension of these two cases to geometries with arbitrary values of p is obvious.

A fundamental question in curve-theory is that of finding whether the curves along which the tangent vector is autoparallel (with respect to

the "connection" defined in the space) are also the curves along which the distance between any two points is extremal. For a Finsler space the two sets of curves coincide. For a Cartan space, however, the autoparallel curves only coincide with the extremal curves under certain conditions to be imposed on the $\vec{A}$ vector.

Let us examine whether, for the contravariant case in general, the coincidence of autoparallels and extremals holds as it does in Finsler spaces. If the curve is represented parametrically by $x = x(t)$, the variation ds of the length element may be written

$$\frac{\delta(ds/dt)}{ds/dt} = \frac{\delta(ds)}{ds} = \tfrac{1}{2}\left(\underset{L}{D}\, g_{ij}\right) l^i l^j.$$

On using (7.7), modifying and noting that $l^i l^j A_{ijm} = pA_m$, we have $\delta(ds) = 0$ provided that

$$(l_i + pA_i)\, \xi^i|_o = 0. \tag{10.1}$$

Since $\xi^i|_o = \dfrac{D\xi^i}{ds} - A^i_{jh}\, \xi^j \dfrac{\varpi^h}{ds}$ the equation (10.1) can be modified to

$$\frac{d}{ds}\left[(l_i + pA_i)\, \xi^h\right] - \xi^h \left[(\delta_h{}^i + pl_h A^i + pA_m A^{mi}_h)\, \frac{\varpi_i}{ds} + p\, \frac{DA^i}{ds}\right] = 0. \tag{10.2}$$

If we are considering the integral $\int ds$ between fixed points, the ξ^i will vanish at the extremities, and the variation of the integral will therefore vanish for all values of ξ^i provided that

(i) the curve is an autoparallel line, and

(ii) the vector $\vec{A}$ remains parallel to itself along the curve.

These conditions correspond to those stated by Cartan for a geometry based on the notion of area.

For a Finsler geometry the equation (10.2) reduces to

$$\frac{d}{ds}\,(l_i\, \xi^i) - \xi^i\, \frac{\varpi_i}{ds} = 0,$$

so that between two fixed points the variation of $\int ds$ will vanish for all ξ^i if the curve is an autoparallel line.

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THE EFFECT OF INSULATING AND CONDUCTING SHIELDS AND PARTLY STOPPED-OFF ELECTRODES ON CURRENT DISTRIBUTION IN ELECTROLYTIC CELLS

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1. *Introduction.*

The ever-increasing application of electrodeposition for wear resistance and salvaging of worn or over-machined parts has made it imperative to study current distribution on anodes and cathodes, especially the latter, and the effects of various devices adopted in order to obtain local control of thickness of deposit. The devices are mostly "tricks of the trade", instinctively adopted or taught by experience in electrodeposition, such as special arrangements of electrodes in relation to one another, interposing insulating shields to reduce current densities in certain areas, or conducting shields or "thieves", etc., but considerable disagreement exists about the effectiveness of the various arrangements. The disagreement is no doubt partly due to different solutions and electrodes employed which produce varying polarisation effects, but the mathematical analysis of the "primary distribution", *i.e.* on the original electrodes which are to be plated, even without considering polarisation, is of great value in clarifying the issues.

In the United States C. Kasper has taken upon himself this work and in a series of papers (1) has dealt with a number of arrangements of anodes and cathodes, given a comprehensive historical survey, and has also treated cases with polarised electrodes. No detailed reference will therefore be made in the present paper to the literature except where definite results have been used or methods are analogous.

Current distribution on the *inside* of a circular cylinder (regarded as cathode, say) of unit height and radius, bounded at top and bottom by insulators, will be studied when an internal electrode of circular section

(insoluble anode, say) with axis parallel to that of the outer cylinder is inserted and solution is poured into the space between. The general assumptions about homogeneity of solution, electrodes, conductivities, etc., necessary to make the subject suitable for the mathematical apparatus employed here are given in detail in Kasper's papers and in any case are known to mathematicians, so they will not be enunciated here. All arrangements considered are actually encountered in practice except that here the interfering shields are placed in special positions and are assumed to be infinitely thin.

The method used is that of conformal mapping and invariably consists in mapping the arrangement described and lying in the complex z-plane upon a part of the w-plane so that the internal electrode becomes a circle concentric with the origin lying within the w-unit circle, and that any insulating shields or stopped-off portions (regarded as slots in the domain) are mapped upon radial slots; the outer electrode is mapped upon the unit circle and any conducting shields are mapped upon part of this. All maps are *schlicht*. If $w = f(z)$ is the mapping function, the current distribution is taken essentially as $i = |f'(z)|_{|z|=1}$, thus absorbing factors relating to total current, etc. (*i.e.* taking total current through the *outer* electrode as 2π amps), which simplifies comparison. Where conducting shields absorb current this has been adjusted by multiplying by a suitable factor, to which reference will be made.

The cases here described incidentally serve as a preliminary for the treatment of the case of two (or more) thin flat plates (anode and cathode) in a rectangular enclosure (tank), which will be dealt with in another paper. The analogous case for an infinitely large tank has been treated by the late A. E. H. Love (2), whose work has been of great assistance to the present author. It may also be mentioned that the conformal maps given may be employed for the treatment of the problem of flexure and torsion of *hollow* circular shafts with various types of cracks, a problem solved for a solid shaft with two diametrically opposite cracks in a recent paper by Wigglesworth (3). The electrostatic problems solved are apparent, and nearly all conformal maps given are new, as is also the simple method for the transformation and reduction of various special hyperelliptic integrals. Some parts are closely related to cases considered by Moulton (4), but the difference and advance beyond these will be obvious.

Once and for all let it be said that, where multivalued functions are used, suitable barriers are introduced to avoid confusing their branches; in some instances these are shown in the diagrams, but where this is not done it is not difficult to decide how they have been chosen.

2. *Plain concentric and excentric electrodes.*

These cases are well known but required here for purposes of comparison. In the concentric case the map is the identity $w = z$, and the

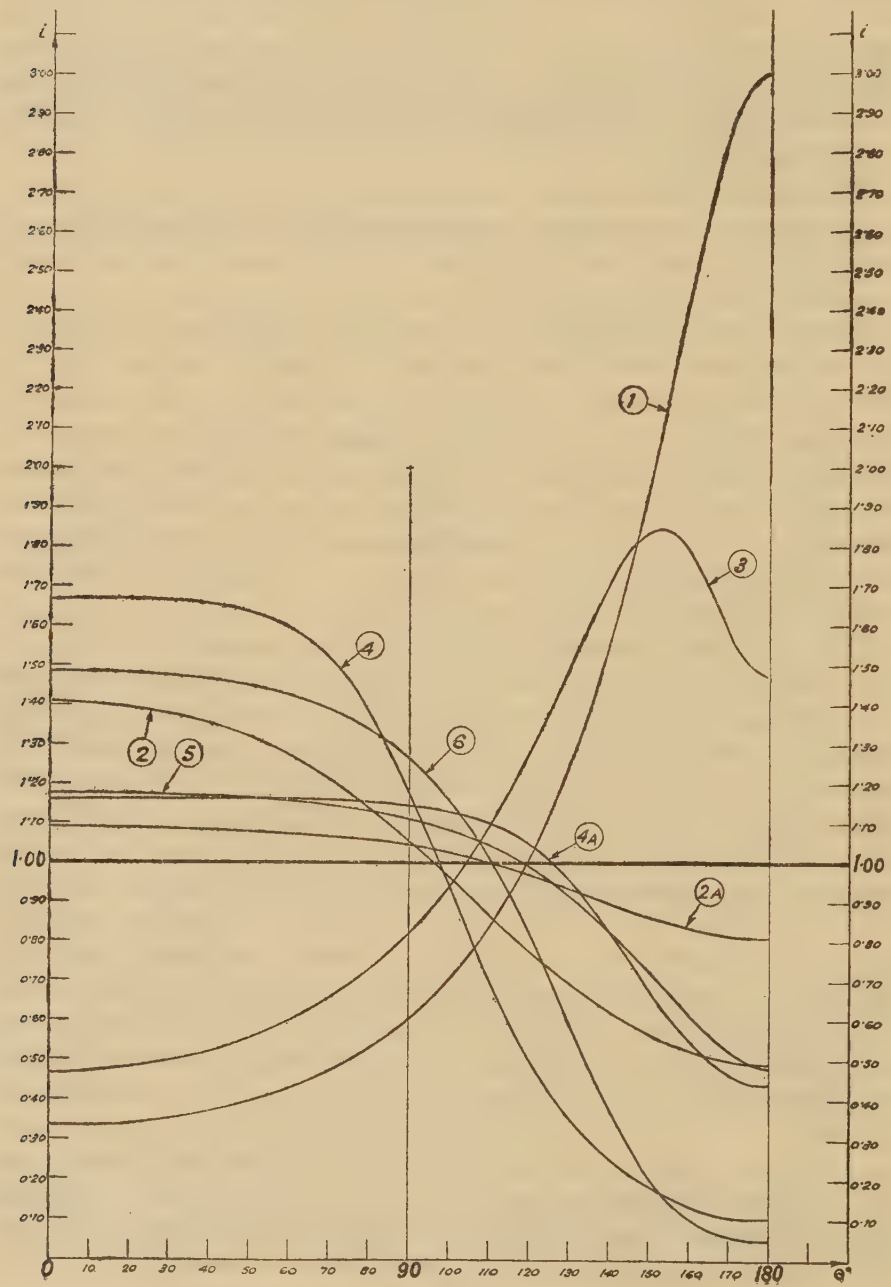

Fig. 6.

current distribution is uniform $|w'|_{|z|=1}=1$. This is represented by the horizontal line through the ordinate $+1$ in the set of curves (Fig. 6).

In the excentric case with the internal electrode displaced in the direction of negative real numbers, if $0>-a=-e^{-g}>-1$ is the right-hand intersection of the common orthogonal circle of the unit circle, the internal electrode circle, and the axis of real numbers (Fig. 1) with the last named, the mapping function is the well-known linear transformation

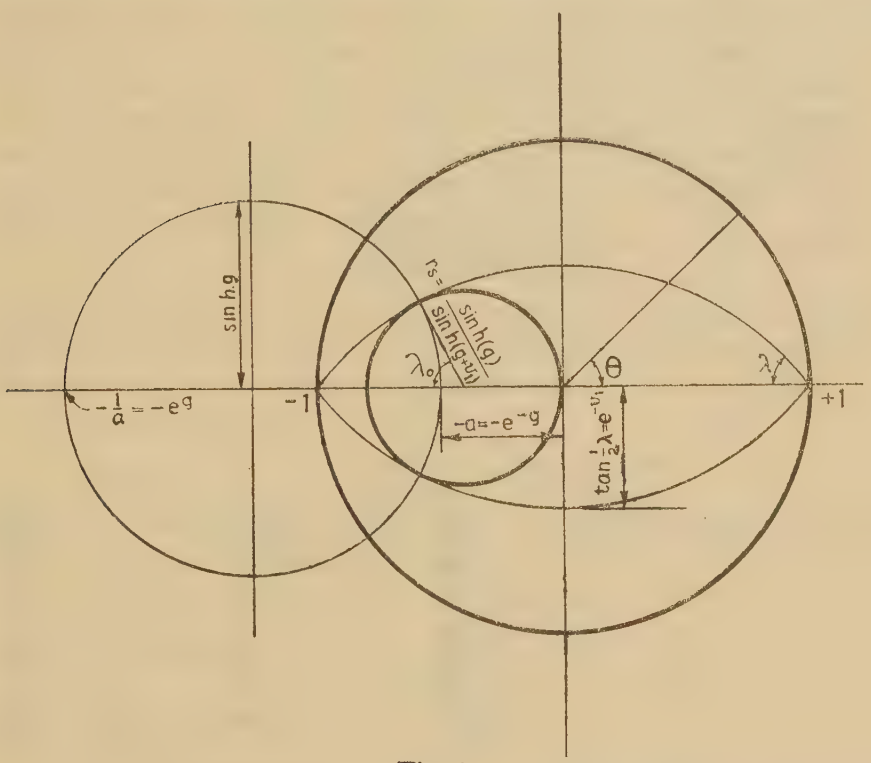

Fig. 1.

of the unit circle into itself, so that $-a$ is transformed into zero:

$$w=\frac{z+a}{1+az}.$$

The potential difference between the electrodes is $v_1=\log\tan\frac{1}{2}\lambda$, with v_1 equal to the (purely imaginary) angle of intersection of the internal with the outer electrode circle (5). The current density is given by

$$i_1=\frac{1-a^2}{1+2a\cos\theta+a^2}, \tag{2.1}$$

where θ is the polar angle in the z-plane. This is plotted in curve (1) for $a = \frac{1}{2}$. In all curves $\lambda = 27{\cdot}30°$ approx. has been chosen, which was convenient for calculation, as will be seen later.

3. *Concentric case with part of internal electrode "stopped-off".*

The arrangement is shown in Fig. 2, the internal electrode having a radius $q = \exp(-\pi K_0'/K_0)$, where K_0 and K_0' are real numbers greater

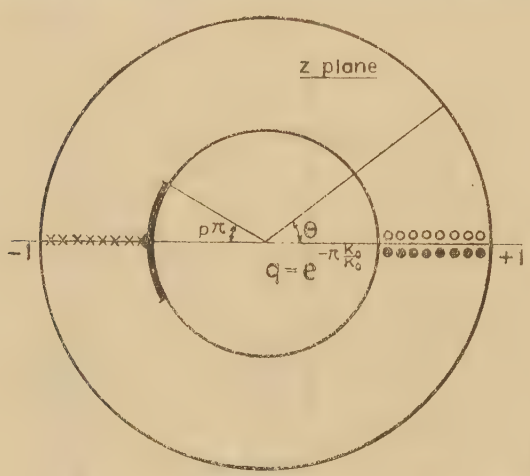

Fig. 2.

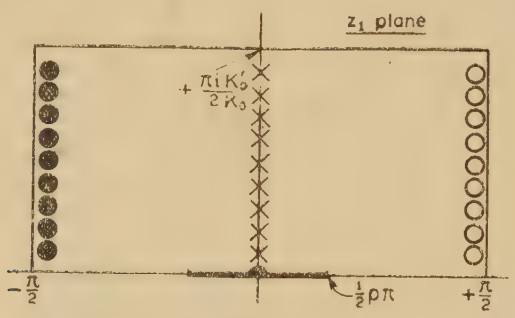

Fig. 2A.

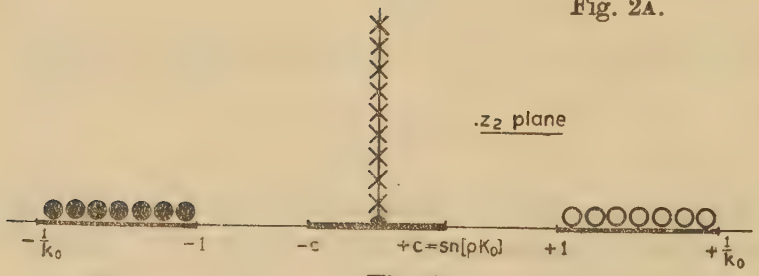

Fig. 2B.

than 0. The portion $\pi(1-\rho)$ to $\pi(1+\rho)$, with $0 < \rho < 1$, is stopped off, *i.e.* insulated with wax or paint in practice. The function (ring slotted

from q to 1, principal branch of logarithm)

$$z_1 = \tfrac{1}{2}i \log z + (\pi i K_0'/K_0) + \tfrac{1}{2}\pi$$

maps the ring $q \leqslant |z| \leqslant 1$ upon the rectangle shown in Fig. 2A. This is mapped upon the upper z_2-half plane in Fig. 2B by

$$z_2 = \mathrm{sn}\left[\frac{2K_0}{\pi} z_1 \,\middle|\, k_0\right],$$

where the modulus k_0 is formed with the quantity q given above. The stopped-off portion of the internal electrode is the part of the axis of real numbers from $-c = -\mathrm{sn}[\rho K_0|k_0]$ to $+c = +\mathrm{sn}[\rho K_0|k_0]$. This half-plane is further mapped upon a z_3-half-plane as shown in Fig. 2C, the stopped-off part now being a short slot cut into the axis of imaginary numbers. The length of this is determined quite easily, but has no practical significance in the present connection. The map is effected by

$$z_3 = (z_2^2 - \mathrm{sn}^2[\rho K_0])^{\frac{1}{2}}/\mathrm{cn}[\rho K_0|k_0],$$

where the upper or lower z_3-half-plane is obtained according to the sign selected for the square root when z_2 has real positive values greater than c. The significant values are given in the diagram.

It is next noted that

$$z_3 = \mathrm{sn}\left[\frac{2K_1}{\pi} z_4 \,\middle|\, k_1\right]$$

maps the rectangle shown in Fig. 2D upon the previous z_3-half-plane with the correspondence indicated, and finally,

$$z_4 = \tfrac{1}{2}i \log z_5 + (\pi i K_1'/2K_1) + \tfrac{1}{2}\pi$$

maps this rectangle upon the ring

$$q_1 = \exp(-\pi K_1'/2K_1) \leqslant |z_5| = |w| \leqslant 1$$

indicated in Fig. 2E.

The modulus k_1 is connected with the original modulus k_0 by the relation

$$k_1 = k_0\, \mathrm{cn}[\rho K_0|k_0]/\mathrm{dn}[\rho K_0|k_0]. \tag{3.10}$$

The relation between the z- and w-rings is therefore

$$\mathrm{cn}^2[\rho K_0|k_0]\,\mathrm{sn}^2\left[\frac{iK_1}{\pi}\log w + K_1 + iK_1' \,\middle|\, k_1\right] = \mathrm{sn}^2\left[\frac{iK_0}{\pi}\log z + K_0 + iK_0' \,\middle|\, k_0\right] - \mathrm{sn}^2[\rho K_0|k_0], \tag{3.1}$$

which reduces to

$$\mathrm{cn}\left[\frac{iK_1}{\pi}\log w \,\middle|\, k_1\right] = \mathrm{cn}\left[\frac{iK_0}{\pi}\log z \,\middle|\, k_0\right]. \tag{3.11}$$

The current density on the outer electrode is now given by*

$$(3.2) \qquad i_2 = \frac{K_0}{K_1}\left\{\frac{1-k_0^2\,\mathrm{sn}^2[K_0\theta/\pi|k_0]}{1-k_0^2(\mathrm{cn}^2\rho K_0/\mathrm{dn}^2\rho K_0)\,\mathrm{sn}^2[K_0\theta/\pi|k_0]}\right\}^{\frac{1}{2}},$$

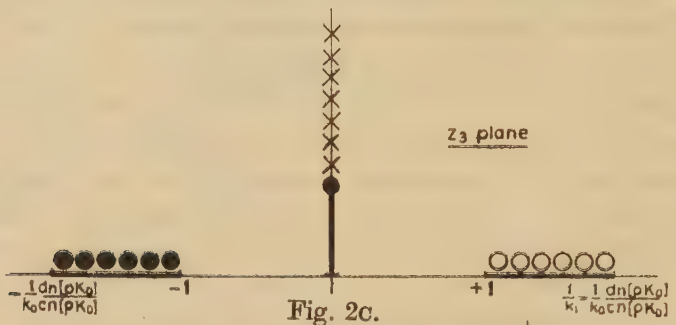

Fig. 2C.

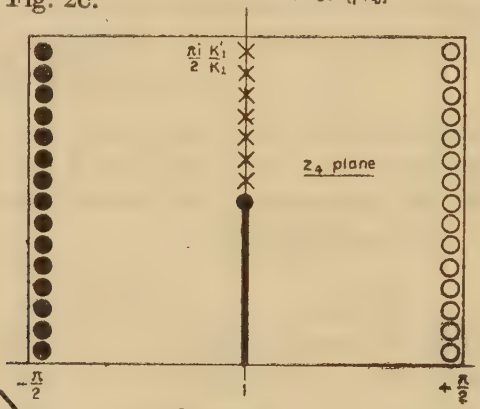

Fig. 2D.

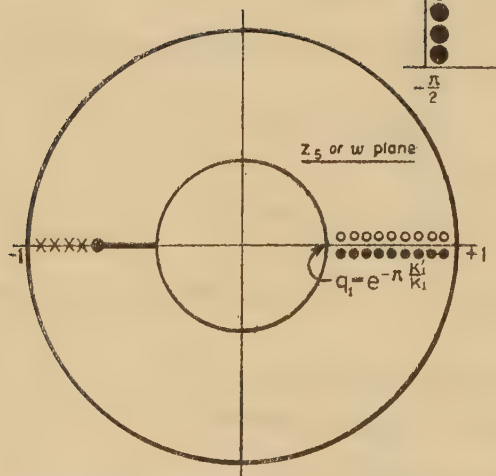

Fig. 2E.

* This arrangement is used instead of writing the function dn in the numerator because calculations were made by the aid of Legendre's tables (see below) and the practical reader realises that then only one set of function values requires to be extracted from them; to the mathematician it will make no difference.

with θ as polar angle in the z-plane. This is plotted in curve (2) for $\rho = \frac{1}{2}$ and $q = \cdot 2429$, which make $k_0 = \sin 83°$. Thus half the internal electrode is stopped off in the graphed example. Plotting is restricted to the interval $0 < \theta < \pi$.

4. *Excentric case with part of internal electrode "stopped-off".*

This case follows immediately from the last; in practice it is only of interest to stop off the internal electrode on that side towards which it

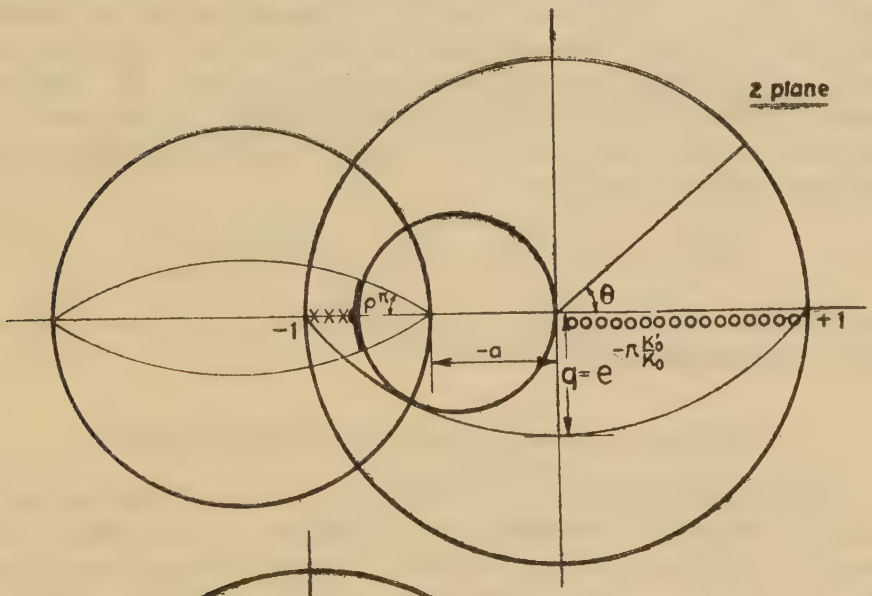

Fig. 3A.

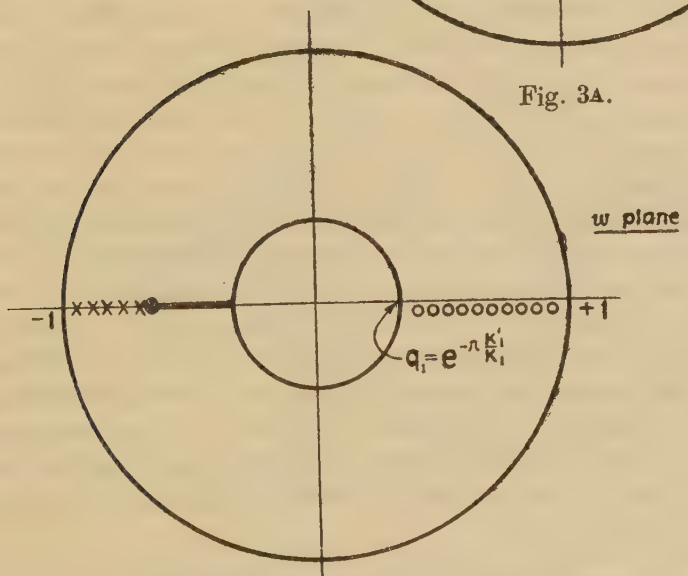

Fig. 3B.

is displaced from the centre. It is only necessary in (3.1) to substitute $(z+a)/(1+az)$ for z; the current distribution is then given by

$$(4.1)\quad i_3 = \frac{K_0}{K_1}\frac{(1-a^2)}{(1+2a\cos\theta+a^2)}\left\{\frac{1-k_0^2\,\mathrm{sn}^2\,[K_0\bar{\theta}/\pi|\,k_0]}{1-k_0^2(\mathrm{cn}^2\rho K_0/\mathrm{dn}^2\rho K_0)\,\mathrm{sn}^2\,[K_0\bar{\theta}/\pi|\,k_0]}\right\}^{\frac{1}{2}}.$$

Here K_0, K_0', K_1 and k_0 refer to the concentric position of the electrodes, so that, with the notation of § 2, $-\pi K_0'/K_0 = \log\tan\frac{1}{2}\lambda = -v_1$. The plotting of curve (3) has been done with the same constants as for curve (2), and with $a = \frac{1}{2}$. The internal electrode is stopped off in the manner indicated in Fig. 3A; in the diagram the stopping off is equivalent to half the equivalent concentric electrode being stopped off. Fig. 3B shows the map produced by the function mentioned. The angle θ is the polar angle of the z-plane of the electrode arrangement, and the angle $\bar{\theta}$ is related to this by

$$(4.2)\qquad \cot\bar{\theta} = \cot\theta\,\frac{1+a^2}{1-a^2} + \frac{2a}{(1-a^2)\sin\theta} = \frac{\cosh g\cos\theta+1}{\sinh g\sin\theta},$$

the previous notation being used. When plotting (4.1) this must be taken into consideration.

5. *Concentric case with insulating shield between electrodes.*

The arrangement of this case is shown in Fig. 4A; an insulating shield is placed between the inner and outer electrodes and concentric with these. In order to determine the effect of this shield upon current distribution on the outer electrode it is required to map the ring in Fig. 4A with the slot through $-q$ and from $\pi(1-\rho)$ to $\pi(1+\rho)$ upon the ring in Fig. 4B so that the image of the shield is a slot lying along a radial line. In general, *i.e.* unless the shield has a special position relative to the two electrodes, this problem involves hyperelliptic integrals, which may reduce to elliptic integrals by transformations of second or higher orders. If, as is shown in Fig. 4A, the shield lies symmetrically with respect to the two electrodes, a single quadratic transformation suffices; this was already used above in § 3 (in passing from the z_2 to the z_3 plane), so that the transformations for the case now under consideration follow immediately from the previous results. If we denote the radius of the *insulating shield* in Fig. 4A by the symbol $q = \exp(-\pi K_0'/K_0)$, exactly as for the *internal electrode* in Fig. 2A and § 3, and therefore the radius of the *present* internal electrode in Fig. 4A by

$$q^2 = \bar{q} = \exp(-2\pi K_0'/K_0) = \exp(-\pi\bar{K}_0'/\bar{K}_0),$$

as shown in the diagram, the formulae (3.10) and (3.11) give the mapping relation between Figs. 4A and 4B, first between the outer rings $q \leqslant |z_0| \leqslant 1$

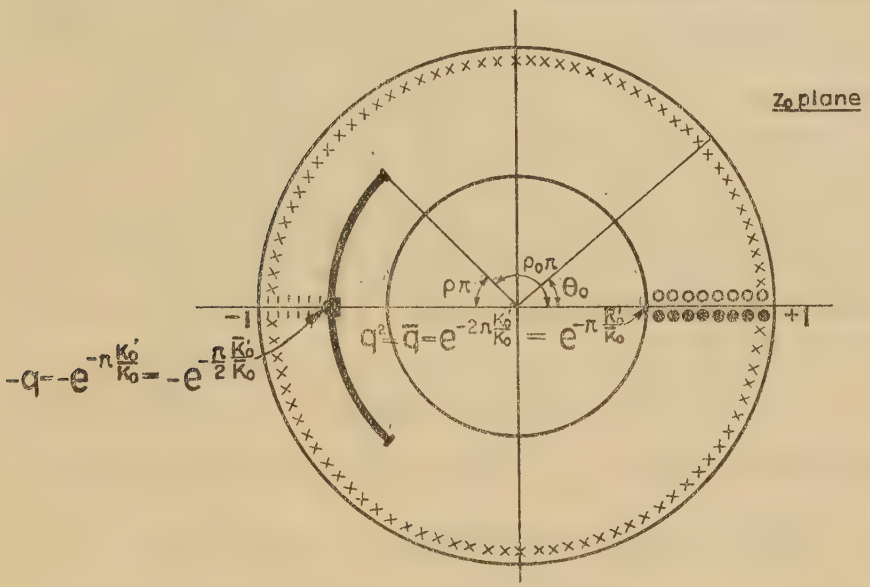

Fig. 4A.

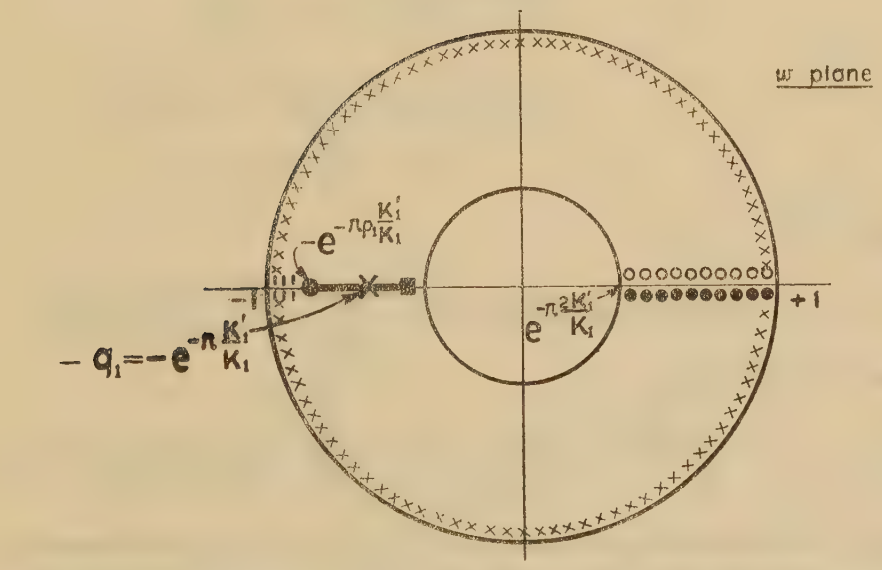

Fig. 4B.

and $q_1 \leqslant |w| \leqslant 1$, respectively, and then by Schwarz' inversion principle the inverses of them in the circles $|z_0| = q$ and $|w| = q_1$, respectively, into

one another. It now becomes evident that the image of the insulating shield in the w-ring in this special case is part of a radial slot which lies symmetrically with respect to the circle $|w| = q_1$. It follows, moreover, that a stopped-off electrode of radius q has the same effect as an insulating shield put into the position of the stopped-off portion together with a plain electrode of radius q^2, the outer electrode having radius unity.

For purposes of comparison curve (4), which appertains to the present arrangement, has been plotted for $q^2 = \cdot 2429$ and $\rho = \frac{1}{2}$; for this value the formulae (3.10), (3.11) and the current distribution formula (3.2) is unsuitable since k_0 has then a value too near unity: a Landen transformation by means of which $\bar{K}_0$, $\bar{K}_0'$, $\bar{k}_0$ are introduced is therefore expedient; because the imaginary period is halved, these numbers are related to K_0, K_0', k_0 by

$$(5.01)\quad 2\bar{K}_0 = (1+k_0')K_0;\quad \bar{K}_0' = (1+k_0')K_0;\quad \bar{k}_0' = 2\sqrt{k_0'}/(1+k_0').$$

From these follows, *e.g.*, $k_0 = 2\sqrt{\bar{k}_0}/(1+\bar{k}_0)$, etc., and it follows that

$$(5.1)\quad \operatorname{cn}\left[\frac{iK_1}{\pi}\log w \,\middle|\, k_1\right] = \operatorname{cn}\left[\frac{iK_0}{\pi}\log z_0 \,\middle|\, k_0\right] \quad [\text{from } (3.11)],$$

$$= \operatorname{cn}\left[\frac{(1+\bar{k}_0)i\bar{K}_0}{\pi}\log z_0 \,\middle|\, \frac{2\sqrt{\bar{k}_0}}{1+\bar{k}_0}\right]^*$$

$$= \frac{\operatorname{cn}\left[\frac{i\bar{K}_0}{\pi}\log z_0 \,\middle|\, \bar{k}_0\right]\operatorname{dn}\left[\frac{i\bar{K}_0}{\pi}\log z_0 \,\middle|\, \bar{k}_0\right]}{1+\bar{k}_0\operatorname{sn}^2\left[\frac{i\bar{K}_0}{\pi}\log z_0 \,\middle|\, k_0\right]},$$

with

$$(5.11)\quad k_1 = k_0\operatorname{cn}[\rho K_0|k_0]/\operatorname{dn}[\rho K_0|k_0] = \frac{2\sqrt{\bar{k}_0}\operatorname{cn}[\rho\bar{K}_0|\bar{k}_0]\operatorname{dn}[\rho\bar{K}_0|\bar{k}_0]}{(1+\bar{k}_0)(1-\bar{k}_0\operatorname{sn}^2[\rho\bar{K}_0|\bar{k}_0])},$$

and the distribution

$$(5.2)\quad i_4 = \frac{(1+\bar{k}_0)\bar{K}_0}{K_1}\left\{1+\frac{4\bar{k}_0(1-\bar{k}_0)^2\operatorname{sn}^2\rho\bar{K}_0\operatorname{sn}^2[\bar{K}_0\theta_0/\pi|\bar{k}_0]}{(1-\bar{k}_0\operatorname{sn}^2\rho\bar{K}_0)^2(1-\bar{k}_0\operatorname{sn}^2[\bar{K}_0\theta_0/\pi|\bar{k}_0])^2}\right\}^{-\frac{1}{2}},$$

where the functions are obtained from (3.2) by the same transformation. This expression is much easier to handle than without the transformation.

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* See, *e.g.*, Cayley, *Elliptic Functions* (2nd ed., 1895), 185 and 180.

6. *Concentric case with radial conducting shield between electrodes.*

In this arrangement, indicated in Fig. 5A, a conducting shield is placed between the electrodes along the axis of negative real numbers, between

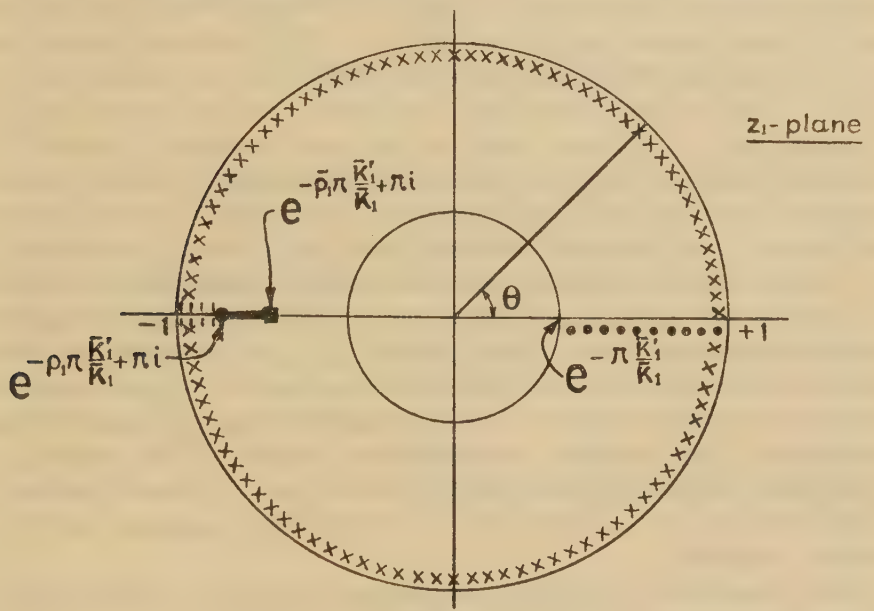

Fig. 5A.

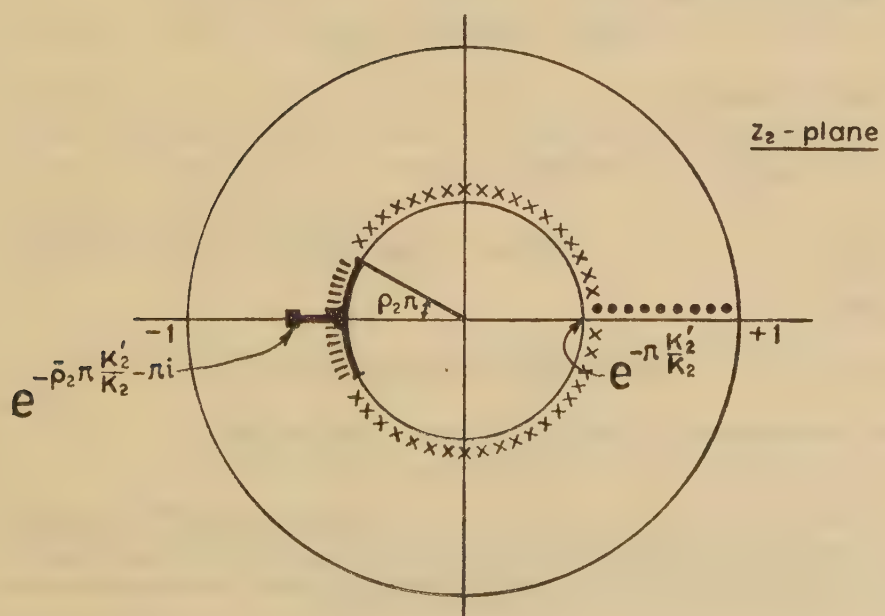

Fig. 5B.

the points $-\exp(-\rho_1 \pi \bar{K}_1'/\bar{K}_1)$ and $-\exp(-\bar{\rho}_1 \pi \bar{K}_1'/\bar{K}_1)$ with

$$0 < \rho_1 < \bar{\rho}_1 < 1,$$

at right angles to the plane of the paper. This conductor is assumed to be at the same potential as the *outer* electrode, which is achieved in practice by connecting the shield and the outer electrode by a wire external to the cell. This is a device frequently employed to reduce current density on certain parts of the outer electrode, but more frequently the shield is placed across the axis of negative real numbers concentric to the electrodes, as was the insulating shield in Fig. 4A. This case will be discussed in the next section; the present development serves as a necessary preliminary.

The general case of Fig. 5A involves, of course, hyperelliptic integrals (as does that of Fig. 4A), so, to restrict the calculations to elliptic integrals, special assumptions on the magnitude of $(\bar{\rho}_1 - \rho_1)$ must be made. The selection is here made in such a manner that one quadratic transformation of the hyperelliptic integral reduces it to an elliptic one, and the practical significance of the selection is geometrically evident. The number ρ_1 is arbitrarily assigned within the above limits, and $\bar{\rho}_1$ remains undetermined until it is geometrically clear how it should be chosen to fulfil the reducing condition.

The arrangement of Fig. 5A is first mapped upon that of Fig. 5B, the relations between the rings $\exp(-\pi \bar{K}_1'/\bar{K}_1) \leqslant |z_1| \leqslant 1$ and

$$\exp(-\pi K_2'/K_2) \leqslant |z_2| \leqslant 1$$

being determined by

$$(6.10) \quad \operatorname{cn}\left[\frac{i\bar{K}_1}{\pi} \log z_1 + \bar{K}_1 \,\middle|\, \bar{k}_1\right] = \operatorname{cn}[\rho_1 i \bar{K}_1' | \bar{k}_1] \operatorname{cn}\left[\frac{iK_2}{\pi} \log z_2 + K_2 + iK_2' \,\middle|\, k_2\right],$$

where

$$(6.11) \quad k_2 = \bar{k}_1 \operatorname{cn}[\rho_1 i \bar{K}_1' | \bar{k}_1] / \operatorname{dn}[\rho_1 i \bar{K}_1' | \bar{k}_1].$$

The correspondence of points is evident in the diagrams*.

* Here, again, the functional relations are not reduced to their simplest form because it is easier to follow through the mapping from one configuration to another if they are left in the form given; furthermore this form indicates more clearly dependence of mapping not only upon z_1, z_2, $\bar{K}_1$, $\bar{K}_1'$, K_2, K_2', but also upon ρ_1.

The z_2-ring is then mapped upon the z_3-ring of Fig. 5c by means of the relation

$$(6.20)\quad \operatorname{cn}\left[\frac{iK_2}{\pi}\log z_2+K_2+iK_2'\,\middle|\,k_2\right] = \operatorname{cn}\left[(1-\bar{\rho}_2)iK_2'\,|\,k_2\right]\operatorname{cn}\left[\frac{iK_3}{\pi}\log z_3+K_3+iK_3'\,\middle|\,k_3\right]$$

with

$$(6.21)\quad k_3 = k_2\operatorname{cn}\left[(1-\bar{\rho}_2)iK_2'\,|\,k_2\right]/\operatorname{dn}\left[(1-\bar{\rho}_2)iK_2'\,|\,k_2\right] = \bar{k}_1\operatorname{cn}\left[\bar{\rho}_1 i\bar{K}_1'\,|\,\bar{k}_1\right]/\operatorname{dn}\left[\bar{\rho}_1 i\bar{K}_1'\,|\,\bar{k}_1\right],$$

the latter equality being obtained from the addition theorem.

The number $\bar{\rho}_2$ is not fixed until $\bar{\rho}_1$ is found from the conditions imposed above. The converse of relation (6.20) is

$$(6.201)\quad \operatorname{cn}\left[\frac{iK_3}{\pi}\log z_3+K_3+iK_3'\,\middle|\,k_3\right] = \operatorname{cn}\left[\bar{\rho}_3 K_3\,|\,k_3\right]\operatorname{cn}\left[\frac{iK_2}{\pi}\log z_2+K_2+iK_2\,\middle|\,k_2\right]$$

with

$$(6.212)\quad k_2 = k_3\operatorname{cn}\left[\bar{\rho}_3 K_3\,|\,k_3\right]/\operatorname{dn}\left[\bar{\rho}_3 K_3\,|\,k_3\right].$$

Since ρ_1 is assigned and $\bar{k}_1$ is known, $\bar{\rho}_2$, ρ_2 and k_2 are thus uniquely determined. ρ_3 and $\bar{\rho}_3$ are then subjected to the relation (see Fig. 5c)

$$(6.213)\quad 1-\rho_3 = 2\bar{\rho}_3.$$

This is the relation which ensures the reduction of the hyperelliptic integral and it also represents a necessary condition.

From the point-correspondence

$$z_2 = \exp[-(\pi K_2'/K_2)+(1-\rho_2)\pi i] \longleftrightarrow z_3 = \exp[-(\pi K_3'/K_3)+\tfrac{1}{2}(1-\rho_3)\pi i]$$

and

$$z_2 = \exp[-(\pi K_2'/K_2)+\pi i] \longleftrightarrow z_3 = \exp[-(\pi K_3'/K_3)+\tfrac{1}{2}(1+\rho_3)\pi i]$$

follows

$$\operatorname{cn}\left[\rho_2 K_2\,|\,k_2\right] = \frac{\operatorname{cn}\left[\tfrac{1}{2}(1+\rho_3)K_3\,|\,k_3\right]}{\operatorname{cn}\left[\tfrac{1}{2}(1-\rho_3)K_3\,|\,k_3\right]}$$

and, from (6.212) and (6.213),

$$k_2 = k_3\frac{\operatorname{cn}\left[\tfrac{1}{2}(1-\rho_3)K_3\,|\,k_3\right]}{\operatorname{dn}\left[\tfrac{1}{2}(1-\rho_3)K_3\,|\,k_3\right]},$$

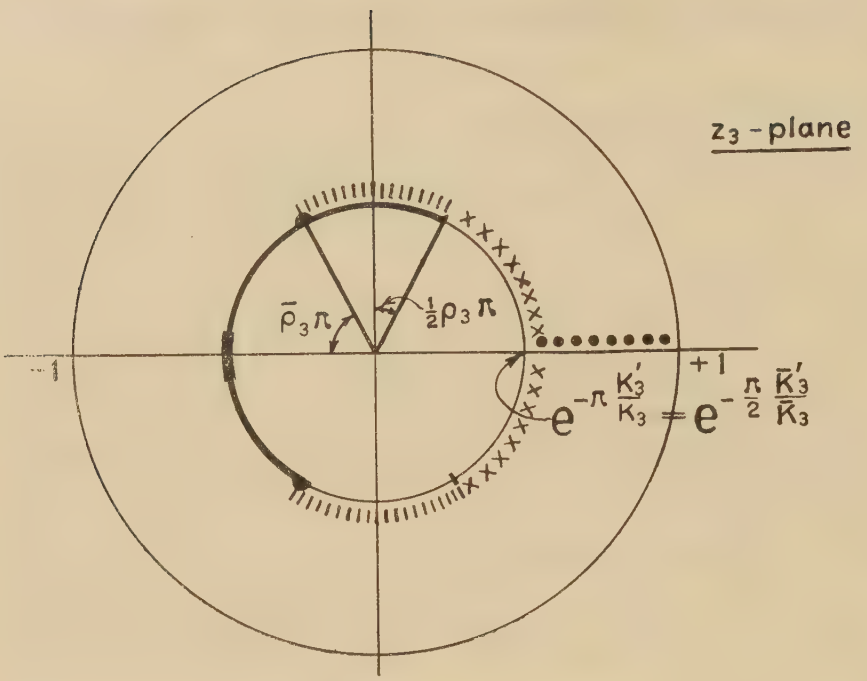
z_3 - plane
$\bar{\rho}_3\pi$
$\frac{1}{2}\rho_3\pi$
-1
+1
$e^{-\pi\frac{K_3'}{K_3}} = e^{-\frac{\pi}{2}\frac{\bar{K}_3'}{\bar{K}_3}}$

Fig. 5C.

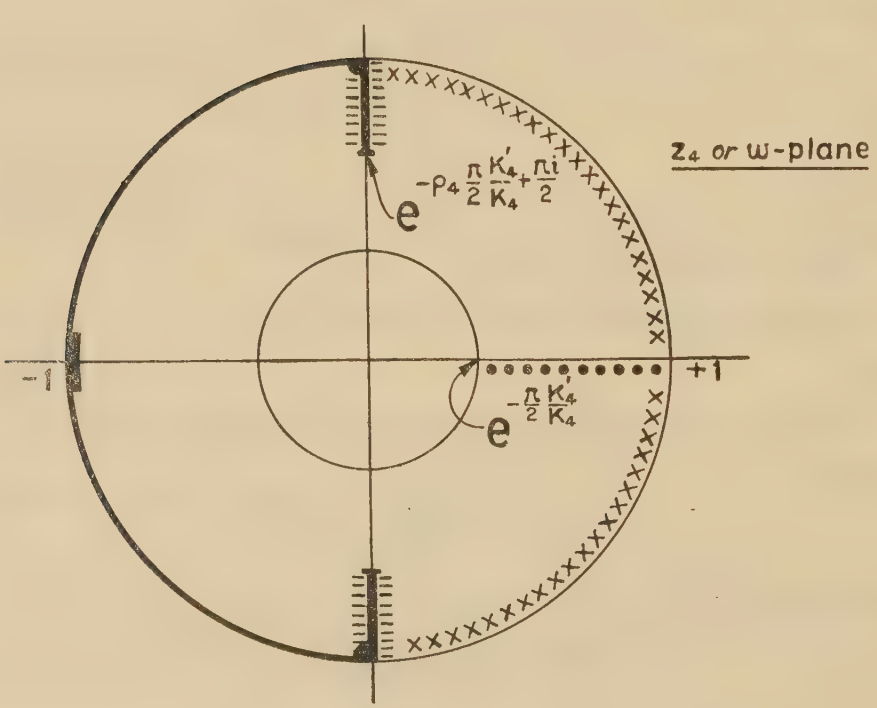
z_4 or w-plane
$e^{-\rho_4\frac{\pi}{2}\frac{K_4'}{\bar{K}_4}+\frac{\pi i}{2}}$
-1
+1
$e^{-\frac{\pi}{2}\frac{K_4'}{K_4}}$

Fig. 5D.

from which

$$(6.214)\qquad \begin{cases} k_3^2 = \bar{k}_1^2 + \bar{k}_1'^2 k_2^2 = k_2^2 + k_2'^2 \bar{k}_1^2, \\ k_3'^2 = \bar{k}_1'^2 k_2'^2. \end{cases}$$

With these equations all data are assembled for numerical calculations, especially since the diagrams indicate clearly the points corresponding to the quantity $\bar{\rho}_1$.

The next step is a Landen transformation. After the determination of k_3 and the associated periods a new set of periods is defined by

$$(6.32)\qquad 2\bar{K}_3 = (1+k_3')K_3; \quad \bar{K}_3' = (1+k_3')K_3'; \quad \bar{k}_3 = \frac{1-k_3'}{1+k_3'},$$

and then the z_3-ring is mapped upon the z_4-ring or w-ring by

$$(6.30)\qquad \operatorname{cn}\left[\frac{2iK_3}{\pi}\log z_3 + \bar{K}_3 + i\bar{K}_3' \,\middle|\, \bar{k}_3\right] = \operatorname{cn}[\rho_3 \bar{K}_3 | \bar{k}_3] \operatorname{cn}\left[\frac{2iK_4}{\pi}\log w + K_4 \,\middle|\, k_4\right],$$

where

$$(6.31)\qquad k_4 = \bar{k}_3 \operatorname{cn}[\rho_3 \bar{K}_3 | \bar{k}_3]/\operatorname{dn}[\rho_3 \bar{K}_3 | \bar{k}_3].$$

The final plane is now reached and the map is indicated in Fig. 5D. The elimination of z_2 and z_3 as well as of intermediate constants leads to the following relation between the z_1- and w-rings:

$$(6.40)\qquad \frac{\operatorname{dn}\left[\frac{2iK_4}{\pi}\log w \,\middle|\, k_4\right]\{[\operatorname{dn}\rho_1 i\bar{K}_1']^{-\frac{1}{2}} + [\operatorname{dn}\rho_1 i\bar{K}_1']^{\frac{1}{2}}\}}{k_4'} = \frac{\bar{k}_1'}{\operatorname{dn}\left[\frac{i\bar{K}_1}{\pi}\log z_1 \,\middle|\, \bar{k}_1\right]\sqrt{\operatorname{dn}\rho_1 i\bar{K}_1'}} + \frac{\operatorname{dn}\left[\frac{i\bar{K}_1}{\pi}\log z_1 \,\middle|\, \bar{k}_1\right]\sqrt{\operatorname{dn}\rho_1 i\bar{K}_1'}}{\bar{k}_1'};$$

and apart from the relations between the various constants already given the following are required for the numerical calculations:

$$(6.41)\qquad \bar{k}_1' = \operatorname{dn}\rho_1 i\bar{K}_1'/\operatorname{dn}\bar{\rho}_1 i\bar{K}_1',$$

which replaces (6.213), and

$$(6.42)\qquad k_4 = \bar{k}_1 k_2/(1+\bar{k}_1' k_2'); \quad k_4' = (\bar{k}_1' + k_2')/(1+\bar{k}_1' k_2').$$

The last, together with (6.11), fixes k_4 in terms of constants of the original z_1-ring.

The current distribution on the outer electrode is now found to be given by

$$(6.50)\quad i_5 = \frac{\bar{K}_1}{\bar{K}_4}\,\frac{[\bar{k}_1'^2+\mathrm{dn}\,\rho_1 i\bar{K}_1'][\mathrm{dn}\,\rho_1 i\bar{K}_1'\,\mathrm{dn}^2\,\bar{K}_1\,\theta/\pi-\bar{k}_1'^2]}{[\mathrm{dn}^2\,\rho_1 i\bar{K}_1'\,\mathrm{dn}^2\,\bar{K}_1\,\theta/\pi-\bar{k}_1'^2]^{\frac{1}{2}}\,[\mathrm{dn}^2\,\rho_1 i\bar{K}_1'\,\mathrm{dn}^2\,\bar{K}_1\,\theta/\pi-\bar{k}_1'^4]^{\frac{1}{2}}}.$$

This is plotted in curve (5) with the same ring constants as before.

If $\rho_1 \to 0$ the outer electrode becomes a circular cylinder that has one rib inside; the quantity $\bar{\rho}_1$ is then no longer tied, but is freely variable in the interval $0 < \bar{\rho}_1 < 1$, and the case may be treated by reversing the procedure of § 3. The arrangement is, but for a complex inversion, that of Fig. 2B, the thick line ending in a dot on the axis of negative real numbers now representing a conductor integral with the internal electrode. Mapping ends here with Fig. 5B or 5C.

It is worth while to examine more closely the case when the conducting shield of Fig. 5A lies in a position symmetrical with respect to the two electrodes, especially as this leads to the case of the next section where it is concentric with and symmetrical to them; the condition to be imposed upon ρ_1 and $\bar{\rho}_1$ is

$$(6.501)\qquad \bar{\rho}_1 = 1-\rho_1.$$

With a given z_1-ring (or $\bar{k}_1$) there is therefore one, and only one, size of such conductor for which the hyperelliptic integral reduces, and this is determined by

$$(6.502)\qquad k_1' = \mathrm{dn}\,\rho_1 i\bar{K}_1'/\mathrm{dn}\,(1-\rho_1)i\bar{K}_1', \quad \text{with} \quad 0<\rho_1<\tfrac{1}{2}.$$

From (6.21) and this relation it then follows that the moduli $\bar{k}_1$ of Fig. 5A and k_3 of Fig. 5C are connected algebraically by

$$(6.503)\qquad [\bar{k}_1^2-k_3^2]^2 = \bar{k}_1^2 k_3'^2.$$

The quantities $\bar{k}_1$ and k_4 are likewise connected algebraically, but the relation is not so simple:

$$(6.504)\quad \bar{k}_1'^4+\bar{k}_1'^4 k_4'^4-\bar{k}_1'^4 k_4'^2-2\bar{k}_1'^3 k_4'^3-2\bar{k}_1'^3 k_4'-2\bar{k}_1' k_4'^3 +6\bar{k}_1'^2 k_4'^2+k_4'^2-2\bar{k}_1' k_4' = 0.$$

Notwithstanding its complexity it is not difficult to solve the biquadratic in k_4' and to arrive at the conclusion (geometrically) that the quantity k_4' of Fig. 5D is given by the following root:

$$(6.505)\qquad k_4' = \bar{k}_1'\,\frac{\sqrt{2}+\sqrt{[1+\bar{k}_1'^2+\bar{k}_1(1+3\bar{k}_1'^2)^{\frac{1}{2}}]}}{\bar{k}_1'^2\sqrt{2}+\sqrt{[1+\bar{k}_1'^2+\bar{k}_1(1+3\bar{k}_1'^2)^{\frac{1}{2}}]}},$$

where the positive sign is taken throughout for the square roots.

For numerical calculations it is of course preferable to express (6.502) in terms of real numbers thus:

$$(6.5002) \qquad k_1' \operatorname{cn}[\rho_1 \bar{K}_1' | \bar{k}_1'] = \operatorname{dn}[\rho_1 \bar{K}_1' | \bar{k}_1'] \operatorname{sn}[\rho_1 \bar{K}_1' | \bar{k}_1'],$$

which permits the determination of ρ_1 from tables and shows that as $\rho_1 \to 0$ also $\bar{k}_1' \to 0$, and as $\rho_1 \to \frac{1}{2}$ also $\bar{k}_1' \to 1$, the geometrical significance of which is evident from the figures.

7. *Conducting shield concentric with and symmetrical to electrodes.*

This arrangement with the special conducting shield in the symmetrical position may be illustrated by Fig. 4A where now the shield represents a conductor kept at the same potential as the outer electrode by a wire external to the cell. It has already been referred to in the previous paragraph and follows immediately, at least in principle. The z_1-ring of Fig. 5A is simply identified with the w-ring of Fig. 4B, from which it is seen that the quantities $\bar{k}_1$, $\bar{K}_1$, $\bar{K}_1'$ of the former are connected with k_1, K_1, K_1' of the latter by the formulae of the Landen transformation

$$(7.01) \qquad 2\bar{K}_1 = (1+k_1')K_1; \quad \bar{K}_1' = (1+k')K_1'; \quad \bar{k}_1 = \frac{1-k_1'}{1+k_1'}.$$

The results of §§ 5 and 6 then become applicable to obtain the map of Fig. 4A upon Fig. 5B, subject to the centre angle subtended by the shield in Fig. 4A satisfying the previously given condition for the reduction of the hyperelliptic integral. This condition must, for numerical calculations, be transplanted from the z_1-ring of Fig. 5A to the z_0-ring of Fig. 4A. If the supplement of half the angle subtended by the shield is denoted by $\pi\rho_0$ and the remaining notation of Fig. 4A is retained, $\pi\rho_0$ is found to be connected with k_0 by

$$(7.02) \qquad \frac{1}{k_0 \operatorname{sn}^4[\frac{1}{2}\rho_0 K_0 | k_0]} - k_0 \operatorname{sn}^4[\tfrac{1}{2}\rho_0 K_0 | k_0] = \frac{2}{k_0 \operatorname{sn}^2[\frac{1}{2}\rho_0 K_0 | k_0]} - 2k_0 \operatorname{sn}^2[\tfrac{1}{2}\rho_0 K_0 | k_0],$$

which is equivalent to (6.501) and (6.502).

The repeated Landen and other transformations carried out make the algebraic operations increasingly involved, and, as was done in § 4, a further Landen transformation introducing $\bar{k}_0$, $\bar{K}_0$, $\bar{K}_0'$ of Fig. 4A is necessary to make numerical calculations reasonably simple. The mapping relation will therefore not be given here, especially since the method of obtaining it is clear and it is not required for the determination of the expression for the current distribution function which may be derived by

combining previous operations. Formally this amounts to multiplying together (5.2) and (6.50) and transplanting (6.50) and its constants to Fig. 4A (note that the ρ_1 in (6.50) refers to Fig. 5A or to Fig. 4B in the present context). Thus the current distribution is found to be expressed by

$$i_6 = (K_0/K_4)\, A\phi[K_0\theta_0/\pi|k_0]\, F[\bar{K}_1\theta/\pi|\bar{k}_1], \tag{7.11}$$

where θ_0 is the polar angle of Fig. 4A and θ is that of Fig. 5A or 4B. The function ϕ is essentially the bracket expression on the right of (5.2), and F the right-hand side of (6.50) divided by

$$(\bar{K}_1/K_4)[\bar{k}_1'^2+\mathrm{dn}\,\rho_1 i\bar{K}_1']/\mathrm{dn}\,\rho_1 i\bar{K}_1'.$$

The constant A is given by

$$A = [\bar{k}_1'^2+\mathrm{dn}\,\rho_1 i\bar{K}_1']/(1+\bar{k}_1)\,\mathrm{dn}\,\rho_1 i\bar{K}_1', \tag{7.03}$$

with

$$\bar{k}_1 = (1-\mathrm{dn}\,\rho_0 K_0)/(1+\mathrm{dn}\,\rho_0 K_0) = \bar{k}_0\,\mathrm{sn}^2[\rho_0\bar{K}_0|\bar{k}_0], \tag{7.04}$$

$$2\,\mathrm{dn}^2\rho_1 i\bar{K}_1' = \{1+\bar{k}_1'^2+\bar{k}_1\sqrt{(1+3\bar{k}_1'^2)}\},$$

the latter equality being equivalent to (7.02). The numbers k_0 and $\bar{k}_0$ are connected by (5.01), and in the square root the positive sign is to be taken. Thus A is entirely determined. The function $F[\bar{K}_1\theta/\pi|\bar{k}_1]$ is transplanted into the z_0-plane by means of

$$\mathrm{sn}\,[\bar{K}_1\theta/\pi|\bar{k}_1] = \frac{1-\sqrt{P[K_0\theta_0/\pi]}}{\mathrm{sn}\,[K_0\theta/\pi](1-\mathrm{dn}\,[\rho_0 K_0])}, \tag{7.05}$$

where

$$P[K_0\theta_0/\pi] = 1-k_0^2\,\mathrm{sn}^2\rho_0 K_0\,\mathrm{sn}^2[K_0\theta_0/\pi];$$

and for real values of the arguments of the functions the square root has a positive sign. The result is

$$F[\bar{K}_1\theta/\pi|\bar{k}_1] = \psi[K_0\theta_0/\pi|k_0] \tag{7.06}$$

$$= \frac{-1+a\,\mathrm{sn}^2[K_0\theta_0/\pi]+\sqrt{P}}{\{-1+b\,\mathrm{sn}^2[K_0\theta_0/\pi]+\sqrt{P}\}\{-1+c\,\mathrm{sn}^2[K_0\theta_0/\pi]+\sqrt{P}\}},$$

where

$$(1+\bar{k}_1')^2 a = 2[1+\bar{k}_1-\{\tfrac{1}{2}(1+k_1'^2)-\tfrac{1}{2}\bar{k}_1\sqrt{(1+3\bar{k}_1'^2)}\}^{\frac{1}{2}}], \tag{7.07}$$

$$\bar{k}_1'^2(1+\bar{k}_1)^2 b = \bar{k}_1[2\bar{k}_1'^2-\bar{k}_1+\sqrt{(1+3\bar{k}_1'^2)}],$$

$$(1+\bar{k}_1)^2 c = \bar{k}_1[2+\bar{k}_1+\sqrt{(1+3\bar{k}_1'^2)}].$$

Hence (7.11) may be written thus:

$$i_6 = (K_0/K_4)\, A\phi[K_0\theta_0/\pi|k_0]\,\psi[K_0\theta_0/\pi|k_0], \tag{7.12}$$

and this has been plotted in curve (6) with $\bar{k}_0$ the same value as in the previous cases, *i.e.* for an internal electrode of radius $q^2 = \bar{q} = \cdot 2429$ approximately.

The expression for k_4 is required for numerical work; it is

$$k_4 = \frac{\bar{k}_1\sqrt{\bar{k}_1}\{\bar{k}_1+\sqrt{(1+3\bar{k}_1'^2)}\}^{\frac{1}{2}}}{\sqrt{2}\,\bar{k}_1'^2+\{1+\bar{k}_1'^2+\bar{k}_1\sqrt{(1+3\bar{k}_1'^2)}\}^{\frac{1}{2}}}, \tag{7.08}$$

and it is worth while noting that

$$k_0^2 = \frac{\bar{k}_1\{2-\bar{k}_1+\sqrt{(1+3\bar{k}'^2)}\}^2}{\{\bar{k}_1+\sqrt{(1+3\bar{k}_1'^2)}\}\{1+\bar{k}_1'^2+\bar{k}_1\sqrt{(1+3\bar{k}_1'^2)}\}}, \tag{7.09}$$

which shows that k_0 and k_4 are connected by an algebraic relation.

The case discussed in this section is very special; only one degree of freedom remains in order to make the method employed applicable. This means that the electrodepositor, in order to use these results, may either choose the internal electrode freely, and then the angle subtended by the conducting shield from the centre is determined; or, this angle may be arbitrarily selected, and then the internal electrode diameter is determined. The method nevertheless serves to indicate the general effect of a conducting shield, and it is suitable for numerical calculations since ordinarily available tables of elliptic integrals (or functions) can be used. The development of this case completes the picture it was intended to convey.

8. *General comments on numerical results and other cases.*

The detailed discussion of the graphs together with the practical implications will be reserved for another place, but a few remarks may appropriately be included in this paper.

All curves are so drawn that the "deviation of current distribution from uniformity" is obvious; the uniformity line is of course represented by the horizontal line through the ordinate unity.

The elementary case of Fig. 1 is the only one distinguished by an invariant voltage-current relationship if the internal electrode is adjusted as shown in the diagram for excentric displacement. Hence the term "equivalent concentric electrode" as used in § 4 is, strictly speaking, not justified; it was, however, convenient in the context.

It was mentioned that the internal electrode radius for all concentric cases was chosen as approximately ·2429; the internal electrode in Fig. 1 is then shrunk to approximately ·1938 for $a = \frac{1}{2}$.

For curve (2) the stopped-off portion of Fig. 2 was taken as half the electrode; for curve (3) again $\rho = \frac{1}{2}$ was taken, but the fraction of the electrode then stopped-off is only about ·4607, and the angle of the stopped-off portion subtended from the centre of the displaced internal electrode is about $2\lambda_0 = 166° 4' 9{\cdot}25''$; the angle of the same portion subtended from the centre of the outer electrode is determined from (4.2) and with $a = \frac{1}{2}$ is found to be about 73° 44′. A glance at curve (3) suggests that, since half this angle enters into it, this is the significant angle for screening.

Curve (4) may be interpreted as applying to the case of Fig. 2 with radius of internal electrode about ·4928 and $\rho = \frac{1}{2}$, as already mentioned.

In curve (5), $\rho_1 = \frac{1}{2}$ was chosen; from (6.41) then follows $\bar{\rho}_1 = {\cdot}9452$ approximately: the conducting shield is a band of width about ·2303 which with its nearest edge is only ·0196 distant from the internal electrode, and with its farthest edge about ·4928.

For curve (6) it was necessary to determine the angle subtended by the conducting shield from the centre; it is found from (7.02) to be 84° 13′ 40·8″; *i.e.* $\rho_0 = {\cdot}7660355$. Curve (4A) represents the case when the same conducting shield is replaced by an insulating one, and curve (2A) the case where the internal electrode is stopped off to the same angle as the shield.

All graphs were computed with the aid of five-figure logarithm tables and the Pearson edition of Legendre's elliptic integrals. The constant ρ_0 was determined from the Landen transform of (7.02) with the aid of seven-figure tables. It was found that $\mathrm{sn}^2[\rho_0 \bar{K}_0 | \bar{k}_0] = {\cdot}9876965$ makes the Landen transform of (7.02) zero to five places of decimals; a slightly smaller figure in the last decimal would have given even closer results, but for the present purpose the value given was considered sufficiently accurate.

The calculations by means of Legendre's tables are very tedious, but it seems that there are no other sufficiently accurate tables of elliptic functions which make the calculations appreciably easier and equally accurate. The selection of a smaller internal electrode diameter would of course have simplified matters, but very much smaller electrodes do do not occur in practical electrodeposition. Even for such a small internal electrode radius as ·2429 the elliptic modulus is $\sin 83°$, and for a radius ·4928, or a little under one-half the outer electrode radius, it is found that $k = {\cdot}9999930(7)$, so that Legendre's tables are useless and Landen transforms or calculations *de novo* are essential. In practice it is by no means rare to encounter $q = \frac{7}{8}$, and $\frac{5}{8}$ or $\frac{3}{4}$ are quite common.

The derivatives of the mapping functions for the conducting shields were multiplied by 2 to obtain i_5 and i_6; this balanced the absorption of

current by the shields. The total current through the cells was here taken as 4π amps, which left 2π amps to pass through the electrode on the outside; small wonder that a conducting shield is frequently termed a "thief".

The author must pay tribute to Mr. R. Weck*, who so ably assisted him in the numerical work.

In a recently published paper (**6**) the author gave numerical results for various other electrode arrangements without, however, stating the methods by means of which they were obtained (since the paper was meant for practical electrodepositors); it will be clear from the foregoing that they are the same as those employed here and that essentially the same class of functions is encountered.

By the same means numerous other arrangements of practical interest can be treated, such as the case of a partly stopped-off internal and a partly stopped-off external electrode; to remain within the realm of elliptic functions special assumptions on symmetry of stopped-off portions, etc., must be made. All the functions given can be used for the treatment of the cases where several parts of the internal electrode are stopped-off at the same time, provided only that these parts are so arranged that their midpoints lie on a regular polygon concentric with the origin; the same applies to shields between the electrodes, and by letting the stopped-off portions or the screens surround the internal electrode more and more closely the character of electrodeposits obtained through narrow slots may be studied.

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INVARIANTS OF SYSTEMS OF QUADRICS

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1. *Introduction.* In a joint paper by Turnbull and Young, published in the *Trans. Cambridge Phil. Soc.**, the linear invariants of 10 quaternary quadrics were discussed. The paper was partly devoted to obtaining an expression for the alternating invariant of 10 quadrics with a minimum number of terms, and partly to a classification of all the invariants of 10 quadrics into conjugate sets, according to their behaviour when the 10 quadrics are subjected to a permutation. This classification was the work of Young†, and was accomplished by the methods of substitutional analysis, of which he was the only master.

In this paper a similar classification is made by S-functional analysis for any system of 9 or less ternary quadrics, or 10 or less quaternary quadrics. The system considered by Young is thus included, and it is found that in one instance his conclusion is incorrect. The set of invariants which he describes as $T_{22222}I_6$ does not in fact exist. The complete sets of invariants of degree less than or equal to 9 in any set of ternary, or of degree less than or equal to 10 in any system of quaternary, quadrics are also specified and enumerated.

2. *The classification of invariants.* In order to illustrate the significance of this classification of the invariants of a system of quadrics, we consider first the simple case of the invariants linear in 4 binary quadrics. We denote these symbolically by α_x^2, β_x^2, γ_x^2, δ_x^2.

The set of concomitants which are linear in each of these 4 quadrics corresponds to the expansion in S-functions‡ of $\{2\}^4$. Since the binary case

* H. W. Turnbull and A. Young (4).

† See obituary notice by H. W. Turnbull, *Journal London Math. Soc.*, 16 (1941), 194.

‡ D. E. Littlewood (3).

is under consideration, partitions with more than two parts can be omitted. We have

$$\{2\}^2 = \{4\}+\{31\}+\{2^2\},$$

$$\{2\}^3 = \{6\}+2\{51\}+3\{42\}+\{33\},$$

$$\{2\}^4 = \{8\}+3\{71\}+6\{62\}+6\{53\}+3\{44\}.$$

Of the S-functions in the last expansion the terms $\{44\}$ correspond to invariants, and from the coefficients it follows that there are exactly 3 linearly independent invariants which are linear in each of 4 quadrics.

The sequence of S-functions $\{2\}$, $\{2^2\}$, $\{42\}$, $\{44\}$ is such that each appears as one term when the preceding S-function is multiplied by $\{2\}$. We obtain thus a sequence of concomitants linear respectively in 1, 2, 3, 4 quadrics. We build a tableau of type $\{44\}$ which, in the building, passes through these consecutive types, *i.e.*,

$$\begin{pmatrix} \alpha, \alpha, & \gamma, \gamma \\ \beta, \beta, & \delta, \delta \end{pmatrix}.$$

This is interpreted symbolically as the product of the determinantal forms obtained from the 4 columns, *e.g.*,

$$(\alpha\beta)^2\,(\gamma\delta)^2.$$

By taking the quadrics in a different order we could have obtained, in the same way, two other invariants, namely,

$$(\alpha\gamma)^2\,(\beta\delta)^2, \quad (\alpha\delta)^2\,(\beta\gamma)^2.$$

These three invariants are linearly independent, and thus represent the complete set of invariants. Since these invariants can be obtained one from another by permuting the quadrics, we might be tempted to take the view that there was only one *kind* of invariant linear in 4 quadrics, all others being obtained from it by permuting the quadrics. Such a view would, however, be superficial, as it would obscure the fact that there is one linear combination of the invariants, namely,

$$(\alpha\beta)^2\,(\gamma\delta)^2+(\alpha\gamma)^2\,(\beta\delta)^2+(\alpha\delta)^2\,(\beta\gamma)^2, \tag{2.1}$$

which occupies a privileged position in that it is the one and only invariant which is symmetric in the 4 quadrics.

We take the two remaining invariants in the form

$$\left.\begin{aligned} &(\alpha\beta)^2\,(\gamma\delta)^2-(\alpha\gamma)^2\,(\beta\delta)^2, \\ &(\alpha\delta)^2\,(\beta\gamma)^2-(\alpha\gamma)^2\,(\beta\delta)^2. \end{aligned}\right\} \tag{2.2}$$

No permutation of the quadrics will alter the invariant (2.1). No permutation operating on an invariant of (2.2) will give anything but a linear combination of these two invariants. On the other hand, no further separation is possible; both invariants of (2.2) may be obtained by the use of substitutional operators from any given linear combination of them.

The invariants of 4 binary quadrics are thus separated into two sets, which we term *primitive sets*. This separation corresponds to the separation of the Frobenius algebra of the substitutional operators permuting the quadrics, into simple subalgebras. The simple subalgebras of the Frobenius algebra for the symmetric group on n symbols are associated* with the partitions of n. Thus each primitive set of invariants linear in n quadrics can be associated with a partition $\{\lambda\}$ of n, which we call the *class* of the primitive set.

The symmetric invariant (2.1) is clearly of class $\{4\}$. The invariants (2.2) are of class $\{2^2\}$.

For the general case we may consider any type of ground form and any type of concomitant. If there is a set of concomitants of type $\{\mu\}$ linear in the coefficients of r ground forms each of type $\{\lambda\}$, these may be separated into primitive sets whose classes are various partitions of r. In this paper, however, we deal only with quadrics, *i.e.*, we take $\{\lambda\} = \{2\}$, and we consider only invariants, *i.e.*, the partition $\{\mu\}$ is of type $\{m^r\}$, where r is the number of variables and m is any positive integer. However, we first prove four theorems concerning the general case.

3. *Four general theorems.*

THEOREM I. *If there exist concomitants of type $\{\mu\}$ and class $\{\nu\}$ which are linear in each of r ground forms each of type $\{\lambda\}$, then $\{\mu\}$ appears in the expansion of $\{\lambda\} \otimes \{\nu\}$.*

The symbol $\otimes$ denotes a "new multiplication", which has been defined in two previous papers†. If the variables are transformed by a matrix A^{-1}, then the coefficients of a ground form of type $\{\lambda\}$ are transformed by the invariant matrix $A^{(\lambda)}$. The products of r coefficients, one from each ground form, are transformed by a matrix which is the direct product of r such invariant matrices. But if we pick out only those linear combinations of products which are not annihilated by that substitutional expression

* See D. E. Littlewood (2), chap. V.

† D. E. Littlewood (1), (3).

permuting the ground forms, which corresponds to the modulus of the subalgebra of the Frobenius algebra corresponding to the partition (ν), then the matrix of transformation will be the invariant matrix corresponding to $\{\nu\}$ of $A^{\{\lambda\}}$, *i.e.*, $[A^{\{\lambda\}}]^{\{\nu\}}$. These must include the coefficients of the concomitants corresponding to $\{\mu\}$. Hence if, expressing $[A^{\{\lambda\}}]^{\{\nu\}}$ as a sum of irreducible invariant matrices, we have

$$[A^{\{\lambda\}}]^{\{\nu\}} = \dot{\Sigma} A^{\{\zeta\}},$$

$\dot{\Sigma}$ denoting a direct sum, so that

$$\{\lambda\} \otimes \{\nu\} = \Sigma \{\zeta\},$$

then $\Sigma\{\zeta\}$ must include the S-function $\{\mu\}$. This proves the theorem.

THEOREM II. *The number of linearly independent concomitants of type $\{\mu\}$ and class $\{\nu\}$ which are linear in each of r ground forms each of type $\{\lambda\}$ is an exact multiple of $f^{(\nu)}$, the degree of the character $\chi^{(\nu)}$ of the symmetric group of order $r!$.*

Consider the Frobenius algebra of the group of substitutions permuting the ground forms. In the subalgebra corresponding to (ν) let

$$e_{11}, e_{12}, \ldots, e_{1f}, e_{21}, \ldots, e_{ff}$$

be the substitutional expressions which form the basis of a matrix representation, so that $e_{ij} e_{kp} = 0$ if $j \neq k$, $e_{ij} e_{jp} = e_{ip}$. Let Φ denote any concomitant of the required type, and of class $\{\nu\}$.

Then since Σe_{ii} is the modulus of the subalgebra, we have $\Sigma e_{ii}\Phi \neq 0$. Suppose that $e_{11}\Phi \neq 0$. Then, since

$$e_{11}\Phi = e_{12} e_{21} \Phi,$$

we must have also $e_{21}\Phi \neq 0$,

and also $e_{21}\Phi \neq e_{11}\Phi$,

because $e_{11} e_{21}\Phi = 0, \quad e_{11} e_{11}\Phi = e_{11}\Phi \neq 0.$

Similarly the f concomitants

$$e_{11}\Phi, \ e_{21}\Phi, \ e_{31}\Phi, \ \ldots, \ e_{f1}\Phi$$

are non-zero and linearly independent. Here $f = f^{(\lambda)}$ is the degree of the character $\chi^{(\lambda)}$.

Similarly, if there are q linearly independent concomitants which can be expressed in the form $e_{11}\Phi$, for some concomitant Φ, then there will be altogether $qf^{(\lambda)}$ concomitants of class $\{\nu\}$. This proves the theorem.

THEOREM III. *If there are $qf^{(\lambda)}$ concomitants as defined in Theorem II, then q is the coefficient of $\{\mu\}$ in the expansion of $\{\lambda\}\otimes\{\nu\}$.*

The theorem is true for $\{\nu\}=\{r\}$, for the concomitants of class $\{r\}$ are those which are symmetric in the ground forms, and if the ground forms are made identical these become the concomitants of degree r in a single ground-form. The concomitants of degree r in a single ground-form of type $\{\lambda\}$ correspond* exactly to the terms in the expansion of $\{\lambda\}\otimes\{r\}$.

Now let $(\tau)=(\tau_1, \tau_2, \ldots, \tau_i)$ be any partition of r, and suppose that the r ground forms are made identical in sets of $\tau_1, \tau_2, \ldots, \tau_i$ respectively. Let $P_{\tau_1}, \ldots, P_{\tau_i}$ be the substitutional expressions which correspond to the sums of the symmetric groups of permutations on the $\tau_1, \tau_2, \ldots, \tau_i$ ground forms respectively. For each set of $f^{(\lambda)}$ linearly independent concomitants of class $\{\nu\}$, the number of linearly independent concomitants that will remain after operating on them with the product $P_{\tau_1}P_{\tau_2}\ldots P_{\tau_i}$ will be equal to the rank of the matrix corresponding to $P_{\tau_1}\ldots P_{\tau_i}$ in the sub-algebra corresponding to $\{\nu\}$. This has been proved to be equal to the coefficient $K_{\nu\tau}$ of $\{\lambda\}$ in the expansion†

$$h_{\tau_1}\ldots h_{\tau_i}=\Sigma K_{\nu\tau}\{\nu\}.$$

If there are $q_\nu f^{(\nu)}$ linearly independent concomitants when all the ground forms are distinct, there will thus be $q_\nu K_{\nu\tau}$ linearly independent concomitants when the ground forms are made equal in groups of $\tau_1, \tau_2, \ldots, \tau_i$.

For the next part of the proof we require the following lemma, a property of the operation $\otimes$ which has not yet been proved.

LEMMA.

$$\{\lambda\}\otimes[\{\mu\}\{\nu\}]=[\{\lambda\}\otimes\{\mu\}][\{\lambda\}\otimes\{\nu\}].$$

This indicates that the operation $\otimes$ is distributive on the right with respect to multiplication as well as addition. Since the spur of a direct product of matrices is equal to the product of the spurs, if

$$\{\mu\}\{\nu\}=\Sigma\{\zeta\},$$

then the direct product of $A^{\{\mu\}}$ and $A^{\{\nu\}}$ will be equivalent to $\dot{\Sigma}A^{\{\zeta\}}$, and thus also the direct product of $[A^{\{\lambda\}}]^{\{\mu\}}$ and $[A^{\{\lambda\}}]^{\{\nu\}}$ will be equivalent to $\dot{\Sigma}[A^{\{\lambda\}}]^{\{\zeta\}}$.

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* D. E. Littlewood (**1**) or (**3**), or (**2**), 206.

† D. E. Littlewood (**2**), 193.

Taking the spurs of the matrices we obtain

$$[\{\lambda\}\otimes\{\mu\}][\{\lambda\}\otimes\{\nu\}] = \Sigma\{\lambda\}\otimes\{\zeta\},$$

which proves the lemma.

Returning to our theorem, it is known* that the total number of concomitants of type $\{\mu\}$ of i ground forms each of type $\{\lambda\}$, of degree $\tau_1, \tau_2, \ldots, \tau_i$ respectively in the coefficients, is given by the coefficient of $\{\mu\}$ in the expansion of

$$[\{\lambda\}\otimes\{\tau_1\}][\{\lambda\}\otimes\{\tau_2\}]\ldots[\{\lambda\}\otimes\{\tau_i\}]$$
$$= \{\lambda\}\otimes[h_{\tau_1}h_{\tau_2}\ldots h_{\tau_i}] = \{\lambda\}\otimes\Sigma K_{\nu\tau}\{\nu\} = \Sigma K_{\nu\tau}\{\lambda\}\otimes\{\nu\}.$$

Thus $q_\nu K_{\nu\tau}$ is the coefficient of $\{\mu\}$ in $\Sigma K_{\nu\tau}\{\lambda\}\otimes\{\nu\}$. Since this is true for all partitions (τ) it follows that q must be the coefficient of $\{\mu\}$ in $\{\lambda\}\otimes\{\nu\}$.

We have proved incidentally the following theorem.

THEOREM IV. *If q is the coefficient of $\{\mu\}$ in $\{\lambda\}\otimes\{\nu\}$, then the number of concomitants of type $\{\mu\}$ and class $\{\nu\}$ which are of degrees $\tau_1, \tau_2, \ldots, \tau_i$ respectively in i ground forms each of type $\{\lambda\}$ is $qK_{\nu\tau}$, where $K_{\nu\tau}$ is the coefficient of $\{\nu\}$ in $h_{\tau_1}h_{\tau_2}\ldots h_{\tau_i}$.*

4. *Formulae in S-functions.* As a preliminary to our examination of the invariants of quadrics we evaluate $\{2\}^n$ for $n \leqslant 5$, and $\{2\}\otimes\{\lambda\}$ for all partitions (λ) of numbers less than or equal to 5.

$$\{2\}^2 = \{4\}+\{31\}+\{2^2\}. \tag{4.1}$$

$$\{2\}^3 = \{6\}+2\{51\}+3\{42\}+\{41^2\}+\{3^2\}+2\{321\}+\{2^3\}. \tag{4.2}$$

$$\begin{aligned}\{2\}^4 = {} & \{8\}+3\{71\}+6\{62\}+3\{61^2\}+6\{53\}+8\{521\}+\{51^3\}+3\{4^2\}\\ & +7\{431\}+6\{42^2\}+3\{421^2\}+3\{3^2 2\}+2\{3^2 1^2\}\\ & +3\{32^2 1\}+\{2^4\}.\end{aligned} \tag{4.3}$$

$$\begin{aligned}\{2\}^5 = {} & \{10\}+4\{91\}+10\{82\}+6\{81^2\}+15\{73\}+20\{721\}+4\{71^3\}\\ & +15\{64\}+30\{631\}+20\{62^2\}+15\{621^2\}+\{61^4\}+6\{5^2\}\\ & +24\{541\}+30\{532\}+21\{531^2\}+20\{52^2 1\}+4\{521^3\}\\ & +16\{4^2 2\}+10\{4^2 1^2\}+10\{43^2\}+24\{4321\}+5\{431^3\}\\ & +10\{42^3\}+6\{42^2 1^2\}+5\{3^3 1\}+6\{3^2 2^2\}+5\{3^2 21^2\}\\ & +4\{32^3 1\}+\{2^5\}.\end{aligned} \tag{4.4}$$

* D. E. Littlewood (3), 330.

The expansion of $\{2\}\otimes\{n\}$ is known*; it includes every partition of $2n$ into even parts only. The expansion of $\{2\}\otimes\{1^n\}$ is also known*; it includes every partition of $2n$ which in Frobenius nomenclature is in one of the forms

$$\binom{r+1}{r}, \quad \binom{r+1,\ s+1}{r,\ s}, \quad \binom{r+1,\ s+1,\ t+1}{r,\ s,\ t}, \quad \ldots.$$

To obtain $\{2\}\otimes\{\lambda\}$ for other partitions we may use the lemma proved in § 3, namely,

$$\{\lambda\}\otimes[\{\mu\}\{\nu\}] = [\{\lambda\}\otimes\{\mu\}][\{\lambda\}\otimes\{\nu\}].$$

When $\{\lambda\}$ is expressed in terms of the h_r's, this formula may be applied to evaluate $\{2\}\otimes\{\lambda\}$. Thus

$$\begin{aligned}\{2\}\otimes\{21\} &= \{2\}\otimes[h_2h_1-h_3]\\ &= [\{2\}\otimes\{2\}][\{2\}\otimes\{1\}]-\{2\}\otimes\{3\}\\ &= [\{4\}+\{2^2\}]\{2\}-\{6\}-\{42\}-\{222\}\\ &= \{6\}+\{51\}+\{42\}+\{42\}+\{321\}+\{222\}-\{6\}-\{42\}-\{222\}\\ &= \{51\}+\{42\}+\{321\}.\end{aligned}$$

Other methods are available. In a previous paper† the following formula was used for the evaluation of $\{\lambda\}\otimes\{n\}$.

If $$\{\lambda\}\otimes\{n\} = \Sigma\{\mu\},$$

then $$[\{\lambda\}\otimes\{n-1\}]\,\Sigma g_{1\xi\lambda}\{\xi\} = \Sigma g_{1\zeta\mu}\{\zeta\},$$

where $g_{1\xi\lambda}$ is the coefficient of $\{\lambda\}$ in $\{1\}\{\xi\}$.

The corresponding formula for $\{\lambda\}\otimes\{\nu\}$ is as follows. If

$$\{\lambda\}\otimes\{\nu\} = \Sigma\{\mu\},$$

then $$[\{\lambda\}\otimes\Sigma g_{1\eta\nu}\{\eta\}]\,\Sigma g_{1\xi\lambda}\{\xi\} = \Sigma g_{1\zeta\mu}\{\zeta\}.$$

Thus, putting $\{\lambda\} = \{2\}$, $\{\nu\} = \{2^2\}$, we have

$$\Sigma g_{1\eta\nu}\{\eta\} = \{21\}, \quad \Sigma g_{1\xi\lambda}\{\xi\} = 1.$$

* D. E. Littlewood (3).

† D. E. Littlewood (3).

Thus

$$\begin{aligned}[\{2\}\otimes\{21\}]\{1\} &= [\{51\}+\{42\}+\{321\}]\{1\}\\ &= \{61\}+2\{52\}+\{511\}+\{43\}+2\{421\}+\{331\}+\{322\}+\{3211\}\\ &= \Sigma g_{1\zeta\mu}\{\zeta\}.\end{aligned}$$

We must then choose the following partitions for $\{\mu\}$ in order that $\Sigma g_{1\zeta\mu}\{\zeta\}$ may equal the expression on the right:

$$\begin{aligned}\{62\} &\to \{61\}+\{52\},\\ \{521\} &\to \{52\}+\{511\}+\{421\},\\ \{44\} &\to \{43\},\\ \{422\} &\to \{421\}+\{322\},\\ \{3311\} &\to \{331\}+\{3211\}.\end{aligned}$$

Thus

$$\{2\}\otimes\{2^2\} = \{62\}+\{521\}+\{44\}+\{422\}+\{3311\}.$$

One way or another we obtain

$$\left.\begin{aligned}\{2\}\otimes\{2\} &= \{4\}+\{2^2\},\\ \{2\}\otimes\{1^2\} &= \{31\}.\end{aligned}\right\} \quad (4.5)$$

$$\left.\begin{aligned}\{2\}\otimes\{3\} &= \{6\}+\{42\}+\{2^3\},\\ \{2\}\otimes\{21\} &= \{51\}+\{42\}+\{321\},\\ \{2\}\otimes\{1^3\} &= \{411\}+\{33\}.\end{aligned}\right\} \quad (4.6)$$

$$\left.\begin{aligned}\{2\}\otimes\{4\} &= \{8\}+\{62\}+\{44\}+\{422\}+\{2^4\},\\ \{2\}\otimes\{31\} &= \{71\}+\{62\}+\{53\}+\{521\}+\{431\}+\{422\}+\{3221\},\\ \{2\}\otimes\{2^2\} &= \{62\}+\{521\}+\{44\}+\{422\}+\{3311\},\\ \{2\}\otimes\{211\} &= \{611\}+\{521\}+\{53\}+\{431\}+\{4211\}+\{332\},\\ \{2\}\otimes\{1^4\} &= \{51^3\}+\{431\}.\end{aligned}\right\} \quad (4.7)$$

$$
\left.\begin{aligned}
\{2\}\otimes\{5\} &= \{10\}+\{82\}+\{64\}+\{622\}+\{442\}+\{42^3\}+\{2^5\},\\
\{2\}\otimes\{41\} &= \{91\}+\{82\}+\{73\}+\{721\}+\{64\}+\{631\}+\{622\}\\
&\quad+\{541\}+\{532\}+\{5221\}+\{442\}+\{4321\}\\
&\quad+\{4222\}+\{32^3\,1\},\\
\{2\}\otimes\{32\} &= \{82\}+\{73\}+\{721\}+\{64\}+\{631\}+2\{622\}+\{541\}\\
&\quad+\{532\}+\{531^2\}+\{52^2\,1\}+\{4^2\,2\}+\{4321\}\\
&\quad+\{42^3\}+\{3^2\,21^2\},\\
\{2\}\otimes\{31^2\} &= \{81^2\}+\{73\}+\{721\}+2\{631\}+\{621^2\}+\{5^2\}+\{541\}\\
&\quad+2\{532\}+\{531^2\}+\{52^2\,1\}+\{4^2\,1^2\}+\{43^2\}+\{4321\}\\
&\quad+\{42^2\,1^2\}+\{3^2\,2^2\},\\
\{2\}\otimes\{2^2\,1\} &= \{721\}+\{64\}+\{631\}+\{62^2\}+\{621^2\}+\{541\}\\
&\quad+\{532\}+\{531^2\}+\{52^2\,1\}+\{4^2\,2\}+\{4321\}\\
&\quad+\{431^3\}+\{3^3\,1\},\\
\{2\}\otimes\{21^3\} &= \{71^3\}+\{621^2\}+\{631\}+\{541\}+\{532\}+\{531^2\}\\
&\quad+\{521^3\}+\{4^2\,1^2\}+\{43^2\}+\{4321\},\\
\{2\}\otimes\{1^5\} &= \{61^4\}+\{531^2\}+\{4^2\,2\}.
\end{aligned}\right\}\qquad(4.8)
$$

5. *Invariants of ternary quadrics.* We proceed to study the invariants of ternary quadrics, or conics. In 3 variables an invariant linear in each conic must correspond to a partition of the form $\{r, r, r\}$. Hence the number of conics must be divisible by 3.

The case of 3 conics is simple. From equations (4.6) we notice that the S-function $\{2^3\}$ appears only in $\{2\}\otimes\{3\}$. Hence there is exactly one invariant linear in each of 3 conics, which is symmetric in the conics.

Turning to the case of 6 conics we determine first the number of invariants. This is equal to the coefficient of $\{4^3\}$ in $\{2\}^6$. Since $\{4^3\}$ appears in the product $\{\lambda\}\{2\}$ only if $\{\lambda\}=\{442\}$, it follows that the coefficient of $\{4^3\}$ in $\{2\}^6$ is the same as the coefficient of $\{442\}$ in $\{2\}^5$, which, by (4.4), is equal to 16. There are thus 16 linearly independent invariants linear in each of 6 conics.

To determine the classes of the irreducible sets of invariants we need to know the distribution of the terms $\{4^3\}$ in expansion of $\{2\}\otimes\{\lambda\}$, for all the partitions (λ) of 6. We observe that the distribution of the term

$\{442\}$ in $\{2\}\otimes\{\lambda\}$, when (λ) is a partition of 5, from (4.8) is given by

$$\{5\}+\{41\}+\{32\}+\{221\}+\{1^5\}.$$

Thus when each of these S-functions is multiplied* by $\{2\}$, one of the S-functions only in the product will give a partition (λ) of 6 for which $\{2\}\otimes\{\lambda\}$ contains $\{444\}$. We know also that $\{444\}$ appears in $\{2\}\otimes\{6\}$. It is clear, then, that the correspondences must be

$$\{6\}\to\{5\}.$$

$$\{42\}\to\{41\}+\{32\},$$

$$\{222\}\to\{221\},$$

$$\{1^6\}\to\{1^5\}.$$

Thus the classes of the invariants linear in each of the six conics are

$$\{6\},\quad \{42\},\quad \{222\},\quad \{1^6\}. \tag{5.1}$$

As a check, the total number of invariants is

$$f^{(6)}+f^{(42)}+f^{(222)}+f^{(1^6)}=1+9+5+1=16.$$

The value of $f^{(\lambda)}$ may be obtained from Young's expression

$$\frac{n!\,\Pi(a_r-a_s-r+s)}{\Pi(a_r+h-r)}$$

or it may be read directly from a table of characters†.

We consider next the invariants linear in each of 9 conics‡. The number of such invariants is the coefficient of $\{6^3\}$ in $\{2\}^9$. The equations (4.3) and (4.4) give us the expansions of $\{2\}^4$ and $\{2\}^5$, and we observe that $\{6^3\}$ appears in the product $\{\lambda\}\{\mu\}$ only if

$$\lambda_1+\mu_3=\lambda_2+\mu_2=\lambda_3+\mu_1=6.$$

Hence by multiplying the coefficients of such pairs of S-functions in (4.3) and (4.4) and adding, we obtain as the number of invariants

$$6.15+3.6+6.30+8.24+3.20+7.30+6.16+3.10=876.$$

To obtain the classes of these 876 invariants we need to know the distribution of $\{6^3\}$ in the expansions of $\{2\}\otimes\{\lambda\}$ when $\{\lambda\}$ is a partition of 9.

---

* Ordinary multiplication. When new multiplication is meant, it will be specially indicated.

† See D. E. Littlewood (2), 265.

‡ These are of less interest than those of 6 or less, since all necessary information is supplied by 6 or less (*cf.* Peano's theorem).

The expansions of $\{2\}\otimes\{n\}$ are known, and we note first the coefficients of $\{6^3\}$ in $\{2\}\otimes\{9\}$, $\{2\}\otimes[\{8\}\{1\}]$, $\{2\}\otimes[\{7\}\{2\}]$, $\{2\}\otimes[\{6\}\{3\}]$ and $\{2\}\otimes[\{5\}\{4\}]$.

The expansion of $\{2\}\otimes\{9\}$ includes $\{6^3\}$, and $\{9\}$ is one of the classes. The expansion of $\{2\}\otimes[\{8\}\{1\}] = [\{2\}\otimes\{8\}]\{2\}$ includes $\{6^3\}$ once, as obtained from the product $\{664\}\{2\}$. Since $\{8\}\{1\} = \{9\}+\{81\}$, it follows that $\{81\}$ is not one of the classes.

Since $\{2\}\otimes[\{7\}\{2\}]$ includes $\{6^3\}$ twice as obtained from the products of $\{662\}\{4\}$ and $\{644\}\{2^2\}$, and $\{2\}\otimes[\{6\}\{3\}]$ includes $\{6^3\}$ three times, from the products $\{66\}\{6\}$, $\{642\}\{42\}$, $\{444\}\{222\}$, it follows that $\{72\}$ and $\{63\}$ are both classes of these invariants. Similarly $\{54\}$ is not.

For the rest, we obtain the number of classes included in $\{2\}\otimes[\{\lambda\}\{\mu\}]$ where (λ) and (μ) are partitions of 4 and 5 respectively by picking out the coefficient of $\{666\}$ in $[\{2\}\otimes\{\lambda\}][\{2\}\otimes\{\mu\}]$, using the equations (4.7) and (4.8). Tabulating these results, we obtain

	$\{5\}$	$\{41\}$	$\{32\}$	$\{311\}$	$\{221\}$	$\{21^3\}$	$\{1^5\}$
$\{4\}$	3	3	4	0	3	0	1
$\{31\}$	2	5	5	5	5	3	1
$\{22\}$	3	4	5	1	4	1	1
$\{211\}$	0	3	3	7	3	4	0
$\{1^4\}$	0	1	1	2	1	1	0

We notice that

$$\{5\}\{31\} = \{81\}+\{72\}+\{711\}+\{63\}+\{621\}+\{531\}$$

contains two of the classes which we have already obtained as $\{72\}$ and $\{63\}$. The other partitions are not classes. But

$$\{4\}\{41\} = \{81\}+\{72\}+\{711\}+\{63\}+\{621\}+\{54\}+\{531\}+\{441\}$$

contains three classes. The third must clearly be $\{441\}$.

Again, $$\{5\}\{2^2\} = \{72\}+\{621\}+\{522\}$$

contains three classes. One is $\{72\}$, and $\{621\}$ is not a class. Hence $\{522\}$ must count for 2 classes. That is to say, the coefficient of $\{6^3\}$ in $\{2\}\otimes\{522\}$ is 2. The circumstance is treated in the manner of a repeated root of an algebraic equation.

The product

$$\{1^4\}\{1^5\} = \{2^4\,1\}+\{2^3\,1^3\}+\{2^2\,1^5\}+\{21^7\}+\{1^9\}$$

contains no classes, and the same is true of

$$\{21^2\}\{1^5\} = \{32211\}+\{321^4\}+\{31^6\}+\{2^3\,1^3\}+\{2^2\,1^5\}+\{21^7\}.$$

But $\{21^3\}\{1^4\}$ contains one class, as does $\{2^2\}\{1^5\}$. It follows that $\{32^3\}$ and $(3^2\,1^3\}$ are both classes. Also $\{5\}\{1^4\}$ gives no class, but $\{4\}\{1^5\}$ gives one. Thus $\{41^5\}$ is a class.

Lastly, since $\{5\}\{211\}$ gives no class, so that $\{5211\}$ is absent, and

$$\{4\}\{221\} = \{621\}+\{522\}+\{5211\}+\{4221\}$$

contains 3 classes, of which $\{522\}$ accounts for 2; clearly the remaining class is $\{4221\}$.

The classes are given by

$$\{9\}+\{72\}+\{63\}+\{441\}+2\{522\}+\{4221\}+\{41^5\}+\{3^2\,1^3\}+\{32^3\} \quad (5.2)$$

As a check, the sum of the degrees of the characters of the symmetric group is

$$1+27+48+84+2\times 120+216+56+120+84 = 876,$$

which is the correct number of invariants.

6. *Quaternary quadrics.* Invariants of quaternary forms are of type $\{r^4\}$. Hence an even number of quadrics is required to form an invariant. There are no invariants linear in 2 quadrics. We consider the invariants which are linear respectively in 4, 6, 8 and 10 quadrics.

In the equations (4.7), the S-function $\{2^4\}$ appears only in $\{2\}\otimes\{4\}$. There is thus one single invariant linear in 4 quadrics, which is symmetric in these ground forms.

For the case of 6 quadrics, we seek first the coefficient of $\{3^4\}$ in $\{2\}^6$. This will be the same as the coefficient of $\{3^3\,1\}$ in $\{2\}^5$, which, by (4.4), is 5. There are thus 5 invariants which are linear in 6 quadrics.

To determine the class, we observe first the distribution of $\{3331\}$ in the equations (4.8). It appears only in $\{2\}\otimes\{221\}$. Thus the class appears in $\{221\}\{1\}$ but not in $\{\lambda\}\{1\}$ for any other partition (λ) of 5. Clearly the class is

$$\{222\}, \quad (6.1)$$

and, as a check, the degree of the corresponding character is $f^{(222)} = 5$.

The number of invariants linear in 8 quadrics is equal to the coefficients of $\{4^4\}$ in $\{2\}^8$. From (4.2) and (4.4) we see that the signi-

ficant terms in the expansion of $\{2\}^3$ are

$$3\{42\}+\{33\}+\{411\}+2\{321\}+\{222\},$$

and the significant terms in the expansion of $\{2\}^5$,

$$16\{442\}+10\{4411\}+10\{433\}+24\{4321\}+10\{4222\}.$$

By multiplication we see that the coefficient of $\{4^4\}$ in $\{2\}^8$ is

$$3\,.\,16+1\,.\,10+1\,.\,10+2\,.\,24+1\,.\,10=126,$$

which is the required number of invariants.

To determine the classes we examine first the expansions of $\{2\}\otimes\{8\}$ and $\{2\}\otimes[\{m\}\{n\}]$, where $m+n=8$. The expansion of $\{2\}\otimes\{8\}$ contains $\{4^4\}$, so, referring to Theorem I, $\{8\}$ is a class. Also $\{4^4\}$ appears in the following products: $\{4^3\,2\}\{2\}$, $\{4^3\}\{4\}$, $\{4^2\,2^2\}\{2^2\}$, $\{4^2\,2\}\{42\}$, $(42^3)\{2^3\}$, $\{4^2\}\{4^2\}$, $\{42^2\}\{42^2\}$, $\{2^4\}\{2^4\}$. Hence the number of classes in $\{7\}\{1\}$, $\{6\}\{2\}$, $\{5\}\{3\}$, $\{4\}\{4\}$ are respectively 1, 2, 2, 3. Thus the classes corresponding to partitions into less than or equal to two parts are $\{8\}$, $\{62\}$, $\{44\}$.

We note that $\{1^5\}\{1^3\}$ contains no class, but $\{1^4\}\{1^4\}$ contains one class, as is indicated by the product $\{431\}\{431\}$. Thus $\{2^4\}$ is a class, but no other class has first part 2.

Again, $\{3\}\{1^5\}$ contains a class, as indicated by the product $\{42\}\{442\}$, but $\{4\}\{1^4\}$ contains none. Thus $\{31^5\}$ is a class.

The product $\{5\}\{21\}$ contains but one class, which is known to be $\{62\}$, thus $\{521\}$ is absent. But $\{4\}\{22\}=\{62\}+\{521\}+\{422\}$ contains two classes. Thus $\{422\}$ is a class.

The list

$$\{8\}+\{62\}+\{44\}+\{422\}+\{2^4\}+\{31^5\} \tag{6.2}$$

gives all the classes. As a check, the number of invariants is $\Sigma f^{(\lambda)}$, *i.e.*,

$$f^{(8)}+f^{(62)}+f^{(44)}+f^{(422)}+f^{(2^4)}+f^{(31^5)}=1+20+14+56+14+21=126.$$

The number of invariants linear in 10 quadrics is equal to the coefficient of $\{5^4\}$ in $\{2\}^{10}$. We make use of (4.4) and remember that $\{5^4\}$ appears in $\{\lambda_1\lambda_2\lambda_3\lambda_4\}\{\mu_1\mu_2\mu_3\mu_4\}$ if, and only if,

$$\lambda_1+\mu_4=\lambda_2+\mu_3=\lambda_3+\mu_2=\lambda_4+\mu_1=5.$$

Then clearly, if $k^{(\lambda)}$ is the coefficient of $\{\lambda\}$ in $\{2\}^5$, the required coefficient is

$$k^{(55)}k^{(55)}+k^{(541)}k^{(541)}+k^{(532)}k^{(532)}+2k^{(5311)}k^{(442)}+2k^{(5221)}k^{(433)}+k^{(4411)}k^{(4411)}$$
$$+k^{(4321)}k^{(4321)}+2k^{(4222)}k^{(3331)}+k^{(3322)}k^{(3322)}$$
$$=6\,.\,6+24\,.\,24+30\,.\,30+2\,.\,21\,.\,16+2\,.\,20\,.\,10+10\,.\,10$$
$$+24\,.\,24+2\,.\,10\,.\,5+6\,.\,6=3{,}396,$$

We determine the manner of separation of these 3,396 invariants into primitive sets as follows.

First we notice from (4.1) and (4.2) that the expansions of $\{2\}^2$ and $\{2\}^3$ contain no partition into four odd parts. Hence $\{2\}\otimes[\{7\}\{1\}^3]$ does not contain a term $\{5^4\}$. There is no class of which the first part is greater than or equal to 7.

Next, from (4.7), $\{2\}\otimes\{2^2\}$ and $\{2\}\otimes\{1^4\}$ each contain a partition into four odd parts. Thus $\{2\}\otimes[\{6\}\{2^2\}]$ and $\{2\}\otimes[\{6\}\{1^4\}]$ each contain $\{5^4\}$. It follows that $\{62^2\}$ and $\{61^4\}$ are classes, and these are the only classes whose first part is 6.

The following table gives the coefficients of $\{5^4\}$ in $\{2\}\otimes[\{\lambda\}\{\mu\}]$, where (λ) and (μ) are partitions of 5.

	$\{5\}$	$\{41\}$	$\{32\}$	$\{311\}$	$\{221\}$	$\{21^3\}$	$\{1^5\}$
$\{5\}$	0	0	1	1	2	1	1
$\{41\}$	0	3	4	6	5	5	1
$\{32\}$	1	4	5	6	6	5	2
$\{311\}$	1	6	6	11	6	6	1
$\{221\}$	2	5	6	6	5	5	2
$\{21^3\}$	1	5	5	6	5	4	1
$\{1^5\}$	1	1	2	1	2	1	2

From the first line of the table we can determine all the classes with first part 5, for if (λ) is any partition of 5, the number of classes in $\{5\}\{\lambda\}$ is known, and every S-function in this product begins with 6 or more except for the one term $\{5, \lambda_1, \lambda_2, \ldots\}$. We can thus determine whether this partition represents a class. It is easily seen that $\{5311\}$ and $\{5221\}$ are the classes with first part 5.

Turning to the other extremity of the partitions of 10, we note first that $\{1^{10}\}$ is a class. The expansions of $\{2\}\otimes\{1^n\}$ are known, and include all partitions of $2n$ which are in Frobenius's nomenclature of one of the types

$$\binom{r+1}{r}, \quad \binom{r+1,\ s+1}{r,\ s}, \quad \binom{r+1,\ s+1,\ t+1}{r,\ s,\ t}, \quad \ldots.$$

We call these *alternating* partitions. $\{5^4\}$ is an alternating partition. Also, if $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an alternating partition, so is $\{5-\lambda_4,\ 5-\lambda_3,\ 5-\lambda_2,\ 5-\lambda_1\}$. We look for alternating partitions in the equations (4.5)–(4.8). The expansion of $\{2\}\otimes\{1^2\}$ contains an alternating partition, and thus $[\{2\}\otimes\{1^2\}][\{2\}\otimes\{1^8\}]$ contains $\{5^4\}$, and $\{1^2\}\{1^8\}$ contains one class. This

is clearly the class $\{1^{10}\}$. Similarly, since $\{2\}\otimes\{1^3\}$ contains 2 alternating partitions, it follows that $\{2^3\,1^4\}$ is a class. From the expansion of $\{2\}\otimes\{31\}$ we deduce that $\{421^4\}$ is a class. There are no other classes with 6 or more parts.

Turning now to the last row of the table, we have given the number of classes in the expansion of $\{\lambda\}\{1^5\}$ for every partition (λ) of 5. As this includes only one S-function with 5 parts, we can determine whether $\{\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_4+1, \lambda_5+1\}$ represents a class or not. It is thus seen that $\{43111\}$ and $\{33211\}$ represent the classes corresponding to partitions with 5 parts.

Next

$$\{32\}\{21^3\} = \{5311\}+\{52111\}+\{4411\}+\{4321\}+2\{43111\}+\{42211\}$$
$$+\{421^4\}+\{33211\}+\{3^2\,1^4\}+\{32^2\,1^3\}$$

includes exactly 5 classes which are the known classes:

$$\{5311\}+2\{43111\}+\{421^4\}+\{33211\}.$$

The remaining partitions are not classes.

The product $\{41\}\{41\}$ now shows that $\{442\}$ is a class, and the product $\{41\}\{221\}$ points to the partition $\{4222\}$. The list is now complete.

The following are the classes of invariants which are linear in 10 quaternary quadrics:

$$\{622\}+\{61^4\}+\{531^2\}+\{52^2\,1\}+\{4^2\,2\}+\{42^3\}+\{3^2\,21^2\}+\{431^3\}$$
$$+\{421^4\}+\{2^3\,1^4\}+\{1^{10}\}. \quad (6.3)$$

As a check, we have

$$\Sigma f^{(\lambda)} = 225+126+567+525+252+300+450+525+350+75+1 = 3396.$$

These are the same as the partitions given by Young*, except that he also gives the partition $\{2^5\}$. This, however, would appear to be an error; the set of invariants which he describes as $T_{22222}I_6$ does not in fact exist. The operator T_{22222} annihilates every invariant of the quadrics†.

7. *Reducibility.* We proceed to distinguish the reducible from the irreducible primitive sets of invariants.

The invariant of 3 conics corresponds to $\{2\}\otimes\{3\}$. The reducible invariants obtained by multiplying two such invariants must correspond

* H. W. Turnbull and A. Young (4).

† Young proves that T_{22222} annihilates certain of the invariants, but the reduction was not pressed far enough.

to terms in the expansion of

$$[\{2\}\otimes\{3\}][\{2\}\otimes\{3\}] = \{2\}\otimes[\{3\}\{3\}]$$
$$= \{2\}\otimes\{6\}+\{2\}\otimes\{51\}+\{2\}\otimes\{42\}+\{2\}\otimes\{33\}.$$

There are no invariants of class $\{51\}$ or $\{33\}$. Thus the reducible invariants of 6 conics belong to the classes $\{6\}$ and $\{42\}$.

The irreducible invariants are thus as follows:

Three conics: $\{3\}$. (7.1)

Six conics: $\{2^3\}$, $\{1^6\}$. (7.2)

It may be noticed that the invariants corresponding to $\{2\}\otimes\{3\}\otimes\{1^2\}$ are identically zero; only those which correspond to $\{2\}\otimes\{3\}\otimes\{2\}$ exist.

For 9 conics the reducible invariants obtained by multiplying 3 invariants of class $\{3\}$ correspond to

$$\{3\}\otimes\{3\} = \{9\}+\{72\}+\{63\}+\{522\}+\{441\}.$$

The invariants $\{3\}\otimes\{21\}$ and $\{3\}\otimes\{1^3\}$ will not exist since the invariants $\{3\}\otimes\{1^2\}$ do not exist. But this would follow of necessity from the commutative property of multiplication.

We also obtain reducible invariants by multiplying the invariants of 3 conics by the irreducible invariants of 6 conics. The classes will thus be contained in the expansions

$$\{3\}\{2^3\} = \{522\}+\{4221\}+\{3222\},$$
$$\{3\}\{1^6\} = \{41^5\}+\{31^6\}.$$

Of these the type $\{31^6\}$ is identically zero, because the conic has only six coefficients.

We are thus left with a single primitive set of irreducible invariants of 9 conics, and the class is

Nine conics: $\{33111\}$. (7.3)

Turning to quaternary quadrics, the invariants of 4 and of 6 quadrics are obviously irreducible.

Four quadrics $\{4\}$. (7.4)

Six quadrics: $\{2^3\}$. (7.5)

The reducible invariants of 8 quadrics are obtained by multiplying two invariants each of 4 quadrics. We thus obtain for the reducible classes

$$\{4\}\otimes\{2\} = \{8\}+\{62\}+\{44\}.$$

There remain the following irreducible classes.

Eight quadrics: $\{422\}$, $\{2222\}$, $\{31^5\}$. (7.6)

A reducible invariant of 10 quadrics can only be the product of an invariant of 4 quadrics and an invariant of 6 quadrics. The reducible classes are thus given by

$$\{4\}\{222\} = \{622\}+\{5221\}+\{4222\}.$$

There remain the following irreducible invariants:

Ten quadrics: $\{61^4\}$, $\{531^2\}$, $\{4^2 2\}$, $\{3^2 21^2\}$, $\{431^3\}$,
$\{421^4\}$, $\{2^3 1^4\}$, $\{1^{10}\}$. (7.7)

8. *The total number of invariants of degree less than or equal to* 10 *of m quadrics.* The concomitants of degree n in less than n quadrics can be found from the concomitants linear in n quadrics by making some of the quadrics identical. We can thus determine all the concomitants of degree n in any set of quadrics if the concomitants linear in n quadrics are known and separated into primitive sets.

Consider a primitive set of class (ν) of invariants linear in n quadrics. The number of invariants is $f^{(\nu)}$. From Theorem IV, § 3, if the quadrics are made identical in sets of $\tau_1, \tau_2, \ldots, \tau_i$, the number of independent invariants is reduced to $K_{\nu\tau}$, where $K_{\nu\tau}$ is the coefficient of $\{\nu\}$ in $h_{\tau_1} h_{\tau_2} \ldots h_{\tau_i}$.

It is known*, however, that these coefficients $K_{\nu\tau}$ are the same as the coefficients in the expansion

$$\{\nu\} = \Sigma K_{\nu\tau} a_1^{\tau_1} a_2^{\tau_2} \ldots a_i^{\tau_i},$$

where $\{\nu\}$ is an S-function of the m quantities $a_1, a_2, \ldots, a_m$.

Let there be given a set of m quadrics. The number of combinations of quadrics from which we can form invariants of degrees respectively $\tau_1, \tau_2, \ldots, \tau_i$, is equal to the number of terms in the monomial symmetric function $\Sigma a_1^{\tau_1} \ldots a_i^{\tau_i}$. Hence the total number of invariants of class (ν) that can be obtained by various combinations of the m quadrics is equal to the number of terms in $\Sigma K_{\nu\tau} a_1^{\tau_1} a_2^{\tau_2} \ldots a_i^{\tau_i} = \{\nu\}$, when expressed in terms of monomial symmetric functions. This number of terms can be obtained by putting $a_1 = a_2 = \ldots = a_m = 1$, and is thus equal to the S-function $\{\nu\}$ of the roots of $(x-1)^m = 0$. This is known†, and is given by the formula,

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* D. E. Littlewood (2), 104. It follows immediately from the reciprocal transformation (6.4; 6).

† D. E. Littlewood (2), 126.

if $(\nu) \equiv (\nu_1, \nu_2, \ldots, \nu_p)$ is a partition of r,

$$\phi_m^{(\nu)} = \frac{\Pi(\nu_r - \nu_s - r + s)}{\Pi(\nu_r + p - r)!} (Q_\nu{}^m) = \frac{f^{(\nu)}}{n!} (Q_\nu{}^m),$$

where $(Q_\nu{}^m)$ is the product of the first ν_i terms from each i-th row of

$$\begin{array}{cccccc} m, & m+1, & m+2, & m+3, & \ldots, \\ m-1, & m, & m+1, & m+2, & \ldots, \\ m-2, & m-1, & m, & m+1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

THEOREM V. *The total number of invariants of degree n and class (ν) of m quadrics is equal to $\phi_m^{(\nu)}$.*

The total number of invariants of degree n is obtained by summing $\phi_m^{(\nu)}$ for all the classes.

We give below the number of invariants of each class for each degree we have examined, for a system of m conics and for a system of m quaternary quadrics. For degree 6 in conics we give the reducible and the irreducible invariants, the total number of invariants and the total number of irreducible invariants. For higher degrees we give only the numbers for each irreducible class.

Conics.

Degree 3; $\{3\}$; $\frac{1}{3!} m(m+1)(m+2)$.

Degree 6; $\{6\}$; $\frac{1}{6!} m(m+1)(m+2)(m+3)(m+4)(m+5)$.

$\{42\}$; $\frac{9}{6!} m^2(m^2-1)(m+2)(m+3)$.

$\{2^3\}$; $\frac{5}{6!} (m+1)\, m^2(m-1)^2(m-2)$.

$\{1^6\}$; $\frac{1}{6!} m(m-1)(m-2)(m-3)(m-4)(m-5)$.

Total; $\frac{1}{6!} (16m^6+30m^5+220m^4-30m^3+484m^2)$.

Total irreducible; $\frac{1}{6!} (6m^6-30m^5+90m^4-210m^3+264m^2-120m)$.

Degree 9; $\{3^2 1^3\}$; $\frac{120}{9!} m^2(m^2-1)(m^2-4)(m+1)(m-3)(m-4)$.

Quadrics.

Degree 4; $\{4\}$; $\frac{1}{4!}\, m(m+1)(m+2)(m+3).$

Degree 6; $\{2^3\}$; $\frac{5}{6!}\,(m+1)\,m^2(m-1)^2\,(m-2).$

Degree 8; $\{422\}$; $\frac{56}{8!}\, m^2(m^2-1)(m^2-4)(m-1)(m+3).$

$\{2^4\}$; $\frac{14}{8!}\,(m+1)\,m^2\,(m-1)^2\,(m-2)^2\,(m-3).$

$\{31^5\}$; $\frac{21}{8!}\, m(m^2-1)(m^2-4)(m-3)(m-4)(m-5).$

Degree 10; $\{61^4\}$; $\frac{126}{10!}\, m(m^2-1)(m^2-4)(m^2-9)(m^2-16)(m+5).$

$\{531^2\}$; $\frac{567}{10!}\, m^2(m^2-1)(m^2-4)(m^2-9)(m+1)(m+4).$

$\{4^2\,2\}$; $\frac{252}{10!}\,(m-2)(m-1)^2\,m^2\,(m+1)^2\,(m+2)^2\,(m+3).$

$\{3^2\,21^2\}$; $\frac{450}{10!}\, m^2\,(m^2-1)^2\,(m^2-4)(m-3)(m-4).$

$\{431^3\}$; $\frac{525}{10!}\, m^2\,(m^2-1)(m^2-4)(m^2-9)(m+1)(m-4).$

$\{421^4\}$; $\frac{350}{10!}\, m^2\,(m^2-1)(m^2-4)(m^2-9)(m-4)(m-5).$

$\{2^3\,1^4\}$; $\frac{75}{10!}\,(m+1)\,m^2\,(m-1)^2\,(m-2)(m-3)(m-4)$
$\times (m-5)(m-6).$

$\{1^{10}\}$; $\frac{1}{10!}\, m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)$
$\times (m-7)(m-8)(m-9).$

9. *The determination of the invariants.* In general, the number of different invariant expressions that can be written down exceeds considerably the number of linearly independent invariants. The linear relations between such invariant forms may be intricate and numerous.

To avoid the difficulties of a full analysis of these linear relations, two methods are available.

First*, it is known that there are exactly the same number of standard Young tableaux obtained from n symbols each used twice, but never in the same column, as there are concomitants of type $\{\lambda\}$, linear in n quadrics. These tableaux give a complete set of linearly independent concomitants of type $\{\lambda\}$.

Thus for 4 binary quadratics the tableaux are

$$\begin{pmatrix} \alpha\alpha & \beta\beta \\ \gamma\gamma & \delta\delta \end{pmatrix}, \quad \begin{pmatrix} \alpha\alpha & \gamma\gamma \\ \beta\beta & \delta\delta \end{pmatrix}, \quad \begin{pmatrix} \alpha\alpha & \beta\gamma \\ \beta\gamma & \delta\delta \end{pmatrix},$$

and the invariants are

$$A = (\alpha\gamma)^2(\beta\delta)^2, \quad B = (\alpha\beta)^2(\gamma\delta)^2, \quad C = (\alpha\beta)(\alpha\gamma)(\beta\delta)(\gamma\delta).$$

The separation of these standard forms into the various conjugate sets can be effected without very much difficulty when the number of invariants is small. Thus, in the above case the symmetric invariant is

$$A+B-C$$

and the two invariants of class $\{2^2\}$ are

$$A-C, \quad B-C.$$

But when the number of invariants is large, as in the case of the 3,396 invariants linear in 10 quadrics, such a separation could only be effected by a methodical general procedure based on substitutional analysis. Methods for this do suggest themselves, but considerable difficulties arise, chiefly because of the third factor M_r in Young's expression for a primitive idempotent $P_r N_r M_r$. These difficulties have not yet been resolved.

Alternatively, we may obtain our complete set of linearly independent concomitants by operating on a strictly limited number of invariant expressions with substitutional operators. Theoretically, this method is very much simpler. The number of invariants in each primitive set is equal to the degree of the corresponding symmetric group character, which in turn is equal to the number of substitutional operators corresponding to standard Young tableaux. Each tableau will thus give us one invariant of the primitive set. These operators automatically annihilate all invariants which belong to different primitive sets.

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* D. E. Littlewood (3).

In practice, however, these operators may contain a large number of terms, which leave us with very complicated expressions for our invariants. Thus, for the alternating invariant of 6 conics, the expression in standard forms involves only two terms,

$$(\alpha\beta\delta)(\alpha\gamma\epsilon)(\beta\gamma\zeta)(\delta\epsilon\zeta)-(\alpha\beta\gamma)(\alpha\delta\zeta)(\beta\delta\zeta)(\gamma\epsilon\zeta),$$

but if we operate on the first term of this with the alternating operator we obtain a set of 60 such terms with alternating signs.

In the case of the alternating invariant of 10 quadrics, the alternating operator gives 30,240 terms, but Turnbull and Young* have shown that the odd 240 terms are sufficient. The expression in standard forms has not yet been obtained, but the indications are that it will contain considerably fewer than 240 terms.

To express the invariants with the aid of the substitutional operators, the only difficulty lies in the choice of a suitable invariant on which to operate. Many invariant expressions will be annihilated by the operator. A general method of determining a suitable invariant on which to operate will be illustrated with respect to the class $\{5311\}$ of invariants linear in 10 quadrics, which contains $f^{(5311)} = 567$ invariants.

We choose a standard Young tableau corresponding to this partition, *e.g.*, using the digits 0 to 9 to represent the 10 quadrics,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & & \\ 8 & & & & \\ 9 & & & & \end{pmatrix}. \qquad (9.1)$$

The order of the symbols gives us the following sequence of partitions:

$$\{1\},\ \{2\},\ \{3\},\ \{4\},\ \{5\},\ \{51\},\ \{52\},\ \{53\},\ \{531\},\ \{5311\}.$$

From the sequence of expansions of

$$\{2\}\otimes\{1\},\ \{2\}\otimes\{2\},\ \{2\}\otimes\{3\},\ \{2\}\otimes\{4\},\ \{2\}\otimes\{5\},\ \{2\}\otimes\{51\},$$

$$\{2\}\otimes\{52\},\ \{2\}\otimes\{531\},\ \{2\}\otimes\{5311\},$$

we choose a sequence of partitions ending with $\{5^4\}$, each of which is one of the terms obtained in multiplying the preceding S-function by $\{2\}$.

* H. W. Turnbull and A. Young, (4).

Such a sequence is

$$\{2\},\ \{4\},\ \{42\},\ \{44\},\ \{442\},\ \{5421\},\ \{5432\},\ \{5542\},\ \{5533\},\ \{5555\}.$$

Corresponding to this sequence we build the Young tableau of type $\{5^4\}$ with each symbol repeated, *i.e.*,

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 5 \\ 2 & 2 & 3 & 3 & 7 \\ 4 & 4 & 6 & 7 & 8 \\ 5 & 6 & 8 & 9 & 9 \end{pmatrix}.$$

We operate on the corresponding invariant

$$(0245)\,(0246)\,(1368)\,(1379)\,(5789) \tag{9.2}$$

with the substitutional operator PN as obtained from the Young tableau (9.1).

To find the other 566 invariants of the same class we may operate on this invariant with the permutations which transform (9.1) into each of the other 566 standard Young tableaux. The third factor, M_r of Young's primitive idempotent in this case may safely be ignored.

If we had chosen a different tableau, instead of (9.1), *e.g.*,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & & \\ 8 & & & & \\ 9 & & & & \end{pmatrix}, \tag{9.3}$$

we would have obtained a different sequence, *e.g.*,

$$\{1\},\ \{2\},\ \{3\},\ \{4\},\ \{41\},\ \{42\},\ \{52\},\ \{53\},\ \{531\},\ \{5311\},$$

and this might have led to quite a different type of invariant, *e.g.*, from the tableau

$$\begin{pmatrix} 0 & 0 & 3 & 3 & 6 \\ 1 & 1 & 4 & 4 & 7 \\ 2 & 2 & 5 & 5 & 8 \\ 6 & 7 & 8 & 9 & 9 \end{pmatrix},$$

we obtain the invariant

$$(0126)\,(0127)\,(3458)\,(3459)\,(6789). \tag{9.4}$$

This is the invariant used by Young* and denoted by I_5. It is simpler than (9.2). Being already symmetric in the quadrics 0, 1, 2, and in the quadrics 4, 5, and also alternating in the quadrics 8, 9, the number of terms obtained when operating on it by the operator corresponding to the tableau (9.3) will not exceed 720. Young gives the invariant as $T_{5311} I_5$, where his operator T_{5311} is the modulus of the Frobenius algebra rather than the primitive idempotent. The number of terms involved would be very much more numerous.

For each irreducible class of invariants enumerated above, we give below an invariant and a tableau. The substitutional operator PN obtained from the tableau operating on the given invariant yields an invariant of the given class. In a few simple cases the invariant is given explicitly.

Conics.

Class $\{3\}$: invariant $(012)(012)$.

$\{2^3\}$: tableau $\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}$; invariant $(012)(013)(245)(345)$.

$\{1^6\}$: invariant $(012)(034)(135)(245)-(013)(024)(125)(345)$.

$\{3^2 1^3\}$: tableau $\begin{pmatrix} 0 & 5 & 6 \\ 1 & 7 & 8 \\ 2 \\ 3 \\ 4 \end{pmatrix}$; invariant $(012)(034)(135)(246)(578)(678)$.

Quadrics.

Class $\{4\}$: invariant $(0123)(0123)$.

$\{2^3\}$: invariant $(0134)(0235)(1245)$.

$\{42^2\}$: tableau $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$; invariant $(0234)(0235)(1467)(1567)$.

$\{2^4\}$: tableau $\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$; invariant $(0124)(0135)(2467)(3567)$.

* *Loc. cit.*

$\{31^5\}$: tableau $\begin{pmatrix} 0 & 1 & 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$; invariant

$$(0135)(0146)(2347)(2567).$$

For ten quadrics the simplest invariant on which to operate is that given as (9.4) and denoted by I_5 in the paper by Turnbull and Young. It is suitable for several of the classes. To determine the classes for which it can be used we investigate its substitutional type as follows.

The invariant is symmetric in the quadrics 0, 1, 2, and also in 3, 4, 5. It is alternating in 6 and 7, and also in 8 and 9. Further, the interchange of the sets 0, 1, 2 with 3, 4, 5 is equivalent to the interchange of 6, 7 with 8, 9. This interchange could be either symmetric or alternating.

If the interchange of 0, 1, 2 with 3, 4, 5 is symmetric, the substitutional type in these 6 quadrics would correspond to $\{3\}\otimes\{2\}$, while the substitutional type in 6, 7, 8, 9 would correspond to $\{1^2\}\otimes\{2\}$. If the interchange is alternating, then in each case the S-function $\{2\}$ would be replaced by $\{1^2\}$.

The complete substitutional type is given by

$$[\{3\}\otimes\{2\}][\{1^2\}\otimes\{2\}]+[\{3\}\otimes\{1^2\}][\{1^2\}\otimes\{1^2\}]$$
$$=[\{6\}+\{42\}][\{2^2\}+\{1^4\}]+[\{51\}+\{33\}][\{21^2\}].$$

The S-functions involved in this expression include $\{622\}$, $\{61^4\}$, $\{5311\}$, $\{5221\}$, $\{442\}$, $\{4222\}$, $\{33211\}$, $\{431^3\}$, $\{421^4\}$. Invariants of all these classes can be obtained by operating on I_5. This is in agreement with Young's conclusion. Invariants of the irreducible classes may be obtained by operating on I_5 with operators obtained from the following tableaux:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 6 \\ 4 & 5 & 7 \\ 8 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 8 \\ 7 & 9 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 8 \\ 7 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}.$$

There remain the classes $\{2^3 1^4\}$ and $\{1^{10}\}$. For the former we give the tableau

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

and the invariant (0126)(0137)(2458)(3459)(6789).

For the class $\{1^{10}\}$ the invariant

$$(0136)\,(0247)\,(1258)\,(3459)\,(6789)$$

is operated on by the alternating operator.

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ON THE CONCOMITANTS OF SPIN TENSORS

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1. Whether it be over the full linear group or over a restricted group of transformations such as the orthogonal group, the problem of finding the concomitants of a tensor is exactly equivalent to the problem of finding the concomitants of the algebraic form of which the given tensor is the tensor of coefficients. This problem has received much attention, and at least the concomitants of low degree in any form or system of ground forms can be obtained without undue difficulty.

Over the orthogonal group, however, a new type of tensor can be defined to which no algebraic form corresponds. These are the 2-valued *spin tensors*, which are exemplified in 4-dimensional space-time by the *wave tensors* of quantum mechanics.

It is our purpose to investigate the problem of finding the concomitants of a spin tensor.

In $n = 2\nu$ or $n = 2\nu+1$ variables there is an irreducible representation of the orthogonal group corresponding to each partition $(\lambda_1, \lambda_2, \ldots, \lambda_\nu)$ into less than or equal to ν parts†. The corresponding character is denoted by $[\lambda] \equiv [\lambda_1, \ldots, \lambda_\nu]$. Also, if $n = 2\nu$ and $\lambda_\nu = 0$, or for all such partitions if $n = 2\nu+1$, there is an associated representation, with the associated character $[\lambda]^*$, which differs from $[\lambda]$ only by a change of sign for an orthogonal transformation of negative determinant. For the rotation group only, if $n = 2\nu$ and $\lambda_\nu \neq 0$, the representation of the orthogonal group separates into two irreducible representations of the rotation group, and the corresponding character $[\lambda]$ is expressible as the sum of two simple characters $[\lambda]_1 + [\lambda]_2$. The two representations are interchanged by a transformation of negative determinant.

† See D. E. Littlewood (3), Chap. XI.

These comprise all the true representations, but in addition there are two valued or spin representations which correspond to partitions into ν parts in which each part is equal to half an odd integer. The simplest case is that of the partition $\left((\frac{1}{2})^\nu\right)$: this is called the *basic spin representation*. The basic spin character is denoted by $\Delta = [(\frac{1}{2})^\nu]$. A tensor which is transformed by the basic spin representation is called a *basic spin tensor*.

In conformity with the properties of true representations, if $n = 2\nu$ the basic spin representation, and similarly each spin representation, separates into two conjugate representations if the group is restricted to proper rotations, in accordance with the equation $\Delta = \Delta_1 + \Delta_2$. For the metric form Σx_i^2, with real tensors and transformations, Δ_1 and Δ_2 are conjugate complex numbers when ν is odd, but both are real if ν is even. With the general metric form $g_{ij}x^i x^j$, the reality or otherwise depends on the signature. For Lorentz transformations in space time, Δ_1 and Δ_2 are conjugate complex numbers.

2. *Reducibility.* The product of two simple tensors is in general a complex tensor, the simple subtensors of which correspond to the terms in the product of the corresponding S-functions. If the original tensors are not independent, some of these subtensors may be identically zero. Thus the product of two tensors of rank unity separates into a symmetric and an antisymmetric tensor in conformity with the equation†

$$\{1\}\{1\} = \{2\} + \{1^2\}.$$

If the two tensors of rank unity are identical, however, the antisymmetric tensor corresponding to $\{1^2\}$ is identically zero.

Nothing would be gained if all the tensors which appear in the product of two concomitant tensors were regarded as reducible, for this would lead to the reducibility of all concomitants except the ground-forms, while the problem of separating the zero from the non-zero tensors from the powers and products of the ground-form tensors would remain.

The product of two tensors of type $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $\{\mu_1, \mu_2, \ldots, \mu_r\}$ respectively always contains a tensor of type $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots, \lambda_r+\mu_r\}$, which is always distinct from zero. This simple tensor is called the *principal part* of the product of the tensors. A concomitant tensor is regarded as reducible only if it is the principal part of the product of two concomitant tensors, or is a linear combination of such principal parts.

† See D. E. Littlewood (5).

This definition of reducibility brings the invariant-theory of tensors into exact correspondence with the invariant-theory of algebraic forms.

For the orthogonal group the definition is similar. The principal part of two tensors of type $[\lambda_1, \lambda_2, \ldots, \lambda_\nu]$ and $[\mu_1, \mu_2, \ldots, \mu_\nu]$ respectively is of type $[\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots, \lambda_\nu+\mu_\nu]$.

This convention is extended to the spin tensors. Thus a basic spin tensor, namely one of type $[(\frac{1}{2})^\nu]$, will necessarily have a concomitant of type $[1^\nu]$ which will be regarded as reducible, being the principal part of the product of the basic spin tensor with itself. This convention is justified by the ensuing work, with the rather trivial modification that, if $n = 2\nu+1$ and ν is odd, then the concomitant which is regarded as reducible turns out to be of the associated type $[1^\nu]^*$ instead of $[1^\nu]$. But the discrimination between associated characters, in any case, is of small consequence when the number of variables is odd; for the orthogonal group is expressible as the direct product of the rotation group and a group of order 2.

3. *Concomitants of a basic spin tensor in* 3 *to* 8 *variables.* We now find the types of the irreducible concomitants of a basic spin tensor for the special cases in which there are 3 to 8 variables.

We consider first the cases in which the number of variables is odd, say $n = 2\nu+1$. It is sufficient to consider a transformation of positive determinant.

In 3 variables, let the characteristic equation of the transforming matrix be

$$(x-1)(x^2-2x\cos\theta+1) = 0,$$

whence we obtain for the symmetric function a_1,

$$a_1 = \{1\} = [1] = 2\cos\theta+1.$$

The basic spin character is given by

$$\Delta = 2\cos\tfrac{1}{2}\theta,$$

whence it is apparent that the latent roots of the corresponding transforming matrix are respectively

$$e^{\frac{1}{2}i\theta}, \quad e^{-\frac{1}{2}i\theta}.$$

The latent roots of the second induced matrix are

$$e^{i\theta}, \quad 1, \quad e^{-i\theta}.$$

Using the symbol $\otimes$ as for the new multiplication of S-functions†, we write

$$\Delta \otimes \{2\} = e^{i\theta} + 1 + e^{-i\theta} = 1 + 2\cos\theta = [1]. \qquad (3.1)$$

This expression for $\Delta \otimes \{2\}$ indicates the types of concomitant of degree 2 of a basic spin tensor. We thus see that there is but a single concomitant which is a tensor of rank unity. By our convention, however, this concomitant is regarded as reducible. A basic spin tensor in 3 variables, then, has no irreducible concomitant other than itself.

For the case of 5 variables the characteristic equation of the transforming matrix is taken as

$$(x-1)(x^2-2x\cos\theta_1+1)(x^2-2x\cos\theta_2+1) = 0,$$

whence

$$\{1\} = a_1 = 2(\cos\theta_1+\cos\theta_2)+1,$$

$$\{1^2\} = a_2 = 4\cos\theta_1\cos\theta_2+2(\cos\theta_1+\cos\theta_2)+2.$$

The basic spin character is

$$\Delta = 4\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2 = (e^{\frac{1}{2}i\theta_1}+e^{-\frac{1}{2}i\theta_1})(e^{\frac{1}{2}i\theta_2}+e^{-\frac{1}{2}i\theta_2}).$$

The latent roots are thus, clearly,

$$e^{\frac{1}{2}i(\theta_1+\theta_2)}, \quad e^{-\frac{1}{2}i(\theta_1+\theta_2)}, \quad e^{\frac{1}{2}i(\theta_1-\theta_2)}, \quad e^{-\frac{1}{2}i(\theta_1-\theta_2)}.$$

Hence

$$\begin{aligned}\Delta \otimes \{2\} &= e^{i(\theta_1+\theta_2)}+e^{i(\theta_1-\theta_2)}+e^{-i(\theta_1-\theta_2)}+e^{-i(\theta_1+\theta_2)}+e^{i\theta_1}+e^{i\theta_2}+e^{-i\theta_1}+e^{-i\theta_2}+2\\ &= 4\cos\theta_1\cos\theta_2+2(\cos\theta_1+\cos\theta_2)+2.\end{aligned}$$

$$\Delta \otimes \{2\} = [1^2]. \qquad (3.2)$$

Again, in 5 variables, the basic spin tensor has no irreducible concomitant other than itself.

For 7 variables, we obtain similarly

$$a_1 = 1+2\Sigma\cos\theta_1,$$

$$a_2 = 2+2\Sigma\cos\theta_1+4\Sigma\cos\theta_1\cos\theta_2,$$

$$a_3 = 3+4\Sigma\cos\theta_1+4\Sigma\cos\theta_1\cos\theta_2+8\cos\theta_1\cos\theta_2\cos\theta_3,$$

and

$$\Delta = \Sigma e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2\pm\theta_3)},$$

† See D. E. Littlewood (**2**), or (**3**), 206, or (**5**).

the summation being with respect to the 8 possible combinations of sign. Thence we obtain

$$\begin{aligned}\Delta \otimes \{2\} &= \Sigma e^{i(\pm\theta_1 \pm\theta_2 \pm\theta_3)} + \Sigma e^{i(\pm\theta_1 \pm\theta_2)} + 2\Sigma e^{\pm i\theta_1} + 4 \\ &= 8\cos\theta_1 \cos\theta_2 \cos\theta_3 + 4\Sigma \cos\theta_1 \cos\theta_2 + 4\Sigma \cos\theta_1 + 4 \\ &= a_3 + 1,\end{aligned}$$

$$\Delta \otimes \{2\} = \{1^3\} + \{0\}. \qquad (3.3)$$

Thus a basic spin tensor in 7 variables has a single irreducible concomitant of degree 2, which is an invariant. The fact that the irreducible concomitant is an invariant would seem to indicate that no irreducible concomitant of higher degree exists, and this is verified by a counting of coefficients.

The complete set of concomitants of degrees 3 and 4 must contain respectively

$$\frac{8.9.10}{1.2.3} = 120$$

and

$$\frac{8.9.10.11}{1.2.3.4} = 330$$

terms. The reducible concomitants of degree 3 are of types

$$[\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}] + [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}]$$

and would thus contain $112+8 = 120$ terms. The reducible concomitants of degree 4 are of types $[2, 2, 2]+[1, 1, 1]+[0]$ and would thus contain $294+35+1 = 330$ terms†. Since these utilise all existent terms, there are no irreducible concomitants of degree 3 or 4.

For an even number of variables we must distinguish between associated representations. However, we consider first a transformation of positive determinant. In 4 variables let the characteristic equation be

$$(x^2 - 2x\cos\theta_1 + 1)(x^2 - 2x\cos\theta_2 + 1) = 0,$$

whence

$$a_1 = [1] = 2(\cos\theta_1 + \cos\theta_2), \quad a_2 = [1^2] = 4\cos\theta_1 \cos\theta_2 + 2.$$

As with 5 variables, we have

$$\Delta = 4\cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2,$$

† See D. E. Littlewood (4).

and

$$\Delta \otimes \{2\} = 4 \cos\theta_1 \cos\theta_2 + 2(\cos\theta_1 + \cos\theta_2) + 2.$$

We thus have

$$\Delta \otimes \{2\} = [1^2] + [1].$$

The character $[1^2]$ is self-associated, but we must examine a transformation of negative determinant in order to decide whether the term [1] should be replaced by [1]*. Let the characteristic equation now be

$$(x^2-1)(x^2-2x\cos\theta+1) = 0.$$

The sum of the latent roots of the matrix of the spin transformation is equal to the spin character, which is zero. The sum of the squares of these latent roots is the spin character corresponding to the square of the transformation, for which the characteristic equation is

$$(x^2-2x+1)(x^2-2x\cos 2\theta+1) = 0,$$

and is thus equal to $4\cos\theta = 2e^{i\theta}+2e^{-i\theta}$. Clearly the latent roots of the spin transforming matrix must be

$$e^{\frac{1}{2}i\theta}, \quad e^{-\frac{1}{2}i\theta}, \quad -e^{\frac{1}{2}i\theta}, \quad -e^{-\frac{1}{2}i\theta}.$$

This gives

$$\Delta \otimes \{2\} = 2e^{i\theta}+2e^{-i\theta}-e^{i\theta}-e^{-i\theta}+2-2 = 2\cos\theta = [1].$$

We thus see that the correct equation, valid for positive and negative determinant, is

$$\Delta \otimes \{2\} = [1^2] + [1]. \qquad (3.4)$$

The concomitant corresponding to $[1^2]$ is, by our convention, reducible, and there is thus one irreducible concomitant of type [1].

For degree 3, the reducible concomitants are $[\frac{3}{2}, \frac{3}{2}]$ and $[\frac{3}{2}, \frac{1}{2}]$. We have†

$$[\tfrac{3}{2}, \tfrac{3}{2}] = \Delta(\{1^2\} - \{1\}),$$

and the corresponding concomitant contains $4(6-4) = 8$ terms. Similarly

$$[\tfrac{3}{2}, \tfrac{1}{2}] = \Delta(\{1\} - \{0\}),$$

and the corresponding concomitant contains $4(4-1) = 12$ terms. The total number of terms corresponding to $\Delta \otimes \{3\}$ is $\frac{4.5.6}{1.2.3} = 20 = 8+12$. There are thus no irreducible concomitants of degree 3 or more.

† See D. E. Littlewood (3), 256.

For 6 variables the characteristic equation

$$(x^2-2x\cos\theta_1+1)(x^2-2x\cos\theta_2+1)(x^2-2x\cos\theta_3+1)=0$$

gives us

$$[1]=2\Sigma\cos\theta_1, \qquad [1^2]=4\Sigma\cos\theta_1\cos\theta_2+3,$$

$$[1^3]=8\cos\theta_1\cos\theta_2\cos\theta_3+4\Sigma\cos\theta_1.$$

The latent roots of the spin transforming matrix consist of the 8 terms

$$e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2\pm\theta_3)},$$

which gives us

$$\Delta\otimes\{2\}=\Sigma e^{i(\pm\theta_1\pm\theta_2\pm\theta_3)}+\Sigma e^{i(\pm\theta_1\pm\theta_2)}+2\Sigma e^{\pm\theta_1}+4,$$

the summations being with respect to all combinations of the suffixes and all combinations of the alternative signs. This gives us

$$\Delta\otimes\{2\}=8\cos\theta_1\cos\theta_2\cos\theta_3+4\Sigma\cos\theta_1\cos\theta_2+4\Sigma\cos\theta_1+4$$

$$=[1^3]+[1^2]+[0].$$

To discriminate between associated representations, we consider the characteristic equation

$$(x^2-1)(x^2-2x\cos\theta_1+1)(x^2-2x\cos\theta_2+1)=0,$$

for which

$$[1]=2\Sigma\cos\theta_1, \quad [1^2]=4\cos\theta_1\cos\theta_2+1, \quad [1^3]=0.$$

The latent roots of the corresponding spin transforming matrix are

$$\pm e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2)},$$

whence

$$\Delta\otimes\{2\}=4\cos\theta_1\cos\theta_2=[1^2]-[0].$$

Thus the correct equation for any orthogonal transformation is

$$\Delta\otimes\{2\}=[1^3]+[1^2]+[0]^*. \tag{3.5}$$

There are thus 2 irreducible concomitants of degree 2 of a basic spin tensor in 6 variables, a linear complex, of type $[1^2]$, and a relative invariant.

Numerical evaluation of terms again shows that there exist no irreducible concomitants of higher degree.

For 8 variables the characteristic equation

$$\Pi(x^2-2x\cos\theta_1+1)=0$$

gives

$$[1] = 2\Sigma \cos\theta_1, \qquad [1^2] = 4\Sigma \cos\theta_1 \cos\theta_2 + 4,$$

$$[1^3] = 8\Sigma \cos\theta_1 \cos\theta_2 \cos\theta_3 + 6\Sigma \cos\theta_1,$$

$$[1^4] = 16 \cos\theta_1 \cos\theta_2 \cos\theta_3 \cos\theta_4 + 8\Sigma \cos\theta_1 \cos\theta_2 + 6.$$

The latent roots of the spin transforming matrix consist of the 16 terms

$$e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2\pm\theta_3\pm\theta_4)},$$

and we have

$$\Delta\otimes\{2\} = \Sigma e^{i(\pm\theta_1\pm\theta_2\pm\theta_3\pm\theta_4)} + \Sigma e^{i(\pm\theta_1\pm\theta_2\pm\theta_3)} + 2\Sigma e^{i(\theta_1\pm\theta_2)} + 4\Sigma e^{\pm i\theta_1} + 8$$

$$= 16 \cos\theta_1 \cos\theta_2 \cos\theta_3 \cos\theta_4 + 8\Sigma \cos\theta_1 \cos\theta_2 \cos\theta_3 + 8\Sigma \cos\theta_1 \cos\theta_2 + 8$$

$$= [1^4] + [1^3] + [1] + 2.$$

To distinguish between associated characters we consider the characteristic equation

$$(x^2-1)(x^2-2x\cos\theta_1+1)(x^2-2x\cos\theta_2+1)(x^2-2x\cos\theta_3+1) = 0,$$

for which

$$[1] = 2\Sigma \cos\theta_1, \qquad [1^2] = 4\Sigma \cos\theta_1 \cos\theta_2 + 2.$$

$$[1^3] = 8 \cos\theta_1 \cos\theta_2 \cos\theta_3 + 2\Sigma \cos\theta_1, \qquad [1^4] = 0.$$

The latent roots of the spin transforming matrix are

$$\pm e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2\pm\theta_3)},$$

whence

$$\Delta\otimes\{2\} = 8 \cos\theta_1 \cos\theta_2 \cos\theta_3 = [1^3] - [1].$$

Evidently the correct formula for the general case is

$$\Delta'\otimes\{2\} = [1^4] + [1^3] + [1]^* + [0] + [0]^*. \qquad (3.6)$$

The irreducible concomitants correspond to $[1^3]$, $[1]^*$, $[0]$ and $[0]^*$, there being, in particular, two invariants, one absolute and the other relative.

The complete set of concomitants of degree 3 contains

$$\frac{16.17.18}{1.2.3} = 816$$

terms. The reducible concomitants are of type

$$[(\tfrac{3}{2})^4], \quad [(\tfrac{3}{2})^3, \tfrac{1}{2}], \quad [(\tfrac{3}{2}), (\tfrac{1}{2})^3], \quad [(\tfrac{1}{2})^4], \quad [(\tfrac{1}{2})^4],$$

and these contain respectively†

$$224, \quad 448, \quad 112, \quad 16, \quad 16$$

terms, which gives the same total, 816. Thus there are no irreducible concomitants of degree greater than 2.

Enumeration of terms for the concomitants of degree 4 indicates that there is but one absolute invariant of this degree. Thus the squares of the two invariants of degree 2 must be connected by a syzygy. This is clearly the only possible syzygy that can exist, for no two reducible concomitants will be of the same type.

These results in 3 to 8 variables are not sufficient to indicate the behaviour in the general case. Fortunately, however, the concomitants of degree 2 can be determined without much difficulty for the general case, and these will be obtained in the next section.

But this does not completely solve the problem. We have seen that in 3 to 8 variables there are no irreducible concomitants of degree greater than 2. In 10 variables, however, we shall find, for the first time, irreducible concomitants of degree 3 and 4.

4. *Concomitants of degree 2 of a basic spin tensor in n variables.* The cases $n = 2\nu$ and $n = 2\nu+1$ must be considered separately. We consider first the case $n = 2\nu$. The characteristic equation is taken first to be

$$\Pi(x^2-2x\cos\theta_1+1) = 0.$$

We make use of the known result‡ that

$$\begin{aligned}\Delta^2 &= \Delta\otimes\{2\}+\Delta\otimes\{1^2\}\\ &= [1^\nu]+[1^\nu]^*+[1^{\nu-1}]+[1^{\nu-1}]^*+\ldots+[0]+[0]^*. \qquad (4.1)\end{aligned}$$

Again, if Z_r denotes the sum of the r-th powers of the latent roots of the spin transforming matrix, we have

$$\Delta\otimes\{2\} = \tfrac{1}{2}(Z_1{}^2+Z_2),$$

$$\Delta\otimes\{1^2\} = \tfrac{1}{2}(Z_1{}^2-Z_2),$$

† By using Theorem VII, p. 256, D. E. Littlewood (3).

‡ D. E. Littlewood (5).

so that $$\Delta\otimes\{2\}-\Delta\otimes\{1^2\}=Z_2.$$

These latent roots are given by

$$e^{\frac{1}{2}i(\pm\theta_1\pm\theta_2\pm\ldots\pm\theta_\nu)}.$$

Hence

$$Z_2=\Sigma e^{i(\pm\theta_1\pm\theta_2\pm\ldots\pm\theta_\nu)}=2^\nu\Pi\cos\theta_1=\Pi(2\cos\theta_1).$$

Comparing coefficients in the characteristic equation for the matrix of transformation of the variables, we have

$$\begin{aligned}\Pi(x^2-2x\cos\theta_1+1)&\equiv 1-a_1x+a_2x^2-\ldots+x^{2\nu}\\&\equiv(-1)^\nu[a_\nu x^\nu-a_{\nu-1}(x^{\nu+1}+x^{\nu-1})+\ldots].\end{aligned}$$

If now we put $x=i$ we obtain

$$(-i)^\nu\Pi(2\cos\theta_1)=(-i)^\nu[a_\nu-2a_{\nu-2}+2a_{\nu-4}-2a_{\nu-6}+\ldots],$$

so that

$$\begin{aligned}Z_2=\Delta\otimes\{2\}-\Delta\otimes\{1^2\}&=\Pi(2\cos\theta_1)=a_\nu-2a_{\nu-2}+2a_{\nu-4}-\ldots\\&=[1^\nu]-2[1^{\nu-2}]+2[1^{\nu-4}]-\ldots.\end{aligned}\qquad(4.2)$$

From (4.1) and (4.2) we obtain

$$\Delta\otimes\{2\}=[1^\nu]+[1^{\nu-1}]+[1^{\nu-3}]+2[1^{\nu-4}]+[1^{\nu-5}]+[1^{\nu-7}]+2[1^{\nu-8}]+\ldots.$$

It remains to distinguish between associated representations, and for this we must consider the characteristic equation

$$(x^2-1)\,\Pi(x^2-2x\cos\theta_1+1)=0.$$

Putting $x=i$ in this, and comparing with the expanded form

$$x^{2\nu}-a_1x^{2\nu-1}+\ldots-1=0,$$

we see that

$$2(-i)^{\nu-1}\Pi(2\cos\theta_1)=(-1)^{\nu-1}i^\nu(2ia_{\nu-1}-2ia_{\nu-3}+2ia_{\nu-5}-\ldots).$$

In this case, since $a_r=-a_{n-r}$, we obtain

$$\Delta\otimes\{2\}+\Delta\otimes\{1^2\}=0,$$

$$\Delta\otimes\{2\}-\Delta\otimes\{1^2\}=z_2=2\,\Pi(2\cos\theta_1).$$

Thus

$$\Delta\otimes\{2\}\,\Pi(2\cos\theta_1)=a_{\nu-1}-a_{\nu-3}+a_{\nu-5}-\ldots.$$

The correct equation giving the concomitants of degree 2 for a basic spin tensor in $n = 2\nu$ variables is therefore

$$\Delta \otimes \{2\} = [1^\nu] + [1^{\nu-4}] + [1^{\nu-4}]^* + [1^{\nu-8}] + [1^{\nu-8}]^* + \ldots$$
$$+ [1^{\nu-1}] + [1^{\nu-3}]^* + [1^{\nu-5}] + [1^{\nu-7}]^* + \ldots. \quad (4.3)$$

For $n = 2\nu+1$ variables we take for the characteristic equation

$$(x-1)\,\Pi(x^2 - 2x\cos\theta_1 + 1) = 0.$$

The spin character is the same as for $n = 2\nu$ variables, and using the same nomenclature we obtain as before

$$Z_2 = \Pi(2\cos\theta_1).$$

By putting $x = i$ in the characteristic equation we obtain

$$\Delta \otimes \{2\} - \Delta \otimes \{1^2\} = \Pi(2\cos\theta)$$
$$= a_\nu - a_{\nu-1} - a_{\nu-2} + a_{\nu-3} + a_{\nu-4} - \ldots.$$

Also it is known that†

$$\Delta^2 = \Delta \otimes \{2\} + \Delta \otimes \{1^2\} = a_\nu + a_{\nu-1} + a_{\nu-2} + a_{\nu-3} + \ldots.$$

We thus have

$$\Delta \otimes \{2\} = a_\nu + a_{\nu-3} + a_{\nu-4} + a_{\nu-7} + a_{\nu-8} + \ldots.$$

It is not difficult to discriminate between associated characters. Since

$$\Delta^2 = \Delta \otimes \{2\} + \Delta \otimes \{1^2\} = [0] + [1]^* + [1^2] + [1^3]^* + \ldots,$$

it follows that

$$\left.\begin{aligned} &\text{for even } \nu, \quad \Delta \otimes \{2\} = [1^\nu] + [1^{\nu-3}]^* + [1^{\nu-4}] + [1^{\nu-7}]^* + \ldots, \\ &\text{for odd } \nu, \quad \Delta \otimes \{2\} = [1^\nu]^* + [1^{\nu-3}] + [1^{\nu-4}]^* + [1^{\nu-7}] + \ldots. \end{aligned}\right\} \quad (4.4)$$

5. *Concomitants under the rotation group for $n = 2\nu$ variables.* Under the rotation group for $n = 2\nu$ variables, the basic spin character Δ separates into two simple components, $\Delta = \Delta_1 + \Delta_2$. The corresponding tensor T separates similarly, $T = T_1 + T_2$. The concomitants of T of degree 2 consist of the concomitants of degree 2 of T_1, the concomitants of degree 2 of T_2, and the simultaneous concomitants of T_1 and T_2 linear in each. We proceed to determine which of the concomitants of T belong to this last class.

† See D. E. Littlewood (3), 255.

The simultaneous concomitants linear in T_1 and T_2 correspond to the expansion of the product of the characters $\Delta_1\Delta_2$. We have

$$\begin{aligned}\Delta_1\Delta_2 &= \tfrac{1}{2}[\Pi(2\cos\tfrac{1}{2}\theta_1)+\Pi(2i\sin\tfrac{1}{2}\theta_1)]\,\tfrac{1}{2}[\Pi(2\cos\tfrac{1}{2}\theta_1)-\Pi(2i\sin\tfrac{1}{2}\theta_1)]\\ &= \tfrac{1}{4}[\Pi(2\cos\tfrac{1}{2}\theta_1)^2-(-1)^\nu\Pi(2i\sin\tfrac{1}{2}\theta_1)^2]\\ &= \tfrac{1}{4}[\Pi(2+2\cos\theta_1)-(-1)^\nu\Pi(2-2\cos\theta_1)]\\ &= \tfrac{1}{4}(a_\nu+2a_{\nu-1}+2a_{\nu-2}+\ldots-a_\nu+2a_{\nu-1}-2a_{\nu-2}+\ldots)\\ &= a_{\nu-1}+a_{\nu-3}+a_{\nu-5}\ldots.\end{aligned}$$

It follows that of the concomitants of T given in (4.3), the following are simultaneous concomitants linear in T_1 and T_2,

$$[1^{\nu-1}]+[1^{\nu-3}]+[1^{\nu-5}]+\ldots. \tag{5.1}$$

The character $[1^\nu]$ separates into 2 conjugate components in the form

$$[1^\nu]=[1^\nu]_1+[1^\nu]_2. \tag{5.2}$$

It is easily seen that, since the concomitants of T_1 alone are conjugate to the concomitants of T_2 alone, the concomitants of T_1 alone are given by

$$[1^\nu]_1+[1^{\nu-4}]+[1^{\nu-8}]+\ldots, \tag{5.3}$$

while the concomitants of T_2 alone are given by

$$[1^\nu]_2+[1^{\nu-4}]+[1^{\nu-8}]+\ldots. \tag{5.4}$$

6. *The case of ten variables.* The absence of irreducible concomitants of degree greater than 2 in the cases considered, combined with the very simple expression for the types of concomitant of degree 2 in n-variables, strongly suggests that this absence will hold for any number of variables, and that the simple expression will thus describe the complete set of irreducible concomitants.

This expectation is not, however, realised. In the case of 10 variables, for the first time, there appear irreducible concomitants of degrees 3 and 4.

By (4.3) we see that the irreducible concomitants of degree 2 of a basic spin tensor in 10 variables correspond to

$$[1^4]+[1^2]^*+[1]+[1]^*+[0].$$

The basic spin tensor has 32 terms, and the concomitants of degree 2 should have $\frac{1}{2}.32.33=528$ terms. The reducible concomitant of type

$[1^5]$ has 252 terms, and the irreducible concomitants enumerated above,

$$210+45+10+10+1=276.$$

The equation $528=252+276$ checks the correctness of our concomitants.

The reducible concomitants of degree 3 correspond to

$$[(\tfrac{3}{2})^5],\quad [(\tfrac{3}{2})^4\tfrac{1}{2}],\quad [(\tfrac{3}{2})^2(\tfrac{1}{2})^3],\quad [(\tfrac{3}{2})(\tfrac{1}{2})^4],\quad [(\tfrac{3}{2})(\tfrac{1}{2})^4],\quad [(\tfrac{1}{2})^5],$$

which are respectively equal to†

$$\Delta(a_5-a_4),\quad \Delta(a_4-a_3),\quad \Delta(a_2-a_1),\quad \Delta(a_1-1),\quad \Delta(a_1-1),\quad \Delta.$$

These concomitants therefore contain

$$32(252-210+210-120+45-10+10-1+10-1+1)=32\times 186$$

terms.

The total number of terms in the concomitants of degree 3 is

$$\frac{32\,.\,33\,.\,34}{1\,.\,2\,.\,3}=32\times 187.$$

We have thus 32 terms unaccounted for. There must exist an irreducible basic spin concomitant of degree 3, the type being

$$[(\tfrac{1}{2})^5]. \qquad (6.1)$$

For the degree 4 the total number of terms is

$$\frac{32\,.\,33\,.\,34\,.\,35}{1\,.\,2\,.\,3\,.\,4}=52360.$$

The reducible concomitants have the types

$$[2^5]+[2^4]+[2^2]+3[2]+[2^4\,1]+[2^2\,1^2]+[2^2\,1^3]+2[21^4]+2[21^3]$$
$$+2[21]+2[1^5]+[1^4]+[1^2]+2[1]+[0]$$

with the number of terms

$$5544+8910+770+3\,.\,54+13860+5940+7392+2\,.\,2100$$
$$+2\,.\,1728+2\,.\,320+2\,.\,252+210+45+2\,.\,10+1=51654.$$

There remains 706 terms which must belong to irreducible concomitants. These irreducible concomitants must be obtainable on multiplying the basic spin concomitant tensor of degree 3 by the ground-form basic spin

† D. E. Littlewood (3), 256, Theorem VII.

tensor, and will thus be included in

$$\Delta^2 = [0]+[0]^*+[1]+[1]^*+[1^2]+[1^2]^*+[1^3]+[1^3]^*+[1^4]+[1^4]^*+[1^5].$$

Of these the last term $[1^5]$ is excluded, since the concomitant is regarded as reducible. To avoid undue calculation we do not distinguish between associated characters. The numbers of terms associated with tensors of type [0], [1], $[1^2]$, $[1^3]$, $[1^4]$ are respectively 1, 10, 45, 120, 210.

Since

$$1+45+2\,.\,120+2\,.\,210 = 706,$$

it is clear that the irreducible concomitants of degree 4 must be of the following types,

$$[0], \quad [1^2], \quad [1^3], \quad [1^3], \quad [1^4], \quad [1^4]. \tag{6.2}$$

Further calculation shows that all concomitants of degree 5 are reducible, these accounting for all of the 376,992 terms, and that there is, in fact, one syzygy of type $[(\frac{3}{2})^4, \frac{1}{2}]$.

The case of 11 variables is very similar to the case of 10 variables. The irreducible concomitants are of the following types:

$$\left.\begin{array}{ll} \text{Degree 2:} & [1],\ [1^2], \\ \text{Degree 3:} & [(\frac{1}{2})^5], \\ \text{Degree 4:} & [0],\ [1^3],\ [1^4], \end{array}\right\} \tag{6.3}$$

The problem of finding the concomitants of degree higher than 2 in more than 11 variables has not been examined.

7. *The construction of the concomitants.* In 4 or in 5 variables Eddington has, in effect but not explicitly, obtained the irreducible concomitants of a basic spin tensor, which he describes as a *wave tensor*. By this "16-fold frame" he has shown† how to express the 16 terms of a mixed wave tensor of rank 2 in terms of space tensors, for 4 variables, as the sum of 2 invariants, one absolute and one relative, a covariant 4-vector, a contravariant 4-vector, and a 6-vector. This is in accordance with the equation

$$\Delta^2 = [0]+[0]^*+[1]+[1]^*+[1^2].$$

In 5 variables the same frame gives one 5-vector, one 10-vector and one invariant, in accordance with the equation

$$\Delta^2 = [0]+[1]^*+[1^2].$$

† A. S. Eddington (1), Chap. II.

Further, he has shown† that if we multiply a covariant wave tensor of rank 2 by a term which he denotes by E_{45}, we obtain a mixed wave tensor which can be expressed as a sum of space vectors as above. The concomitants of degree 2 of a wave tensor of rank 1 clearly form a symmetric wave tensor of rank 2. Because this tensor is symmetric, it has 10 instead of 16 terms, and we find that in 5 variables the invariant and the 5-vector, and in 4-variables the two invariants and the contravariant 4-vector, are identically zero. The remaining 10-vector, or the 6-vector and the covariant 4-vector, represent the concomitants of the basic spin tensor, or wave tensor.

The procedure can be generalized for any number of variables, and we shall proceed to obtain the concomitants of degree 2 of a basic spin tensor in any number of variables.

For $n = 2\nu$ or $n = 2\nu+1$ variables the spin-transforming matrix has 2^ν rows and columns. We take a set of $(2\nu+1)$ anticommuting matrices,

$$J_1,\ J_2,\ \ldots,\ J_{2\nu+1},$$

which satisfy $$J_r^2 = (-1)^\nu.$$

These can be obtained as follows. If

$$i', j', k';\ i'', j'', k'';\ \ldots;\ i^{(\nu)}, j^{(\nu)}, k^{(\nu)}$$

represent the basal elements of ν independent, mutually commuting quaternion algebras, then the $2\nu+1$ quantities,

$$i',\ j',\ k'i'',\ k'j'',\ k'k''i''',\ \ldots,\ k'k''\ldots k^{(\nu)},$$

are anticommuting. We take as the matrix representation of the direct product of these ν quaternion algebras, the direct product of the matrix representations of each, in which

$$i = \begin{bmatrix} \sqrt{(-1)} & \\ & -\sqrt{(-1)} \end{bmatrix},\quad j = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix},\quad k = \begin{bmatrix} & \sqrt{(-1)} \\ \sqrt{(-1)} & \end{bmatrix}.$$

Then the $2\nu+1$ matrices corresponding to the above $2\nu+1$ quantities, each multiplied if necessary by $\sqrt{(-1)}$ to make its square $(-1)^\nu$, form the required $2\nu+1$ matrices.

The matrices clearly form pairs of one real and one imaginary matrix, so that there will be ν real matrices and ν imaginary matrices, while the last one, corresponding to $k'k''\ldots k^{(\nu)}$, which is equal to the product of

† *Ibid.*, 103.

the other 2ν matrices, and is omitted if there are 2ν rather than $2\nu+1$ variables, will be real if ν is an even number, but imaginary if ν is odd.

If ν is even we suppose that

$$J_1, J_2, \ldots, J_\nu$$

represent the imaginary matrices, the remaining $J_{\nu+1}, \ldots, J_{2\nu+1}$ being real. If ν is odd, we suppose that

$$J_1, J_2, \ldots, J_\nu$$

represent the real matrices, $J_{\nu+1}, \ldots, J_{2\nu+1}$ being imaginary.

In either case, since $J_r^2 = (-1)^\nu$ it will be seen that $J_1, \ldots, J_\nu$ are skew symmetric matrices, while $J_{\nu+1}, \ldots, J_{2\nu+1}$ are symmetric. We put

$$\theta = J_1 J_2 \ldots J_\nu.$$

Then we shall have, for each J_r, real or imaginary,

$$J_r \theta = \theta \bar{J}_r = (-1)^\nu \theta \tilde{J}_r,$$

where $\bar{J}_r$ denotes the conjugate complex of J_r, and $\tilde{J}_r$ the transpose of J_r. Also

$$(-1)^\nu \theta^{-1} J_r = \tilde{J}_r \theta^{-1}$$

Let T denote the spin-transforming matrix, and let

$$T J_r = J_r' T,$$

so that also, by transposing this equation,

$$\tilde{J}_r \tilde{T} = \tilde{T} \tilde{J}_r',$$

where

$$J_r' = \Sigma a_{rs} J_s.$$

Then clearly we shall have

$$J_r' \theta = (-1)^\nu \theta \tilde{J}_r'.$$

Hence

$$\begin{aligned} \theta^{-1} \tilde{T} \theta T J_r &= \theta^{-1} \tilde{T} \theta J_r' T \\ &= (-1)^\nu \theta^{-1} \tilde{T} \tilde{J}_r' \theta T \\ &= (-1)^\nu \theta^{-1} \tilde{J}_r \tilde{T} \theta T \\ &= J_r \theta^{-1} \tilde{T} \theta T. \end{aligned}$$

Thus $\theta^{-1} \tilde{T} \theta T$ commutes with each J_r. Now the J_r's together with their products form a basis for the matrix algebra. It follows that

$\theta^{-1}\tilde{T}\theta T$ commutes with every matrix and is thus a scalar multiple of the unit matrix. Thus

$$\tilde{T}\theta T = \alpha\theta,$$

where α is numerical. It is not difficult to show that $\alpha = 1$. Since we take $|T| = \pm 1$, by considering the determinants of the matrices we see that $\alpha^{2^\nu} = 1$, so that α cannot vary continuously. Since $\alpha = 1$ when T is the unit matrix, it follows that $\alpha = 1$ for all transformations of positive determinant. If $n = 2\nu$, we must prove it also for one transformation of negative determinant, and it will follow for all cases. The case $T = j^{(\nu)}$ does lead to $\alpha = 1$. We thus have

$$\tilde{T}\theta T = \theta.$$

Now if $w_1, w_2, \ldots, w_r$, with $r = 2^\nu$, are the r components of a basic spin tensor, and $w_1', w_2', \ldots, w_r'$ are the r components of a second basic spin tensor, we put

$$\psi = \theta[w_s' w_t],$$

s and t denoting the row and column of the matrix from which a typical element is taken.

After transformation we have

$$\begin{aligned}\psi' &= \theta\tilde{T}[w_s' w_t]\, T \\ &= T^{-1}\theta[w_s' w_t]\, T \\ &= T^{-1}\psi T.\end{aligned}$$

If now $$\psi = \lambda_0 + \Sigma\lambda_i J_i + \Sigma\lambda_{ij} J_i J_j + \ldots,$$

where $\lambda_0, \lambda_i, \lambda_{ij}, \ldots$ are bilinear functions of the components w_r, w_r' of the two basic spin tensors, then the transformation will have the effect of replacing each J_r by

$$J_r' = \Sigma\alpha_{rs} J_r.$$

Thus the sets of quantities

$$\lambda_0, \lambda_i, \lambda_{ij}, \ldots$$

which are bilinear in the components of the two basic spin tensors will transform like tensors of the respective types

$$[0],\ [1],\ [1^2],\ [1^3],\ \ldots.$$

These are the concomitants linear in the two basic spin tensors, and the result conforms with the equations

$$\Delta^2 = [0]+[0]^*+[1]+[1]^*+[1^2]+\ldots \quad \text{if} \quad n = 2\nu,$$

$$\Delta^2 = [0]+[1]^*+[1]^2+[1^3]^*+\ldots \qquad \text{if} \quad n = 2\nu+1.$$

The number of concomitants is smaller for $n = 2\nu+1$ than for $n = 2\nu$ by reason of the identical relation

$$J_{2\nu+1} = J_1 J_2 \ldots J_{2\nu},$$

which can be used to simplify a product of the J_r's so that not more than ν factors are used.

Some of these concomitants are symmetric in the two ground forms, some are antisymmetric. The symmetric concomitants are those which correspond to the expansions (4.3) and (4.4), and when the ground forms are made identical these give the concomitants of degree 2 of a single ground form.

The construction of the concomitants of degree higher than 2 in 10 or more variables would appear to present a formidable problem and this will not be attempted here.

8. *The concomitants of non-basic spin tensors.* The preceding work has been concerned entirely with the case when the ground form is a *basic* spin tensor. We proceed to indicate very briefly how the results obtained, together with known properties of the symbol $\otimes$, may be used to determine the concomitants of other types of spin tensor. We illustrate with a ground form of type $[\frac{3}{2}, \frac{1}{2}]$ in 5 variables.

We make use of the formula†

$$(AB)\otimes\{n\} = \Sigma[A\otimes\{\lambda\}][B\otimes\{\lambda\}]$$

summed for all the partitions (λ) of n. Thus

$$\begin{aligned}[\{1\}\Delta]\otimes\{2\} &= [\{1\}\otimes\{2\}][\Delta\otimes\{2\}]+[\{1\}\otimes\{1^2\}][\Delta\otimes\{1^2\}] \\ &= \{2\}\{1^2\}+\{1^2\}[\{1\}+\{0\}] \\ &= \{31\}+\{211\}+\{21\}+\{1^2\}+\{1^2\} \\ &= [31]+[2]+[1^2]+[21]+[1^2]+[21]+[1]+[1^2]+[1^2].\end{aligned}$$

† D. E. Littlewood (5).

Now

$$[\{1\}\Delta]\otimes\{2\}=\left[[\tfrac{3}{2},\tfrac{1}{2}]+\Delta\right]\otimes\{2\}$$
$$=[\tfrac{3}{2},\tfrac{1}{2}]\otimes\{2\}+[\tfrac{3}{2},\tfrac{1}{2}]\Delta+\Delta\otimes\{2\}.$$

We have

$$[\tfrac{3}{2},\tfrac{1}{2}]\Delta=[\{1\}-\{0\}]\Delta\Delta$$
$$=[\{1\}-\{0\}][\{1^2\}+\{1\}+\{0\}]$$
$$=\{21\}+\{1^3\}+\{2\}+\{1^2\}+\{1\}-\{1^2\}-\{1\}-\{0\}$$
$$=[21]+[1]+[1^3]+[2].$$

Also
$$\Delta\otimes\{2\}=[1^2]+[1]+[0].$$

We thus have

$$[\tfrac{3}{2},\tfrac{1}{2}]\otimes\{2\}=[31]+[21]+2[1^2].$$

As a check, we note that

$$[\tfrac{3}{2},\tfrac{1}{2}]=[\{1\}-\{0\}]\Delta$$

contains $(5-1)4=16$ terms, so that $[\frac{3}{2},\frac{1}{2}]\otimes\{2\}$ should contain

$$\frac{16\,.\,17}{1\,.\,2}=136 \text{ terms.}$$

Of the terms on the right hand [31] contains 81 terms, [21] contains 35, and $[1^2]$ contains 10 terms. The equation

$$136=81+35+2\,.\,10$$

provides the check.

The above method is valid in all cases, but with increasing degrees or numbers of variables the work very soon becomes very laborious.

9. *The concomitants of a double wave tensor.* Eddington† makes use of *double wave tensors.* There are, for these, two sets of 5 variables, each transformed by an independent orthogonal transformation. The group of transformations is thus the direct product of two orthogonal groups, each on 5 variables. Every character is thus expressible as a product of two characters, each of an orthogonal group on 5 variables. We use a dash to denote the characters relative to the second set of 5 variables.

† A. S. Eddington (1), 154.

The double wave tensor is a basic spin tensor relative to each set of 5 variables, and the corresponding character may be written $\Delta\Delta'$, where Δ and Δ' denote basic spin characters relative to each set of 5 variables.

The concomitants of degree 2 correspond to

$$\begin{aligned}[\Delta\Delta']\otimes\{2\} &= [\Delta\otimes\{2\}][\Delta'\otimes\{2\}]+[\Delta\otimes\{1^2\}][\Delta'\otimes\{1^2\}]\\ &= [1^2][1^2]'+([1]+[0])([1]'+[0]')\\ &= [1^2][1^2]'+[1][1]'+[1]+[1]'+[0].\end{aligned}$$

Thus these concomitants of degree 2 consist of a double 10-vector, a double 5-vector, a 5-vector in each set of 5 variables, and an invariant. The number of terms involved is

$$\frac{16.17}{1.2}=136=100+25+5+5+1.$$

The concomitants of degree 3 will have

$$\frac{16.17.18}{1.2.3}=816 \text{ terms.}$$

The reducible concomitants of degree 3 will be of type

$$[\tfrac{3}{2},\tfrac{3}{2}][\tfrac{3}{2},\tfrac{3}{2}]', \quad [\tfrac{3}{2},\tfrac{1}{2}][\tfrac{3}{2},\tfrac{1}{2}]', \quad [\tfrac{3}{2},\tfrac{1}{2}]\Delta', \quad \Delta[\tfrac{3}{2},\tfrac{1}{2}]' \text{ and } \Delta\Delta'.$$

The number of terms involved will be

$$20.20+16.16+16.4+4.16+4.4=800.$$

The remaining 16 terms clearly must belong to an irreducible concomitant of degree 3 which is a double wave tensor, *i.e.*, of type $\Delta\Delta'$.

Of degree 4, the total number of terms in the concomitants, is

$$\frac{16.17.18.19}{1.2.3.4}=3876.$$

The reducible concomitants are of type

$$[2^2][2^2]',\ [21][21]',\ [21][11]',\ [11][21]',\ [11][11]',\ [2][2]',\ [2][1]',$$
$$[1][2]',\ [1][1]',\ [2],\ [1],\ [1]',\ [2]',\ [1],\ [1]',\ [0],$$

and, from the irreducible concomitant of degree 3 and the ground form tensor, $[11][11]'$.

The number of terms involved is

$$1225+1225+400+400+100+196+70+70+25+14+14+25$$
$$+5+5+1+100=3875.$$

There remains one term which clearly belongs to an irreducible invariant of degree 4. There are no more irreducible concomitants.

Summing up, the irreducible concomitants of a double wave tensor are as follows:

Of degree 2, a double 10-vector, a double 5-vector, a 5-vector in each group of variables and an invariant.

Of degree 3, a double wave tensor.

Of degree 4, an invariant.

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ON POSITIVE RIESZ AND ABEL TYPICAL MEANS

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1.1. It is known* that the strength of Riesz's method of summability (R, λ_n, κ) increases, for fixed κ, as the rate of increase of the sequence $\{\lambda_n\}$ decreases. But more is true than is implied by that statement. For it is proved in the two papers cited that, if $\lambda_n = \phi(\mu_n)$, where $\phi(u)$ is an increasing function tending to infinity with u, then summability (R, μ_n, κ) implies summability (R, λ_n, κ) provided that $\phi(u) = O(u^{\Delta})$ for large u, where Δ is any positive number, and that $\phi(u)$ satisfies certain other conditions which ensure that its growth should be "reasonably regular". Roughly speaking, then, we can say that summability (R, μ_n, κ) implies summability (R, λ_n, κ) not only when λ_n tends to infinity more slowly than μ_n, but, more generally, provided that λ_n does not tend to infinity very much more rapidly than μ_n. For example, the different methods of summability (R, n^p, κ) given by taking different values of p are, for a given κ, equivalent.

Now, when we say that two methods of summability, A and B, say, are such that summability (A) implies summability (B), this is, of course, the same as saying that if the means (A) of a given series possess the property of tending to a finite limit, then the means (B) must possess the same property (and that the limits of the two sets of means must be the same). It may or may not happen that the same thing holds when we replace the property of tending to a finite limit by some other property, such as that of being all positive. In this connection, I have recently pointed out† that the means $(R, n^p, 1)$ of the series

$$\tfrac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta$$

* G. H. Hardy, "The second theorem of consistency for summable series", *Proc. London Math. Soc.* (2), 15 (1916), 72–88; K. A. Hirst, "On the second theorem of consistency in the theory of summation by typical means", *ibid.* (2), 33 (1932), 353–366.

† B. Kuttner, "Note on the Riesz means of a Fourier series", *Journal London Math. Soc.*, 18 (1943), 148–154. This will be referred to in what follows as R.F.S.

are, for any θ, all positive if $p \leqslant 1$, but that, when $p \geqslant 2$, the means (R, n^p, κ) fail to possess this property however large κ may be. This result suggests the possibility of the truth of a theorem to the general effect that if the means (R, μ_n, κ) of a given series are all positive, then so are the means (R, λ_n, κ), provided that λ_n tends to infinity less rapidly than μ_n (and is a "reasonably regular" function of μ_n). On the other hand, it shows that, unlike the result in the corresponding "limit" problem, such a proposition is false when λ_n tends to infinity more rapidly than μ_n.

A special case of a theorem of the type indicated was obtained in R.F.S., where it was shown* that, if $\lambda_n = \mu_n{}^r$, where $0 < r < 1$, and if the means (R, μ_n, κ) of a given series are all positive, then so are the means (R, λ_n, κ), provided that κ is an integer. But this is a particularly simple case, the proof of such a theorem for integral values of κ being very much easier than that of the general result.

In §§ 2.1, 2.2 of the present paper we show that, if $\phi(u)$ is an increasing function which tends to infinity less rapidly than u, and which satisfies certain other conditions, if $\lambda_n = \phi(\mu_n)$, and if the means (R, μ_n, κ) of a given series are all positive, then so are the means (R, λ_n, κ). The restrictions which I have had to impose on the function $\phi(u)$ in order to obtain a proof of such a result are rather severe. Still, the conditions imposed are satisfied in the two simplest and most interesting cases; namely the cases $\phi(u) = u^r$ $(0 < r < 1)$ and $\phi(u) = \log u$.

In §§ 3.1, 3.2 we consider the analogous problem for the corresponding Abel means. It may be remarked that in the "limit" problem, far less is true for the Abel means than for the corresponding Riesz means. For example, if $0 < p < q$, summability (A, n^q) neither implies nor is implied by summability (A, n^p). But this is only because a series may be summable (A, n^q), and at the same time the series defining its (A, n^p) means may not converge, so that the definition of summability (A, n^p) breaks down. Miss Cartwright has proved† that if a series is summable (A, n^q), then it is summable (A, n^p), provided that the series defining the (A, n^p) means converges. On the other hand the converse, unlike the corresponding theorem for Riesz summability, is false. Further, Miss Cartwright has stated‡ the more general result that if $\lambda_n = \mu_n{}^r$, where $0 < r < 1$, then summability (A, μ_n) implies summability (A, λ_n), subject to a similar

* The result was stated only for the special case in which $\mu_n = n^p$, where $p > 0$. But it is easily seen that the proof applies in the general case.

† M. L. Cartwright, "The relation between the different types of Abel summation", *Proc. London Math. Soc.* (2), 31 (1930), 81–96.

‡ *Loc. cit.* (§ 1.5).

condition. The theorem that a corresponding result holds when $\lambda_n = \log \mu_n$ is due to Hardy*.

In the present paper we shall prove that, if $\phi(u)$ satisfies conditions similar to those imposed in dealing with the Riesz means, if, as before, $\lambda_n = \phi(\mu_n)$, and if the means (A, μ_n) of a given series are all positive, then so are the means (A, λ_n), subject (as in the corresponding "limit" results) to the condition that the series defining the means (A, λ_n) converges. It will be seen in § 3.1 that, if we assume that it is permissible to invert the order of two infinite integrations, the result is an immediate corollary of the theorems of Hausdorff and of Bernstein on absolutely and on completely monotonic functions†. The justification of this inversion, however, gives a certain amount of trouble.

1.2. We shall use throughout the following notation. Let

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

be any given series, and let $\{\mu_n\}$ be any sequence of numbers increasing to infinity, with $\mu_0 \geqslant 0$. Let $\phi(u)$ be an increasing function of u which tends to infinity as $u \to \infty$. We write

$$\lambda_n = \phi(\mu_n),$$

$$A(u) = A_\mu(u) = \sum_{\mu_n \leqslant u} a_n, \tag{2}$$

$$A_\lambda(u) = \sum_{\lambda_n \leqslant u} a_n. \tag{3}$$

As the expression (2) occurs frequently in the work, we omit the suffix μ to simplify the writing, but we retain the suffix λ in (3) in order to distin-

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* G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.* (2), 8 (1910), 301–320 (319).

† F. Hausdorff, "Summationsmethoden und Momentenfolgen, I and II", *Math. Zeitschrift*, 9 (1921), 74–109 and 280–299; S. Bernstein, "Sur les fonctions absolument monotones", *Acta. Math.*, 52 (1929), 1–66. An account of the theory is given by D. V. Widder, *The Laplace transform* (Princeton, 1941), Chapter 4. Following Widder, we call a function "absolutely monotonic" in a given (possibly infinite) interval if it has, in that interval, derivatives of all orders which are everywhere positive or zero, and "completely monotonic" if the derivatives of odd order are negative or zero, and the derivatives of even order positive or zero. Clearly a necessary and sufficient condition that $f(x)$ should be completely monotonic in (a, b) is that $f(-x)$ should be absolutely monotonic in $(-b, -a)$, so that theorems on completely monotonic functions reduce to theorems on absolutely monotonic functions, and *vice versa*, by a trivial change of variable.

guish it from (2). If k is any positive number we write, as is usual,

$$A^{(k)}(u) = \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} A(t)\, dt = \frac{1}{\Gamma(k+1)} \sum_{\mu_n \leqslant u} a_n (u-\mu_n)^k, \tag{4}$$

and use a similar notation with $A(u)$ replaced by $A_\lambda(u)$. Further, we write, for $s > 0$,

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\mu_n s}, \tag{5}$$

$$f_\lambda(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}, \tag{6}$$

whenever these series converge. It is well known that, if the series (5) converges for all $s > 0$, then the integral

$$s \int_0^\infty A(u) e^{-su}\, du \tag{7}$$

also converges for all $s > 0$, and is equal to $f(s)$. A corresponding result applies to the series (6).

The results which we shall prove are:

THEOREM 1. *Let $\kappa = n+\theta$, where n is a positive or zero integer, and where $0 < \theta \leqslant 1$. Suppose that, for $u > \mu_0$, $\phi(u)$ possesses $n+2$ derivatives, and that*

$$\phi(u) \geqslant 0, \tag{8}$$

$$(-1)^{\nu+1} \phi^{(\nu)}(u) \geqslant 0 \quad (\nu = 1, 2, \ldots, n+2). \tag{9}$$

Suppose also that $\phi(u)$ is continuous for $u \geqslant \mu_0$, and that $A^{(\kappa)}(u) \geqslant 0$ for all u. Then $A_\lambda^{(\kappa)}(u) \geqslant 0$ for all u.

*If, further**,

$$\phi''(u) < 0, \tag{10}$$

for $u > \mu_0$, then $A_\lambda^{(\kappa)}(u) > 0$ unless $A^{(\kappa)}(t)$ is identically zero for $\phi(t) \leqslant u$.

THEOREM 2. *Suppose that $\phi(u)$ satisfies the conditions of Theorem 1 for all values of n. Suppose that $f(s) \geqslant 0$ for all $s > 0$, and that the series (6) converges for all $s > 0$. Then $f_\lambda(s) \geqslant 0$ for all $s > 0$.*

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* *I.e.*, if $\phi'(u)$ is strictly decreasing, instead of being only supposed non-increasing, which is all that is implied by (9).

It may be remarked that the special case of Theorem 2 in which $\phi(u) = \log u$ is an immediate corollary of a known theorem*. The theorem in question, stated in the notation of the present paper, is that, if $\phi(u) = \log u$, and if the series (6) converges for all $s > 0$, then

$$f_\lambda(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} f(x)\, dx.$$

1.3. Before proceeding further, we make a few comments on the theorems. We first observe that, in Theorem 2, the convergence of (6) for all $s > 0$ implies that of (5) for all $s > 0$, so that the tacit assumption that $f(s)$ exists is justified. To prove this, we observe that, since $\phi'(u)$ is non-increasing, there is a constant K such that $\phi'(u) < K$ for sufficiently large u. Then, if we put $s = t$ in (6) and $s = Kt$ in (5), the ratio of the $n+1$-th term in (5) to the $n+1$-th term in (6) is

$$\exp\{-\mu_n Kt + \lambda_n t\} = \exp\{-t[K\mu_n - \phi(\mu_n)]\},$$

which is ultimately decreasing. Thus the convergence of (6) for $s = t$ implies that of (5) for $s = Kt$, which clearly establishes the result stated.

In considering summability by the typical means associated with the sequence $\{\lambda_n\}$, it is usual and convenient, though not absolutely essential, to suppose that $\lambda_0 \geqslant 0$. It is in order to ensure that this condition should be satisfied that we suppose that $\phi(u)$ satisfies equation (8).

We next note that the conditions of Theorem 1 (and hence also of Theorem 2) imply that $\phi(u)$ does not tend to infinity more rapidly than u. For, if X is any constant greater than μ_0, we have, for $u > X$,

$$\phi'(u) \leqslant \phi'(X),$$

and hence

$$\phi(u) \leqslant \phi(X) + (u - X)\phi'(X).$$

We remark that in Theorem 1 the amount which we have to assume about $\phi(u)$ increases with κ; the number of derivatives whose existence is asserted, and whose sign is specified, tends to infinity as $\kappa \to \infty$. This is to be expected, and a corresponding remark applies to Hirst's results†. The important cases are, of course, those in which $\phi(u)$ is such that the conditions of Theorem 1 are satisfied for all values of κ; the conditions of Theorem 2 are then also satisfied. It may easily be verified that this

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* O. Perron, "Zur Theorie der Dirichletschen Reihen", *Journal für Math.*, 134 (1908), 95–143 (131–137); G. H. Hardy, "On a case of term-by-term integration of an infinite series", *Messenger of Math.*, 39 (1910), 136–139.

† *Loc. cit.*

is the case* when $\phi(u)$ is of any of the forms u^r, $(\log u)^r$, $(\log\log u)^r$, ..., where $0 < r \leqslant 1$, or, more generally, when $\phi(u)$ is any linear combination of functions of these forms. A somewhat less simple example is the case in which

$$\phi(u) = \int_A^u t^{-a} (\log t)^{-b} (\log\log t)^{-c} \ldots dt,$$

where $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, ..., where a, b, c, ... are such that $\phi(u) \to \infty$ as $u \to \infty$ (*i.e.*, $a \leqslant 1$; if $a = 1$, then $b \leqslant 1$, etc.), where A is such that the repeated logarithms which occur are positive for $t \geqslant A$, and where $\mu_0 \geqslant A$. But whereas in Hirst's paper the conditions imposed are satisfied by any logarithmico-exponential function ("L function"), which tends to infinity, but not too rapidly, this is not the case here. In many cases where $\phi(u)$ is an L function which tends to infinity less rapidly than u, we find that $\phi(u)$ satisfies the conditions of Theorem 1 for a fixed value of κ, provided that μ_0 is sufficiently large, but that, whatever the value of μ_0, it is false that $\phi(u)$ satisfies the conditions for all κ. Suppose, to take a simple example, that $\phi(u) = (\log u)^2$. Writing

$$c_0 = 0; \quad c_\nu = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + 1/\nu \quad (\nu = 1, 2, 3, \ldots),$$

it may be proved by induction that, for $\nu \geqslant 1$,

$$(-1)^{\nu+1} \phi^{(\nu)}(u) = 2(\nu-1)\,! \,(\log u - c_{\nu-1}) u^{-\nu},$$

which is positive (or zero) for $u \geqslant \exp(c_{\nu-1})$. It is clear that, for a given κ, the conditions of Theorem 1 are satisfied provided that

$$\mu_0 \geqslant \exp(c_{n+1}),$$

n being defined as in Theorem 1. But, since $c_n \to \infty$ as $n \to \infty$, we see that, however large μ_0 may be, the conditions of Theorem 1 fail to be satisfied for all κ. Roughly speaking, we can say that every derivative ultimately has the "right" sign, but that, the greater the order of the derivative, the greater u has to be before the sign is "right". In theorems such as Hirst's we are concerned, in effect, with the behaviour of $\phi(u)$ as $u \to \infty$, so that if the behaviour of $\phi(u)$ is ultimately "correct", this is all that is required. But in theorems such as those of the present paper, we are concerned with the behaviour of $\phi(u)$ for all $u \geqslant \mu_0$, so that this is no longer the case.

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* Subject, of course, to the condition that in the cases in which $\phi(u)$ involves logarithms or repeated logarithms, μ_0 should be sufficiently large for these logarithms or repeated logarithms to be positive (or zero) for $u \geqslant \mu_0$.

Finally, we remark that the result of Theorem 2 enables us to fill up a gap in R.F.S. It was there proved that the proposition "if $f(\theta)$ is everywhere positive, then the Abel means (A, n^p) of the Fourier series of $f(\theta)$ are all everywhere positive" is true for $p = 2$ and for $p \leqslant 1$, but false for $p > 2$. The case $1 < p < 2$ was, however, left unsettled. By using Theorem 2 of the present paper, it is clear that the truth of the proposition for all $p < 2$ can be deduced at once* from its truth for $p = 2$.

1.4. We now make a slight simplification of the theorems. If $\mu_0 > 0$, we observe that, if we write

$$\mu_n' = \mu_n - \mu_0, \quad A'(u) = \sum_{\mu_n' \leqslant u} a_n, \quad \bar{f}(s) = \sum_{n=0}^{\infty} a_n e^{-\mu_n' s},$$

then

$$A'(u) = A(u+\mu_0); \quad \bar{f}(s) = e^{\mu_0 s} f(s).$$

Thus $A'^{(\kappa)}(\bar{u})$ is everywhere positive if $A^{(\kappa)}(u)$ is, and, similarly, $\bar{f}(s)$ is everywhere positive if $f(s)$ is. Further, $\lambda_n = \bar{\phi}(\mu_n')$ where $\bar{\phi}(u) = \phi(u+\mu_0)$, so that $\bar{\phi}(u)$ satisfies the same conditions for $u \geqslant 0$ as $\phi(u)$ satisfies for $u \geqslant \mu_0$. It is therefore clear that, in both Theorems 1 and 2, there is no loss of generality in supposing that $\mu_0 = 0$. We will therefore suppose that $\mu_0 = 0$ throughout the rest of the paper.

2.1. We now come to the proof of Theorem 1. We first observe that (9) implies that

$$\phi'(u) > 0 \quad (u > 0). \tag{11}$$

For if, for any u_0, $\phi'(u_0) = 0$, then, since $\phi'(u)$ is non-negative and non-increasing, $\phi'(u) = 0$ for $u > u_0$, which contradicts the assumption that $\phi(u) \to \infty$ as $u \to \infty$. We also observe that, for small u,

$$A^{(k)}(u) = O(u^k); \tag{12}$$

this fact, together with Lemma 1, ensures the convergence at the origin of certain integrals which will occur, and also the vanishing at the origin of certain integrated terms.

We require a number of lemmas. In all of these it is assumed that, for $u > 0$, $\phi(u)$ possesses $n+2$ derivatives, and satisfies equations (8) and (9). Equation (10) is assumed only where this is specifically stated.

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* [*Note added in proof.*] Since this paper was written, I have proved a stronger result. See *Journal London Math. Soc.*, 19 (1944), 77–84.

LEMMA 1. *Let ν be a positive integer. Then*

(i) *If $\nu \leqslant n+2$, $\phi^{(\nu)}(u)u^{\nu-1}$ is integrable in the neighbourhood of the origin.*

(ii) *If $\nu \leqslant n+1$, $\phi^{(\nu)}(u) = o(u^{-\nu})$ for small u.*

The proof is by induction. It is evident that (i) is true for $\nu = 1$. Hence the general result will follow if we show that:

(*a*) *If* (i) *is true for a given value of ν, then so is* (ii), *provided that $\nu \leqslant n+1$, and*

(*b*) *If* (i) *and* (ii) *are true for a given value of ν, then* (i) *is true for $\nu+1$.*

In order to prove (*a*), we observe that, since $\phi^{(\nu)}(u)$ is of constant sign, (i) implies that

$$\int_0^u |\phi^{(\nu)}(t)|\, t^{\nu-1}\, dt \to 0$$

as $u \to 0$. But, since $|\phi^{(\nu)}(t)|$ is a non-increasing function of t whenever $\nu \leqslant n+1$, we have

$$\int_0^u |\phi^{(\nu)}(t)|\, t^{\nu-1}\, dt \geqslant |\phi^{(\nu)}(u)| \int_0^u t^{\nu-1}\, dt = u^{\nu}|\phi^{(\nu)}(u)|/\nu;$$

and the result is now evident.

Moreover,

$$\int_u^T \phi^{(\nu+1)}(t)\, t^{\nu}\, dt = \phi^{(\nu)}(T)\, T^{\nu} - \phi^{(\nu)}(u)\, u^{\nu} - \nu \int_u^T \phi^{(\nu)}(t)\, t^{\nu-1}\, dt,$$

and (*b*) follows at once on making $u \to 0$.

COROLLARY. *$[\phi'(u)]^{n+1}u^n$ is integrable in the neighbourhood of the origin.*

For, by (ii),

$$[\phi'(u)]^{n+1}u^n = o[\phi'(u)].$$

LEMMA 2. *Let n be a positive integer or zero, and $0 < \theta < 1$. Let**

$$F(w) = \int_0^w A^{(n)}(u)\, \Phi(u)\, du, \tag{13}$$

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* It follows from Lemma 1, corollary, and equation (12) that the integral defining $F(w)$ converges at the origin.

where

$$\Phi(u) = \Phi(u, w) = [\phi(w) - \phi(u)]^{\theta-1} [\phi'(u)]^{n+1}. \tag{14}$$

Then

$$F(w) = \lim_{\eta \to 0} \frac{1}{\Gamma(1-\theta)} \int_0^{w-\eta} A^{(n+\theta)}(t)\, G(\eta, t)\, dt,$$

where

$$G(\eta, t) = G(\eta, t;\ w) = \Phi(w-\eta)(w-\eta-t)^{-\theta} - \int_t^{w-\eta} (u-t)^{-\theta} \Phi'(u)\, du. \tag{15}$$

It is evident that

$$F(w) = \lim_{\eta \to 0} F_\eta(w),$$

where

$$F_\eta(w) = \int_0^{w-\eta} A^{(n)}(u)\, \Phi(u)\, du.$$

Now, integrating by parts,

$$F_\eta(w) = A^{(n+1)}(w-\eta)\, \Phi(w-\eta) - \int_0^{w-\eta} A^{(n+1)}(u)\, \Phi'(u)\, du.$$

But

$$A^{(n+1)}(u) = \frac{1}{\Gamma(1-\theta)} \int_0^u (u-t)^{-\theta} A^{(n+\theta)}(t)\, dt.$$

Hence

$$\Gamma(1-\theta)\, F_\eta(w) = \Phi(w-\eta) \int_0^{w-\eta} (w-\eta-t)^{-\theta} A^{(n+\theta)}(t)\, dt - \int_0^{w-\eta} \Phi'(u)\, du \int_0^u (u-t)^{-\theta} A^{(n+\theta)}(t)\, dt.$$

On inverting the order of integration in the second integral, we obtain

$$\Gamma(1-\theta)\, F_\eta(w) = \int_0^{w-\eta} A^{(n+\theta)}(t)\, G(\eta, t)\, dt,$$

and the lemma follows at once.

LEMMA 3. *Let w be any fixed positive number. Suppose that, in the interval $(0, w)$, $f(t)$ is bounded and that $f(t) \geqslant 0$. Suppose also that, for some $\lambda > 0$,*

$$|f(w) - f(t)| = o(w-t)^\lambda \tag{16}$$

as $t \to w$. Let $G(\eta, t)$ be any function which satisfies the following conditions:

(a) $$\int_{w-\epsilon}^{w-\eta} G(\eta, t)\, dt \geqslant 0$$

for all sufficiently small ϵ, η $(\eta < \epsilon)$;

(*b*) $$\int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}|G(\eta, t)|\,dt$$

is bounded for all sufficiently small ϵ, η;

(*c*) *for any* $T<w$,

$$G(\eta, t)\to G(t)$$

as $\eta\to 0$, *uniformly in* $0<t<T$, *where* $G(t)\geqslant 0$. *Then*

$$\varliminf_{\eta\to 0}\int_0^{w-\eta} f(t)\,G(\eta, t)\,dt\geqslant 0.$$

If (*c*) *is replaced by the stronger condition*:

(*c'*) *that condition* (*c*) *is satisfied, but with* $G(t)>0$, *then the conclusion can be replaced by*

$$\varliminf_{\eta\to 0}\int_0^{w-\eta} f(t)\,G(\eta, t)\,dt>0,$$

except when $f(t)=0$ *almost everywhere in* $(0, w)$.

We deduce at once from (*c*) and from the boundedness of $f(t)$ that, for any fixed ϵ,

$$\lim_{\eta\to 0}\int_0^{w-\epsilon} f(t)\,G(\eta, t)\,dt=\int_0^{w-\epsilon} f(t)\,G(t)\,dt\geqslant 0.$$

Now it follows at once from (*b*) and equation (16) that

$$\int_{w-\epsilon}^{w-\eta} f(t)\,G(\eta, t)\,dt$$

differs from $$f(w)\int_{w-\epsilon}^{w-\eta} G(\eta, t)\,dt$$

by an amount arbitrarily small for sufficiently small ϵ (uniformly in η). The first part of the lemma now follows with the aid of (*a*). The second part is an immediate corollary.

Lemma 4. *Let* $0<\theta<1$. *Then*

$$\int_T^W\{(W-x)^{\theta-1}-(W-T)^{\theta-1}\}(x-T)^{-1-\theta}\,dx=\frac{1}{(W-T)\,\theta}.$$

If α, β are complex numbers such that $\Re\alpha>0$, $\Re\beta>0$, it is clear that

$$\int_T^W\{(W-x)^{\alpha-1}-(W-T)^{\alpha-1}\}(x-T)^{\beta-1}\,dx=(W-T)^{\alpha+\beta-1}\left\{\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}-\frac{1}{\beta}\right\}.$$

By analytic continuation, this result continues to hold for $\Re\alpha > 0, \Re\beta > -1$, the term $\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ being interpreted as meaning 0 when $\alpha+\beta = 0$. Putting $\alpha = \theta$, $\beta = -\theta$, the result follows.

LEMMA 5. *If, for any fixed w, $G(\eta, t)$ is defined by equation* (15), *then conditions* (*a*), (*b*), (*c*) *of Lemma* 3 *are satisfied, condition* (*b*) *being satisfied for any $\lambda > 0$. If, further, $\phi(u)$ satisfies* (10), *then condition* (*c'*) *is satisfied.*

Proof of (*a*). We have

$$\int_{w-\epsilon}^{w-\eta} G(\eta, t)\,dt = \Phi(w-\eta)\int_{w-\epsilon}^{w-\eta}(w-\eta-t)^{-\theta}\,dt - \int_{w-\epsilon}^{w-\eta} dt \int_{t}^{w-\eta}(u-t)^{-\theta}\Phi'(u)\,du$$

$$= \Phi(w-\eta)\int_{w-\epsilon}^{w-\eta}(w-\eta-t)^{-\theta}\,dt - \int_{w-\epsilon}^{w-\eta}\Phi'(u)\,du\int_{w-\epsilon}^{u}(u-t)^{-\theta}\,dt$$

$$= \frac{1}{1-\theta}\left\{\Phi(w-\eta)(\epsilon-\eta)^{1-\theta} - \int_{w-\epsilon}^{w-\eta}\Phi'(u)[u-(w-\epsilon)]^{1-\theta}\,du\right\}.$$

On integrating by parts, this expression becomes

$$\int_{w-\epsilon}^{w-\eta}\Phi(u)[u-(w-\epsilon)]^{-\theta}\,du,$$

and, since $\Phi(u)$ is clearly positive, the result is evident.

In order to prove (*b*) and (*c*), it is convenient to express $G(\eta, t)$ in a slightly different form. Let ξ be any number lying between t and $w-\eta$. Dividing the integral which occurs in (15) into two portions, and integrating by parts that between ξ and $w-\eta$, we have

$$G(\eta, t) = \Phi(\xi)(\xi-t)^{-\theta} - \theta\int_{\xi}^{w-\eta}(u-t)^{-1-\theta}\Phi(u)\,du - \int_{t}^{\xi}(u-t)^{-\theta}\Phi'(u)\,du. \quad (17)$$

Proof of (*b*). In (17), we take $\xi = \frac{1}{2}(w-\eta+t)$. We deduce at once that

$$\int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}|\,G(\eta, t)\,|\,dt \leqslant I_1 + \theta I_2 + I_3,$$

where

$$I_1 = 2^{\theta}\int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}\,(w-\eta-t)^{-\theta}\Phi[\tfrac{1}{2}(w-\eta+t)]\,dt,$$

$$I_2 = \int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}\,dt\int_{\frac{1}{2}(w-\eta+t)}^{w-\eta}(u-t)^{-1-\theta}\Phi(u)\,du,$$

$$I_3 = \int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}\,dt\int_{t}^{\frac{1}{2}(w-\eta+t)}(u-t)^{-\theta}\,|\,\Phi'(u)\,|\,du.$$

We next observe that since, by equation (11), $\phi'(w) \neq 0$, we must have, for sufficiently small $w-u$,

$$\phi(w)-\phi(u) \geqslant c(w-u),$$

where $c > 0$, and hence

$$\Phi(u) = O(w-u)^{\theta-1}, \tag{18}$$

$$\Phi'(u) = O(w-u)^{\theta-2}. \tag{19}$$

Thus

$$I_1 = O\left\{\int_{w-\epsilon}^{w-\eta} (w-t)^{\lambda}(w-\eta-t)^{-\theta}(w+\eta-t)^{\theta-1}\,dt\right\}.$$

Putting $t = w-x$, we have

$$I_1 = O\left\{\int_{\eta}^{\epsilon} x^{\lambda}(x-\eta)^{-\theta}(x+\eta)^{\theta-1}\,dx\right\}$$

$$= O\left\{\eta^{\lambda+\theta-1}\int_{\eta}^{2\eta}(x-\eta)^{-\theta}\,dx + \int_{2\eta}^{\epsilon} x^{\lambda-1}\,dx\right\} = O(1).$$

Again, it follows from (18) that

$$I_2 = O\left\{\int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}\,dt\int_{\frac{1}{2}(w-\eta+t)}^{w-\eta}(u-t)^{-1-\theta}(w-u)^{\theta-1}\,du\right\}.$$

Putting $t = w-x$, $u = w-v$, we have

$$I_2 = O\left\{\int_{\eta}^{\epsilon} x^{\lambda}\,dx\int_{\eta}^{\frac{1}{2}(x+\eta)}(x-v)^{-1-\theta}v^{\theta-1}\,dv\right\}$$

$$= O\left\{\int_{\eta}^{\epsilon} x^{\lambda}(x-\eta)^{-1-\theta}\,dx\int_{\eta}^{\frac{1}{2}(x+\eta)} v^{\theta-1}\,dv\right\},$$

since $x-v \geqslant \frac{1}{2}(x-\eta)$ for all relevant x, v. In the range $\eta \leqslant x \leqslant 2\eta$, we now use the result that the inner integral is $O[\eta^{\theta-1}(x-\eta)]$; for $x > 2\eta$, we use the result that it is $O(x+\eta)^{\theta} = O(x^{\theta})$. We deduce that

$$I_2 = O\left\{\eta^{\theta-1+\lambda}\int_{\eta}^{2\eta}(x-\eta)^{-\theta}\,dx + \int_{2\eta}^{\epsilon} x^{\lambda-1}\,dx\right\} = O(1).$$

Finally, we have, by (19),

$$I_3 = O\left\{\int_{w-\epsilon}^{w-\eta}(w-t)^{\lambda}\,dt\int_{t}^{\frac{1}{2}(w-\eta+t)}(u-t)^{-\theta}(w-u)^{\theta-2}\,du\right\}.$$

Putting $t=w-x$, $u=w-v$ as before, we deduce that

$$I_3 = O\left\{\int_\eta^\epsilon x^\lambda dx \int_{\frac{1}{2}(\eta+x)}^x (x-v)^{-\theta} v^{\theta-2} dv\right\}.$$

Since $v \geqslant \frac{1}{2}x$ for all relevant x, v, we now deduce that

$$I_3 = O\left\{\int_\eta^\epsilon x^{\lambda+\theta-2} dx \int_{\frac{1}{2}(\eta+x)}^x (x-v)^{-\theta} dv\right\}$$

$$= O\left\{\int_\eta^\epsilon x^{\lambda-1} dx\right\} = O(1).$$

The result is now evident.

Proof of (c). It is evident from (17) and (18) that $G(\eta, t) \to G(t)$ as $\eta \to 0$ uniformly in $0 < t < T$, where

$$G(t) = \Phi(\xi)(\xi-t)^{-\theta} - \theta \int_\xi^w (u-t)^{-1-\theta} \Phi(u)\, du - \int_t^\xi (u-t)^{-\theta} \Phi'(u)\, du.$$

It thus only remains to prove that $G(t) \geqslant 0$. In order to do this, we put $\xi = t+\epsilon$, and make $\epsilon \to 0$. Since, as is evident, $\Phi(t+\epsilon)-\Phi(t) = O(\epsilon)$, we deduce that

$$G(t) = \lim_{\epsilon \to 0} \left\{\Phi(t)\, \epsilon^{-\theta} - \theta \int_{t+\epsilon}^w (u-t)^{-1-\theta} \Phi(u)\, du\right\}. \tag{20}$$

We now make the substitution $\phi(u) = x$. Suppose that we then have $u = \psi(x)$, so that ψ is the inverse function of ϕ. Since $\phi(u)$ is an increasing function of u, $\psi(x)$ is an increasing function of x. Further, it is evident that

$$\psi'(x) = \frac{dx}{du} = \frac{1}{\phi'(u)};$$

since $\phi'(u)$ is a non-increasing function of u, we deduce that $\psi'(x)$ is a non-decreasing function of x. Let

$$\phi(t) = T, \quad \phi(w) = W, \quad \phi(t+\epsilon) = T+\eta.$$

Then

$$\epsilon = \psi(T+\eta) - \psi(T) = \eta\psi'(T) + O(\eta^2).$$

Finally, by (14),

$$\Phi(u) = [\phi(w)-\phi(u)]^{\theta-1} [\phi'(u)]^{n+1} = (W-x)^{\theta-1} [\psi'(x)]^{-(n+1)}.$$

Substituting these results in (20), we have

$$G(t) = \lim_{\eta \to 0} \Big\{ (W-T)^{\theta-1} [\psi'(T)]^{-(n+1+\theta)} \eta^{-\theta} - \theta \int_{T+\eta}^{W} (W-x)^{\theta-1} [\psi'(x)]^{-n} [\psi(x)-\psi(T)]^{-1-\theta} dx \Big\}. \quad (21)$$

Now, since $\psi'(x)$ is non-decreasing, it is clear that

$$\psi(x)-\psi(T) \geqslant (x-T)\psi'(T), \quad (22)$$

$$\psi'(x) \geqslant \psi'(T). \quad (23)$$

We deduce from (21) that

$$\begin{aligned} G(t) &\geqslant \lim_{\eta \to 0} [\psi'(T)]^{-(n+1+\theta)} \Big\{ (W-T)^{\theta-1} \eta^{-\theta} - \theta \int_{T+\eta}^{W} (W-x)^{\theta-1} (x-T)^{-1-\theta} dx \Big\} \\ &= \lim_{\eta \to 0} [\psi'(T)]^{-(n+1+\theta)} \Big\{ (W-T)^{\theta-1} \eta^{-\theta} - \theta (W-T)^{\theta-1} \int_{T+\eta}^{W} (x-T)^{-1-\theta} dx \\ &\qquad - \theta \int_{T+\eta}^{W} \{(W-x)^{\theta-1} - (W-T)^{\theta-1}\} (x-T)^{-1-\theta} dx \Big\} \\ &= [\psi'(T)]^{-(n+1+\theta)} \Big\{ (W-T)^{-1} \\ &\qquad - \theta \int_{T}^{W} \{(W-x)^{\theta-1} - (W-T)^{\theta-1}\} (x-T)^{-1-\theta} dx \Big\} \\ &= 0, \end{aligned}$$

by Lemma 4.

Finally, we observe that, if $\phi(u)$ satisfies (10), then $\psi'(x)$ is (strictly) increasing, so that, in (22) and (23) the symbol "$\geqslant$" may be replaced by "$>$". We can easily deduce from this that, in this case, $G(t) > 0$, and the proof of the lemma is completed.

LEMMA 6. *Let $0 < \theta < 1$. Suppose that $A^{(n+\theta)}(t) \geqslant 0$ for all t. Let $F(w)$ be defined as in Lemma 2. Then $F(w) \geqslant 0$ for all w. If, further, $\phi(u)$ satisfies* (10), *then $F(w) > 0$ unless $A^{(n+\theta)}(t)$ is identically zero in $(0, w)$.*

If we put $f(u) = A^{(n+\theta)}(u)$, it is clear that equation (16) of Lemma 3 is satisfied, provided that λ is sufficiently small. In fact $A^{(n+\theta)}(u)$ satisfies a Lipschitz condition of order θ if $n = 0$ and of order 1 if $n \geqslant 1$. The result therefore follows at once from Lemmas 2, 3 and 5.

2.2. We are now in a position to prove Theorem 1. We have

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) = \frac{1}{\Gamma(\kappa)} \int_0^{\phi(w)} \big(\phi(w)-t\big)^{\kappa-1} A_\lambda(t)\, dt,$$

Since, as is evident,

$$A_\lambda\left(\phi(u)\right) = A(u),$$

we obtain, on making the substitution $t = \phi(u)$,

$$A_\lambda^{(\kappa)}\left(\phi(w)\right) = \frac{1}{\Gamma(\kappa)} \int_0^w [\phi(w)-\phi(u)]^{n+\theta-1} A(u)\, \phi'(u)\, du. \tag{24}$$

In what follows we use c for any positive constant (possibly a different constant at each occurrence), and write

$$(-1)^{\nu+1} \phi^{(\nu)}(u) = \phi_\nu(u) \quad (\nu = 1, 2, \ldots, n+2),$$

so that $\phi_\nu(u)$ is positive (or zero) for all u. Since the differentiation of a term $[\phi_\nu(u)]^m$ yields $-m[\phi_\nu(u)]^{m-1}\phi_{\nu+1}(u)$, and since the differentiation of $[\phi(w)-\phi(u)]^p$ yields $-p[\phi(w)-\phi(u)]^{p-1}\phi_1(u)$, which is negative provided that $p > 0$, it is easily proved by induction that, for $m \leqslant n$,

$$(-1)^m \frac{d^m}{du^m} \{[\phi(w)-\phi(u)]^{n+\theta-1}\phi'(u)\}$$

$$= c[\phi(w)-\phi(u)]^{n+\theta-1-m}[\phi_1(u)]^{m+1} + \sum_{\nu=1}^{m} [\phi(w)-\phi(u)]^{n+\theta-1-m+\nu} K_\nu^{(m)}(u), \tag{25}$$

where $K_\nu^{(m)}(u)$ denotes a sum of terms of the form

$$c[\phi_1(u)]^{a_1}[\phi_2(u)]^{a_2}\ldots[\phi_{\nu+1}(u)]^{a_{\nu+1}}, \tag{26}$$

$a_1, a_2, \ldots, a_{\nu+1}$ being positive or zero integers which satisfy*

$$a_1 + 2a_2 + \ldots + (\nu+1)\, a_{\nu+1} = m+1. \tag{27}$$

We next note that it follows from (27) and Lemma 1 (ii) that

$$K_\nu^{(m)}(u) = o(u^{-(m+1)}). \tag{28}$$

We now integrate (24) by parts n times. Using equation (12), we deduce from (28) that the integrated terms vanish at the origin; and, using this fact, the convergence at the origin of each integral which occurs follows from that of the preceding one (and could, in any case, be deduced from

* They also satisfy $a_1 + a_2 + \ldots + a_{\nu+1} = m+1-\nu$, but we do not need this equation. The result is a special case of a much more general theorem due to Faà da Bruno; see de la Valleé Poussin, *Cours d'analyse infinitésimale*, 1, 6 ed., 89. A similar special case is used by Hirst, *loc. cit.* (Lemma A).

(27) with the aid of both parts of Lemma 1). We obtain

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) = c\int_0^w A^{(n)}(u)[\phi(w)-\phi(u)]^{\theta-1}[\phi_1(u)]^{n+1}\,du + \sum_{\nu=1}^{n} c\int_0^w A^{(n)}(u)[\phi(w)-\phi(u)]^{\theta-1+\nu}K_\nu^{(n)}(u)\,du.$$

Every term except the first may now be integrated by parts again. We deduce that

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) = c\int_0^w A^{(n)}(u)[\phi(w)-\phi(u)]^{\theta-1}[\phi_1(u)]^{n+1}\,du + \sum_{\nu=1}^{n+1} c\int_0^w A^{(n+1)}(u)\,[\phi(w)-\phi(u)]^{\theta-2+\nu}K_\nu^{(n+1)}(u)\,du. \quad (29)$$

Now since $A^{(n+\theta)}(u) \geqslant 0$ for all u, it is clear that $A^{(n+1)}(u)$ must have the same property. We may therefore deduce from (29) that

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) \geqslant c\int_0^w A^{(n)}(u)\,[\phi(w)-\phi(u)]^{\theta-1}[\phi_1(u)]^{n+1}\,du. \quad (30)$$

If $\theta < 1$, the expression on the right of (30) is the same as the function $F(w)$ defined in equation (13) of Lemma 2. The theorem therefore follows at once from Lemma 6. If $\theta = 1$, equation (30) becomes

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) \geqslant c\int_0^w A^{(n)}(u)[\phi_1(u)]^{n+1}\,du.$$

On integrating by parts again, we obtain

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) \geqslant cA^{(n+1)}(w)[\phi_1(w)]^{n+1} + c\int_0^w A^{(n+1)}(u)[\phi_1(u)]^n\,\phi_2(u)\,du \geqslant 0.$$

Finally, if $\phi(u)$ satisfies equation (10), we have $\phi_2(u) > 0$. Using (11) we see that, in this case, the last equation may be replaced by

$$A_\lambda^{(\kappa)}\big(\phi(w)\big) > 0,$$

except when $A^{(n+1)}(u)$ is identically zero for $u < w$. The proof of Theorem 1 is now complete.

3.1. The proof of Theorem 2 is based on the following known result*.

* See Hausdorff, *l.c.*, Bernstein, *l.c.*, or Widder, *l.c.*

LEMMA 7. *Let $f(u)$ be completely monotonic in $(0, \infty)$. Then there is a non-decreasing function $\chi(t)$, such that*

$$f(u) = \int_0^\infty e^{-ut} d\chi(t),$$

for $u > 0$, and

$$f^{(n)}(u) = (-1)^n \int_0^\infty t^n e^{-ut} d\chi(t).$$

Further, if the expression on the left of any of these equations tends to a finite limit as $u \to 0$, the corresponding equation continues to hold for $u = 0$.

It is easily verified that, if $\phi(u)$ satisfies the conditions of Theorem 2, then, for any fixed $s > 0$, $\exp\left(-s\phi(u)\right)$ is completely monotonic. Thus there is a non-decreasing function $\chi_s(t)$, say, such that, for $u > 0$,

$$\exp\left(-s\phi(u)\right) = \int_0^\infty e^{-ut} d\chi_s(t); \tag{31}$$

$$H_s(u) = s\phi'(u) \exp\left(-s\phi(u)\right) = \int_0^\infty te^{-ut} d\chi_s(t), \tag{32}$$

and such that equation (32) may again be differentiated formally. We also observe that, since $\phi(u)$ tends to a finite limit as $u \to 0$, equation (31) still holds when $u = 0$. This assertion includes the result that the integral on the right converges; thus $\chi_s(t)$ is bounded, or, what is the same thing (since it is non-decreasing) of bounded variation, in $(0, \infty)$.

Since the series (6) converges for all $s > 0$, we have

$$f_\lambda(s) = s \int_0^\infty A_\lambda(x) e^{-sx} dx,$$

for $s > 0$. Putting $x = \phi(u)$, and bearing in mind that

$$A_\lambda\left(\phi(u)\right) = A(u),$$

we obtain

$$\begin{aligned} f_\lambda(s) &= s \int_0^\infty A(u) \phi'(u) \exp\left(-s\phi(u)\right) du \\ &= \int_0^\infty A(u) du \int_0^\infty te^{-ut} d\chi_s(t), \end{aligned} \tag{33}$$

by (32). If it is permissible to invert the order of integration, we have

$$f_\lambda(s) = \int_0^\infty t\, d\chi_s(t) \int_0^\infty A(u) e^{-ut} du = \int_0^\infty f(t)\, d\chi_s(t), \tag{34}$$

and, since $\chi_s(t)$ is non-decreasing, the theorem is evident.

It thus only remains to justify the inversion in the order of integration. To this end, we first show that

$$t\int_X^\infty A(u)\,e^{-ut}du \to 0 \tag{35}$$

as $X\to\infty$, uniformly in t for $t\geqslant c>0$, where c is a constant. To establish this, we observe that, since the integral (7) converges for all $s>0$,

$$R(u)=\int_u^\infty A(v)\,e^{-vc}dv$$

exists, and $R(u)\to 0$ as $u\to\infty$. Thus

$$\begin{aligned}\left|t\int_X^\infty A(u)\,e^{-ut}du\right| &= \left|t\int_X^\infty e^{-u(t-c)}dR(u)\right| \\ &= \left|-te^{-X(t-c)}R(X)+t(t-c)\int_X^\infty e^{-u(t-c)}R(u)\,du\right| \\ &\leqslant 2te^{-X(t-c)}\max_{u\geqslant X}|R(u)|.\end{aligned}$$

Now $te^{-X(t-c)}$ is a decreasing function of t for $t>1/X$. Supposing, as we may, that $X\geqslant 1/c$, it follows that this function is decreasing for $t>c$. We deduce from this that

$$te^{-X(t-c)}\leqslant c$$

for $t\geqslant c$, and (35) is now evident. We note also that, since $A(u)$ is bounded for u in any finite interval, we have

$$\left|t\int_0^X A(u)\,e^{-ut}du\right| \leqslant t\int_0^X |A(u)|\,e^{-ut}du = O(1), \tag{36}$$

for any fixed X, whence it follows with the aid of (35) that, for $t\geqslant c$,

$$\int_0^\infty A(u)\,e^{-ut}du = O\left(\frac{1}{t}\right). \tag{37}$$

Since, as has already been observed, $\chi_s(t)$ is of bounded variation in $(0,\infty)$, it follows from (37) that, if $c>0$, the repeated integral

$$\int_c^\infty t\,d\chi_s(t)\int_0^\infty A(u)\,e^{-ut}du$$

converges, and from (35) that

$$\int_c^\infty t\,d\chi_s(t)\int_X^\infty A(u)\,e^{-ut}du\to 0$$

as $X \to \infty$. Thus

$$\int_c^\infty t\,d\chi_s(t) \int_0^\infty A(u)\,e^{-ut}du = \lim_{X\to\infty} \int_c^\infty t\,d\chi_s(t) \int_0^X A(u)\,e^{-ut}du. \tag{38}$$

Moreover, it clearly follows from the second half of (36) that the repeated integral on the right of (38) is absolutely convergent. We may therefore invert the order of integration, when we obtain

$$\int_c^\infty t\,d\chi_s(t) \int_0^\infty A(u)\,e^{-ut}du = \int_0^\infty A(u)\,du \int_c^\infty te^{-u}\,d\chi_s(t), \tag{39}$$

the equation including the assertion that both sides converge. If we can now prove that*

$$\int_0^\infty A(u)\,du \int_0^c te^{-ut}d\chi_s(t) \to 0 \tag{40}$$

as $c \to 0$, it will clearly follow from (39) that the expressions (33) and (34) are equal, and the proof of the theorem will be completed.

We now write†

$$G(u, c) = \frac{1}{H_s(u)} \int_0^c te^{-ut}d\chi_s(t),$$

for $c > 0$, $u \geqslant 0$, where $H_s(u)$ is given by (32); it is evident that

$$0 \leqslant G(u, c) \leqslant 1, \tag{41}$$

and that, for fixed u,

$$G(u, c) \to 0 \tag{42}$$

as $c \to 0$. We next show that, for fixed c, $G(u, c)$ is a non-decreasing function of u. For, using (32) and the equation obtained from (32) by formal differentiation, we see that

$$\frac{\partial}{\partial u} G(u, c) = \frac{1}{[H_s(u)]^2} \left\{ -\int_0^\infty te^{-ut}d\chi_s(t) \int_0^c v^2 e^{-uv}d\chi_s(v) + \int_0^c te^{-ut}d\chi_s(t) \int_0^\infty v^2 e^{-uv}d\chi_s(v) \right\}$$

for $u > 0$. Cancelling out the part common to both integrals, and then altering the notation in the second integral by interchanging t and v, we

* We know that the repeated integral on the left of (40) converges, since it is the difference of the expressions on the right of (33) and of (39).

† If $H_s(u) \to \infty$ as $u \to 0$, we take $G(0, c)$ to mean 0.

obtain

$$\frac{\partial}{\partial u} G(u, c) = \frac{1}{[H_s(u)]^2} \int_c^\infty te^{-ut} d\chi_s(t) \int_0^c (t-v) ve^{-uv} d\chi_s(v) \geqslant 0,$$

which proves the result stated. We now write

$$J_s(u) = \int_u^\infty A(v) H_s(v) dv;$$

by (33) this integral converges, and hence tends to 0 as $u \to \infty$. Thus the expression on the left of (40) is equal to

$$\int_0^\infty A(u) G(u, c) H_s(u) du = -\int_0^\infty G(u, c) dJ_s(u)$$

$$= G(0, c) J_s(0) + \int_0^\infty J_s(u) dG(u, c). \qquad (43)$$

Now, for all u, $|J_s(u)| \leqslant A$, say, where A is a constant. Further, if ϵ is any assigned positive quantity, $|J_s(u)| \leqslant \epsilon$ for $u \geqslant X$, say. Thus the modulus of the expression (43) is less than or equal to

$$A|G(0, c)| + A \int_0^X |dG(u, c)| + \epsilon \int_X^\infty |dG(u, c)|$$

$$= A|G(0, c)| + A \int_0^X dG(u, c) + \epsilon \int_X^\infty dG(u, c)$$

[since $G(u, c)$ is non-decreasing]

$$\leqslant AG(X, c) + \epsilon,$$

by (41). Equation (40) now follows at once with the aid of (42), and the proof of the theorem is thus complete.

3.2. In this paragraph we make some further comments on the simplest case of Theorem 2—that in which $\phi(u) = u^r$, where $0 < r < 1$. It will have been observed, in the general case, that the essential point of the proof is that, for any $s > 0$, we can write $\exp\left(-s\phi(u)\right)$ in the form

$$\int_0^\infty e^{-ut} d\chi_s(t),$$

where $\chi_s(t)$ is non-decreasing. In the case $\phi(u) = u^r$, it is clear that, if

$$\exp(-u^r) = \int_0^\infty e^{-ut} d\chi_1(t), \qquad (44)$$

then

$$\exp(-su^r) = \int_0^\infty \exp(-us^{1/r}t)\,d\chi_1(t)$$

$$= \int_0^\infty e^{-ut} d\chi_1(ts^{-1/r}), \tag{45}$$

by an obvious substitution. Thus, instead of having to consider the behaviour of $\exp\left(-s\phi(u)\right)$ for general values of s, it is, in this case, enough to take $s=1$. If we assume for the moment that $\chi_1(t)$ is an indefinite integral of a function $\psi(t)$, say, then (44) is equivalent to

$$\exp(-u^r) = \int_0^\infty e^{-ut}\psi(t)\,dt, \tag{46}$$

where $\psi(t) \geqslant 0$ for all t.

In what follows we shall obtain an explicit expression for a function $\psi(t)$ satisfying (46) in the form of an integral which is obviously positive. This will clearly yield an alternative proof of the special case of Theorem 2 which we are now considering. We first show that (46) is satisfied if we take

$$\psi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(zt-z^r)\,dz, \tag{47}$$

where z^r is made single valued in the complex plane cut along the negative real axis by taking it real and positive on the positive real axis, and where the contour passes to the right of the origin. To this end, we first observe that, if we put $z=\rho e^{i\theta}$ and if t is real, then

$$|\exp(zt-z^r)| = \exp(t\rho\cos\theta - \rho^r\cos r\theta). \tag{48}$$

If we define $\psi(t)$ by (47) for negative as well as for positive values of t, it follows at once from (48) that, if $t<0$, the integral (47) can be made arbitrarily small by taking the contour to lie to the right of a circle with centre at the origin, and of sufficiently large radius. Since the integral is independent of the contour, we deduce that $\psi(t)=0$ for $t<0$. Hence, if $u \geqslant 0$,

$$\int_0^\infty e^{-ut}\psi(t)\,dt = \int_{-\infty}^\infty e^{-ut}\psi(t)\,dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^\infty e^{-ut}dt\int_{c-i\infty}^{c+i\infty}\exp(zt-z^r)\,dz, \tag{49}$$

provided that either side converges. Now, again using (48), it is clear that, in (47), c may be given any non-negative value, and the contour

taken as a straight line. Taking $c = u$, and writing $z = u + iy$, the expression (49) becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} e^{ity} \exp\left[-(u+iy)^r\right] dy.$$

Applying Fourier's repeated integral theorem, sufficient conditions for which are clearly satisfied, to $\exp\left[-(u+iy)^r\right]$ regarded as a function of y, we deduce that (49) converges to $\exp(-u^r)$, so that (46) is satisfied.

Now, writing $z = \rho e^{i\theta}$, we see, on equating the imaginary part of $zt - z^r$ to zero, that this function is real provided that either z lies on the positive real axis, or that

$$\rho^{1-r} = \frac{\sin r\theta}{t \sin\theta}; \tag{50}$$

equation (50) clearly gives a positive value of ρ for $-\pi < \theta < \pi$, and this value of ρ tends to infinity as $\theta \to \pm\pi$. If we denote by C the curve given by (50) for $-\pi < \theta < \pi$, it is clear that the contour in (47) may be displaced to the curve C. We deduce at once that

$$\psi(t) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \exp\left(t\rho\cos\theta - \rho^r \cos r\theta\right) \frac{dz}{d\theta}\, d\theta,$$

where $\rho = \rho(\theta)$ is given by equation (50), and where $z = z(\theta) = \rho e^{i\theta}$. Since the exponential term is an even function of θ, and since

$$z(\theta) - z(-\theta) = 2i\rho \sin\theta,$$

we deduce that

$$\begin{aligned}\psi(t) &= \frac{1}{\pi}\int_0^{\pi} \exp\left(t\rho\cos\theta - \rho^r\cos r\theta\right) \frac{d(\rho\sin\theta)}{d\theta}\, d\theta \\ &= \frac{r}{\pi t(1-r)}\int_0^{\pi} \exp\left(t\rho\cos\theta - \rho^r\cos r\theta\right) \frac{\rho^r \sin(1-r)\theta}{\sin\theta}\, d\theta \\ &= \frac{r}{\pi(1-r)}\, t^{-1/(1-r)} \int_0^{\pi} k(\theta) \exp\left(-t^{-r/(1-r)} k(\theta)\right) d\theta, \end{aligned} \tag{51}$$

where $$k(\theta) = (\sin\theta)^{-1/(1-r)} (\sin r\theta)^{r/(1-r)} \sin(1-r)\theta.$$

It is clear that the integrand in (51) is everywhere positive, so that $\psi(t) > 0$ for $t > 0$.

It may be remarked that this gives us slightly more than the results of Hausdorff and Bernstein, which show only that $\chi_s(t)$ is non-decreasing, and not that it is (strictly) increasing. It clearly follows that, in the special case $\phi(u) = u^r$, the conclusion in Theorem 2 may be replaced by

the strict inequality $f_\lambda(s) > 0$, except in the trivial case in which $f(s)$ is identically zero.

In the special case $r = \frac{1}{2}$, equation (51) reduces to

$$\psi(t) = \frac{1}{4\pi t^2} \int_0^\pi \sec^2 \tfrac{1}{2}\theta \exp\left(-\frac{\sec^2 \frac{1}{2}\theta}{4t}\right) d\theta$$

$$= \frac{1}{2\pi t^2} \exp\left(-\frac{1}{4t}\right) \int_0^\infty \exp\left(-\frac{v^2}{4t}\right) dv$$

(putting $\tan \frac{1}{2}\theta = v$)

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}} t^{-\frac{3}{2}} \exp\left(-\frac{1}{4t}\right), \tag{52}$$

so that the result just obtained reduces to a familiar one quoted by Miss Cartwright*.

In the special case in which $r = 1/n$, where n is any integer greater than 1, we can, as has been pointed out by Hausdorff†, obtain a different expression for an obviously positive function $\psi(t)$ satisfying (46) by using the result‡, due to Liouville, that

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \exp\left(-x_1 - x_2 - \dots - x_{n-1} - \frac{u}{n^n x_1 x_2 \dots x_{n-1}}\right)$$

$$\times x_1^{(1/n)-1} x_2^{(2/n)-1} \dots x_{n-1}^{\{(n-1)/n\}-1} dx_1 dx_2 \dots dx_{n-1}$$

$$= (2\pi)^{\frac{1}{2}(n-1)} n^{-\frac{1}{2}} \exp(-u^{1/n}). \tag{53}$$

For if, in (53), we put

$$n^n x_1 x_2 \dots x_{n-1} = 1/t,$$

and express x_1 in terms of the other variables, we see at once that (46) is satisfied when

$$\psi(t) = n^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}(n-1)} t^{-(1/n)-1}$$

$$\times \int_0^\infty \int_0^\infty \dots \int_0^\infty \exp\left(-x_2 - x_3 - \dots - x_{n-1} - \frac{1}{n^n t x_2 x_3 \dots x_{n-1}}\right)$$

$$\times x_2^{(1/n)-1} x_3^{(2/n)-1} \dots x_{n-1}^{\{(n-2)/n\}-1} dx_2 dx_3 \dots dx_{n-1}. \tag{54}$$

---

* *L.c.*, equation 2 . 62 (page 88).

† *Loc. cit.*, 91.

‡ See, *e.g.*, Bertrand, *Calcul intégral*, 475.

Thus $\psi(t)$ is expressed as a multiple integral of an everywhere positive function. We remark that, when $n = 2$, the expression (54) reduces* to (52). In the general case, the integral (54) is of a type which has been considered by Steen†.

3.3. It is perhaps worth pointing out that the "limit" result corresponding to the case $\phi(u) = u^r$, $0 < r < 1$ of Theorem 2, which, as already stated, is due to Miss Cartwright, can be deduced at once from the results of §3.1. That is to say, we can deduce the theorem:

If $\phi(u) = u^r$, where $0 < r < 1$, and if the series (6) *converges for all $s > 0$, then, if the series* (1) *is summable (A, μ_n), it is summable (A, λ_n) to the same sum.*

In order to prove this theorem we may suppose, without loss of generality, that the sum (A, μ_n) of the given series is 0 (otherwise we can alter the first term). The proof of Theorem 2 shows that equation (34) is satisfied. But it follows from (45) that we may take

$$\chi_s(t) = \chi_1(ts^{-1/r}), \tag{55}$$

and thus

$$f_\lambda(s) = \int_0^\infty f(t)\, d\chi_1(ts^{-1/r}).$$

Now $f(t) \to 0$ as $t \to 0$, so that, given any ϵ, there is a T such that $|f(t)| < \epsilon$ for $0 < t < T$. But $f(t)$, being the sum of a Dirichlet series convergent for $t > 0$ must be bounded for $t \geqslant c > 0$; thus we have $|f(t)| < A$, say, for all $t > 0$. Further, $\chi_1(t)$ is of bounded variation in $(0, \infty)$. Thus

$$|f_\lambda(s)| \leqslant \epsilon \int_0^T |d\chi_1(ts^{-1/r})| + A \int_T^\infty |d\chi_1(ts^{-1/r})|$$

$$= \epsilon \int_0^{Ts^{-1/r}} |d\chi_1(t)| + A \int_{Ts^{-1/r}}^\infty |d\chi_1(t)|.$$

We deduce that

$$\overline{\lim_{s \to 0}}\, |f_\lambda(s)| \leqslant \epsilon \int_0^\infty |d\chi_1(t)|,$$

and, since ϵ is arbitrary, the result follows.

---

* In the general case, (54) is an $(n-2)$-ple integral; thus when $n = 2$, it is no integral at all, *i.e.* the variables of integration $x_2, x_3, \ldots$ are absent, and $\psi(t)$ is equal to the integrand as it stands.

† S. W. P. Steen, "Divisor functions", *Proc. London Math. Soc.* (2), 31 (1930), 47–80 [60, equation (3.22)].

In the general case, however, when $\phi(u)$ is *any* function satisfying the conditions of Theorem 2, we have no equation analogous to (55). Thus, since the argument of § 3.1 does not give us any information as to the behaviour of $\chi_s(t)$ when s varies, it does not throw any light on the corresponding "limit" problem.

3.4. I am indebted to Prof. Hardy for drawing my attention to the importance of the results of Hausdorff and Bernstein in connection with Theorem 2. I had originally made no use of these results, and had consequently only been able to prove Theorem 2 in the special case $\phi(u) = u^r$. My original proof was essentially the same as that indicated in § 3.2; the proof now given in § 3.2 has, however, been very much simplified as a result of suggestions made by Prof. Hardy.

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ON ARITHMETICAL PROPERTIES OF QUARTIC SURFACES

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Addendum.

1. In six papers which have recently appeared in the *Journal of the London Math. Soc.*, I have initiated a systematic study of the arithmetical properties of cubic surfaces*. The arithmetical properties of quartic surfaces may be expected to present new and more difficult problems, since the geometry of these surfaces is not a mere extension of the geometry of cubics. In fact, in the first of the eight sections into which this paper is divided, I prove the impossibility of extending the tangent and chord process, and also indicate some arithmetical questions concerning the existence of unicursal curves on a quartic surface and having no counterpart for cubics.

I therefore limit the present investigation to some special quartic surfaces. The remaining sections deal in fact chiefly with rational quartic surfaces which contain rational lines†. This condition can be expressed analytically, by the vanishing of a projective invariant Θ, of degree 320 (Section II).

* I shall deal more completely with this subject in a forthcoming work.

† The word *rational*, when applied to an algebraic surface, or curve, or point-set, will imply throughout this paper, as in my previous ones, that these entities can be defined by one or more algebraic equations with rational coefficients. Likewise, a *rational* polynomial will mean a polynomia with rational coefficients.

Let F denote a rational quartic surface containing a rational line. Section III gives a simple geometrical process leading in general to a certain number (possibly an infinity) of unicursal curves of F, each containing an infinity of rational points. This is tantamount to finding rational parametric solutions for the Diophantine equation

$$xf(x, y, z)+yg(x, y, z) = 0, \tag{1}$$

where f and g are non-homogeneous cubic polynomials with rational coefficients.

Other geometric processes can generally be applied when F contains two rational lines, incident or skew.

The cases in which the processes indicated fail are classified and studied in Sections IV and V. Here, in particular, I also give the complete solution, involving two parameters, of certain quartic Diophantine equations representing ruled surfaces (§ 13).

The methods indicated may sometimes be unsuccessful, not on account of their inadequacy, but owing to the arithmetical impossibility of the Diophantine equations involved. This is proved by various examples, given in Section VI.

Then several types of quartic Diophantine equations are investigated and solved in Section VII. The equation involving 27 parameters considered in § 24 is of special interest, since it includes in particular Euler's equation $x^4+y^4 = z^4+w^4$, and has a discontinuous infinity of rational parametric solutions. One of these reduces, for Euler's equation, to the well-known solution of the 7th degree*.

The last section contains a systematic study of the Diophantine equation

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4 = a'z^4+4b'z^3w+6c'z^2w^2+4d'zw^3+e'w^4.$$

All the cases to which our method applies are obtained and investigated. Particular attention is paid to the equation $ax^4+by^4+cz^4+dw^4 = 0$.

The table of contents above gives a more detailed idea of the content of the paper.

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* This seems to be the only solution to have been noticed explicitly. *Cf.* the long list of arithmetical papers on the subject quoted in L. E. Dickson, *History of the theory of numbers*, Vol. II (Washington, 1920), pp. 644–647, to which a very recent one may be added, due to P. S. Dyer, *Journal London Math. Soc.*, 18 (1943), 2–4. The present paper incidentally gives a geometric interpretation of the last one, thus answering a question put to me by Prof. L. J. Mordell.

I. *Rational points and unicursal curves on a rational quartic surface.*

2. We observe first an important difference between the arithmetical theory of cubics and quartics, on comparing the following theorem with the recent result that a rational cubic surface cannot have a simple rational point without containing an infinity of rational points*.

The quartic surface

$$(x^2+y^2+z^2+z+1)^2 = z^4+1 \tag{2}$$

contains only one rational point at a finite distance (the origin), and only one rational point at infinity (on the z-axis), and both these points are simple.

On putting in (2),

$$x = x_1/x_4, \quad y = x_2/x_4, \quad z = x_3/x_4,$$

with x_1, x_2, x_3, x_4 integral and not all zero, we obtain

$$(x_1^2+x_2^2+x_3^2+x_3x_4+x_4^2)^2 = x_3^4+x_4^4. \tag{3}$$

Since the Diophantine equation $X^4+Y^4 = Z^2$ has no solutions in positive integers†, we see that (3) has no integral solution with $x_3x_4 \neq 0$. Hence its only rational solutions are $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 = x_4 = 0$, which correspond to the two points indicated above. Each of them is a simple point of the quartic (2) or (3).

It is easily seen that this surface has, as its only multiple points, the double tacnodal points

$$x_1 : x_2 : x_3 : x_4 = 1 : \pm i : 0 : 0,$$

and contains an elliptic pencil of conics passing through these two points, so that it is birationally equivalent to an elliptic ruled surface. Moreover, the intersection of the surface with the tangent plane at the origin, *i.e.* with $z = 0$, has multiplicity two (and not greater than two) at this point.

The result for equation (2) clearly shows that

There is nothing analogous, for the quartic surface (2) *or* (3), *of the well-known tangent plane and chord process for cubics*‡.

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* *Cf.* B. Segre, "A parametric solution of the indeterminate cubic equation $z^2 = f(x, y)$", *Journal London Math. Soc.*, 18 (1943), 226–233. A *simple* point of a surface $F(x, y, z) = 0$ is a point (x, y, z) at which F vanishes, but some of its partial derivatives of the first order do not vanish. A point at which F and all its partial derivatives of order less than k vanish is said to have the *multiplicity* k.

† *Cf., e.g.*, L. E. Dickson, *Introduction to the theory of numbers* (Chicago University Press, 1929), 42–43, or G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (Oxford, Clarendon Press, 1938), 190–191.

‡ On this process, *cf., e.g.*, B. Segre, "A note on arithmetical properties of cubic surfaces", *Journal London Math. Soc.*, 18 (1943), 24–31, § I.

This property can be extended to *all algebraic surfaces of order* $n \geqslant 4$, *having no multiple points*; but the proof of this involves a rather deep knowledge of algebraic geometry. Since we never require the extension throughout the paper, we give the proof at the end as an addendum.

3. A prospective study of some arithmetical properties of a general quartic surface would be as follows. I have recently shown that the general quartic F contains a finite number $c_h > 0$ of unicursal curves of order $4h$ ($h = 1, 2, 3, \ldots$), and no unicursal curves of other orders*. When F is rational, the algebraic equation E_h, of degree c_h in a single variable, on which the determination of the unicursal curves of order $4h$ lying on F may be made to depend, has rational coefficients. If E_h has a rational root (and for this only arithmetical conditions on F are required), then we obtain a corresponding rational unicursal curve C on F, of order $4h$. It is known that the problem of determining the simple rational points lying on C can be reduced to that of finding the rational points lying on an irreducible rational conic†. Then there are either no solutions or an infinity of solutions; and a classical result shows that the conditions for the second alternative are purely arithmetic.

Thus we are led to rational quartic surfaces containing an infinity of rational points, and subject to conditions which are purely arithmetic, so that they do not imply that any projective invariant of the surface must vanish‡. These surfaces could be classified according to the values of h for which the corresponding E_h has rational solutions, the number of such solutions, etc. Concrete examples could easily be obtained, *e.g.* for $h = 1$, but a thorough investigation cannot yet be given, since at present nothing is known about the equation E_h, not even its degree c_h for $h > 1$. It is therefore convenient to confine our study to quartic surfaces subject to some geometric restrictions. We do so in the following section.

II. *Surfaces having the invariant* $\Theta = 0$.

4. We begin by proving that

A quartic surface F *contains lines if, and only if,* $\Theta = 0$, *where* Θ *is a projective invariant of* F, *of degree* 320.

* *Cf.* B. Segre, "A remark on unicursal curves lying on the general quartic surface", *Quart. J. of Math.* (Oxford), 15 (1944), 24–25.

† *Cf.* M. Noether, *Math. Ann.*, 3 (1871), 16, and D. Hilbert and A. Hurwitz, *Acta Math.*, 14 (1891), 217.

‡ It is generally surmised that a non-singular plane quartic curve can contain only a finite number of rational points.

A line in space depends on four parameters, and five conditions are necessary for it to lie on F. On eliminating the parameters from these conditions we obtain a single equation $\Theta = 0$, where Θ is a projective invariant of F, given by a rational polynomial in the coefficients of F.

The degree of this polynomial equals the number of surfaces, in a general pencil of quartics, which contain a line. Such a line is clearly a quadrisecant of the base curve of the pencil, and conversely. Hence the required degree is given by the number of quadrisecants of the curve of intersection of two general quartic surfaces, and so is in fact 320*.

5. A non-ruled quartic surface F with the invariant $\Theta = 0$ contains a finite positive number of lines, in general only one; and then this line is rational, if F is rational. The determination of these lines depends upon an algebraic equation in a single variable†, with rational coefficients when F is rational, in such a way that the rational lines of F correspond to the rational roots of this equation. Hence

All the rational lines lying on a given rational non-ruled quartic F can be determined by means of a finite number of arithmetical operations. For the existence of such lines, $\Theta = 0$ is a necessary (but not always sufficient) condition.

The rational points of the rational lines lying on F can consequently all be determined, and give what we shall call the *trivial solutions* of the corresponding Diophantine equation. Our next purpose will be, when possible, to derive from them new *non trivial solutions*.

III. *The R-process for surfaces containing a rational line.*

6. Let F be a rational quartic surface containing a simple rational line, C say. If P is a general rational point of C, the plane π touching F at P contains C, and intersects F residually in a rational reducible or irreducible cubic curve, Γ say, passing through P. Thus in general we can apply to Γ the classical tangent and chord process, and so deduce

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* This follows from a well-known general result, for which *cf.*, *e.g.*, F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Vol. III (Bologna, Zanichelli, 1924), 526, 476.

† The degree of this equation cannot exceed 64 if F is non-singular. *Cf.* B. Segre, "The maximum number of lines lying on a quartic surface", *Quart. J. of Math.* (Oxford), 14 (1943), 86–96. This paper will be referred to later on as M.N.

from P a certain number (possibly an infinity) of rational points distinct from P. This number is greater than zero, except in the particular cases when Γ has an inflexion or a multiple point at P; and we shall study these cases in detail later on. When P moves on C, and so π turns around C, each of the points of Γ thus rationally deduced from P, and not lying on C, will describe a *rational unicursal curve*, which we shall say is obtained from C on applying the *R-process*. Any such curve contains, by construction, an infinity of rational points; hence, if it is not a line, it will give a non-trivial rational parametric solution of the Diophantine equation representing F.

On assuming C as z-axis, and $(0, 0, t)$ as the coordinates of P, the equation of F takes the form (1) of § 1, and the R-process simply amounts to putting

$$y = \lambda x \quad [\lambda = -f(0, 0, t)/g(0, 0, t)]. \quad (4)$$

Then (1) reduces to the following non-homogeneous cubic equation in x, z,

$$f(x, \lambda x, z) + \lambda g(x, \lambda x, z) = 0, \quad (5)$$

where λ is given rationally by (4) in terms of the parameter t. Thus (5) has the solution $x = 0, z = t$. In general, solutions of (5) expressing x, z rationally in terms of t, and having $x \neq 0$, can be deduced from this one by the classical method. Any such solution and (4) then give a rational parametric solution of (1), which is generally non-trivial.

We add that any rational point P' of F, but not of C, lies on one, and only one, cubic Γ, *i.e.* the residual intersection of F with the plane CP', and this is rational. Hence, in general, the tangent and chord process can be applied to Γ and P'. This shows how, *from a non-trivial solution of* (1), *other non-trivial solutions can generally be deduced.*

7. The first step in applying the R-process (§ 6) consists in determining the tangential point of P on Γ. If this is a uniquely determined point Q, then the locus of Q, when P varies on C, is a rational unicursal curve, C_1 say, which we shall call *the R-transform of C.*

In general, the line PQ has three-point contact with F at P, two points arising from Γ and one from C. Moreover, the locus of PQ is an irreducible algebraic ruled surface R, which is of order 8 and contains C with multiplicity 5 (*cf.* M.N., § 6). The curve C_1 is the intersection of R and F, residual to C and to any (rational or irrational) lines incident with C and lying on F. Let $k(\geqslant 0)$ be the number of these lines; each of them is generally a simple line for both the surfaces F and R, which do not touch at every point of such a line. On the other hand, the multi-

plicity of C in the intersection of F and R is $5+2=7$, since each generator PQ of R osculates F at the point P where it meets C. Hence, in general, *the order of the R-transform C_1 of C is*

$$4\,.\,8-k-7=25-k.$$

This result may require modification in special cases. We shall now consider three of them briefly.

(i) If F varies continuously so as to acquire a (possibly irrational) double point O lying on C, then the quadric cone Ω touching F at O becomes a component of R. Hence the intersection of F and Ω residual to C and to the h ($\geqslant 0$) lines (rational or irrational) of F, distinct from C, meeting C at O, breaks off from the intersection of F and R, and does not belong to C_1. Consequently, in general, each double point of F lying on C *diminishes the order of C_1 by* $2\,.\,4-1-h=7-h$ *units.*

(ii) If there is a (possibly irrational) plane π through C, which intersects F in C and in a cubic having a double point at a point of C simple for F, then π is a simple component of R. Hence π diminishes *the order of C_1 by* 3 *units.*

(iii) If there is a (possibly irrational) plane π through C, which intersects F in C and in three lines meeting on C, then π is a double component of R, so that *each of these three lines has to be counted twice in the number k.*

IV. *Surfaces containing two rational lines. The T-transformations.*

8. If a rational quartic surface F contains two incident rational simple lines, C and D say, then the plane CD is rational, and meets F again in a rational conic, which is generally irreducible. When well-known *arithmetical conditions* are satisfied, this conic contains an infinity of rational points. Then we obtain a non-trivial *rational parametric solution of the second degree* for the Diophantine equation representing F.

9. Let us now consider a rational quartic surface F, containing two skew rational simple lines C, D*. Then we define a *birational transformation of F into itself* by associating two points P_1, P_2 of F (generally distinct and lying neither on C nor on D) when the line P_1P_2 meets both

* Some of the results which follow could easily be extended to the case when the lines C, D are quadratic conjugate, *i.e.*, when the pair (C, D), but neither C nor D, is rational. However, we do not require this extension in the present paper.

C and D. This transformation is obviously involutory, *i.e.* symmetrical in P_1, P_2; it will be denoted by T_{CD}, and called the T-transformation of F defined by C, D. It is clear that T_{CD} transforms any rational point of F into a rational point of F.

It may happen that T_{CD} is induced on F by a homography (*i.e.* a linear substitution upon the homogeneous coordinates), and then the transform of any trivial solution of F by T_{CD} is still a trivial solution. In this case the homography also must be involutory in character, and so, from a well-known elementary result, must be the *harmonic biaxial homology of axes C and D*. This is tantamount to saying that two arbitrary points P_1 and P_2 of F, corresponding in T_{CD}, must separate harmonically the intersections of the line P_1P_2 with C and D, a condition clearly involving C and D symmetrically. We then say that the lines C, D are *associate*, and prove the following theorem.

Let F be a quartic surface containing two skew rational simple lines C, D, and suppose that F has no multiple points on one of them, say C. Then C and D are associate if, and only if, the R-transform of C is D counted three times.

An arbitrary point P of C is simple for F, and so F has at it a uniquely defined tangent plane. We denote this plane by π, and its point of intersection with D by Q. It is obvious that both π and Q are rational if P is rational. Moreover, the correspondence between the lines C, D which associates P and Q is a (3, 1) *algebraic correspondence*. For, when Q is given arbitrarily on D, the plane $\pi = CQ$ cannot touch F along C, since then F would have multiple points on C*. Hence π intersects F in C and in a cubic Γ not containing C, and so intersecting C at three points corresponding to Q, say P, P', P''. When Q varies on D, these three points describe a g_3^1 on C, which is free from fixed points, since a fixed point of g_3^1 would be a multiple point of F. Hence P, P', P'' are generally distinct.

If C, D are associate, the harmonic homology of axis C and centre Q transforms Γ into itself. Hence, from a well-known property of plane cubics, Q is an inflexion of Γ, and C is its harmonic line, so that Q is the tangential point of each of the points P, P', P''. This proves that the R-transform of C (as defined in § 7) is the line D counted three times.

Conversely, if the R-transform of C is D counted three times, then each point Q of D must be the tangential point of each of the corresponding

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* *Cf.* the remark made later on, in § 13, about the surface (12).

points P, P', P'' of C. Hence the cubic Γ has an inflexion at Q, and is transformed into itself by the harmonic homology of axis C and centre Q. This proves that F is transformed into itself by the harmonic homology of axes C and D, *i.e.* that C, D are associate.

As a corollary of the theorem just proved, we see that *a rational line of a quartic containing no multiple points of the quartic has at most one associate line.*

10. The (3, 1) algebraic correspondence between C, D defined in § 9 is irreducible, since F has no multiple points on C. Hence the locus of the line PQ joining two corresponding points is an irreducible ruled surface, S say, of order $3+1=4$, which has C and D as directrices of respective multiplicities 1 and 3.

The line C has to be counted with multiplicity 3 or 2 in the intersection of F and S, according as a general generator PQ of S osculates or simply touches F at the point P where it meets C, *i.e.* (§ 9) according as C, D are associate or not. In the first case PQ has all its four intersections with F at P and Q, so that the intersection of F and S, residual to C, D, consists of a certain number of lines incident with both C and D.

In any case the line D has to be counted with multiplicity 3 (and not greater than 3) in the intersection of F and S, *i.e.* these two surfaces cannot touch along D, since then one and only one of the three generators of S, containing a general point Q of D, would lie in the plane (through D) touching F at Q. This generator could then be rationally separated from the other two, which is impossible, owing to the irreducibility of S.

A rational or irrational line incident with C and D, and lying on F, is obviously a generator of S. Such a line is simple for F, since F has no multiple points on C, and for S, since otherwise S would contain the plane joining it with the triple line D; and is generally not a line of contact of F and S, because then F would have multiple points on it. Let us denote by p the number of these lines. From the previous results on the intersection of F and S we see that, when C, D are associate, $4 \cdot 4 = 3+3+p$, whence $p = 10$. If, on the contrary, C, D are not associate, then F and S intersect in C, D and the p lines considered above, and in a residual irreducible curve meeting the generators of S only once. This curve is then clearly the *transform C' of C by T_{CD}*. The order q of C' is given by $4 \cdot 4 = 2+3+p+q$, *i.e.* $q = 11-p$.

We now prove that q cannot be 1 or 2, so that $p \leqslant 8$. For if q were 1 or 2 then C' would be a line, distinct from C, D, or a conic. The ruled quartic surface S would have C, D, C' as directrices, and so it would have a component in common with the ruled surface generated by

a line incident with C, D, C'. But this is impossible, since S is irreducible and contains D with multiplicity 3, while, according as $q = 1$ or $q = 2$, the ruled surface above would be a quadric, or a quartic containing D with multiplicity 2. Hence:

Let C, D be two skew rational simple lines of a quartic F, and suppose that F has no multiple points on at least one of them, C say. Then C and D are associate if, and only if, there are ten (possibly irrational) lines on F incident with C and D. When C, D are not associate, the number p of lines of F incident with both C and D is not greater than eight, and the transform of C by T_{CD} is a rational unicursal curve C', of order $11-p \geqslant 3$.*

There is no difficulty in extending the previous discussion so as to include the possibility of F having multiple points on both C and D. For instance, if F has a double point on C, then the plane joining this point with D breaks off from the surface S defined in § 9, the residuum being an irreducible cubic surface, having C and D as directrices of respective multiplicities 1 and 2, etc.†

11. If F is a rational quartic surface containing two skew rational lines C, D, we can introduce projective homogeneous rational coordinates x_1, x_2, y_1, y_2 in space, in such a way that the equations of C and D are

$$x_1 = x_2 = 0, \quad y_1 = y_2 = 0, \tag{6}$$

respectively. Then the equation of F takes the form

$$\sum_{i,\,j=1}^{2} x_i y_j \sigma_{ij}(x_1, x_2, y_1, y_2) = 0, \tag{7}$$

where the σ's are rational quadratic polynomials, and (7) has (6) as trivial solutions. The line C is simple for F, except only when both the binary cubics

$$\left.\begin{aligned} y_1\sigma_{11}(0, 0, y_1, y_2)+y_2\sigma_{12}(0, 0, y_1, y_2) \\ y_1\sigma_{21}(0, 0, y_1, y_2)+y_2\sigma_{22}(0, 0, y_1, y_2) \end{aligned}\right\} \tag{8}$$

---

* This result agrees with §§ 9, 7. For the R-transform of C is then of order 3 from § 9. On the other hand, on applying § 7 with $k = 10$, and noticing that the particularity (ii) now arises four times, we obtain for this order the same value $25-10-4.3 = 3$ as before.

† A deeper treatment of T-transformations is given by B. Segre, "On the quartic surface $x_1^4+x_2^4+x_3^4+x_4^4 = 0$", *Proc. Cambridge Phil. Soc.*, 40 (1943), 121–145, § 16. This paper will be referred to later on as Q.S.

vanish identically. Moreover, F has multiple points on C if, and only if, the resultant of these cubics vanishes. Likewise D is simple for F, except only when both the binary cubics

$$\left.\begin{aligned} x_1\sigma_{11}(x_1, x_2, 0, 0)+x_2\sigma_{21}(x_1, x_2, 0, 0) \\ x_1\sigma_{12}(x_1, x_2, 0, 0)+x_2\sigma_{22}(x_1, x_2, 0, 0) \end{aligned}\right\} \tag{9}$$

vanish identically.

Let us write for shortness

$$\sum_{i,j=1}^{2} x_i y_j \sigma_{ij}(\lambda x_1, \lambda x_2, y_1, y_2) = \lambda^2 L+\lambda M+N,$$

where L, M, N are rational quartic polynomials in x_1, x_2, y_1, y_2, defined by this identity. Then the T_{CD}-transformation of F amounts to associating two solutions $(\lambda_1 x_1, \lambda_1 x_2, y_1, y_2)$ and $(\lambda_2 x_1, \lambda_2 x_2, y_1, y_2)$ of (7) when λ_1 and λ_2 are the two roots of the equation

$$\lambda^2 L+\lambda M+N = 0 \tag{10}$$

in λ. If one of these roots is $\lambda = 1$, the other is $\lambda = N/L$. Hence

If (x_1, x_2, y_1, y_2) denotes any rational solution of (7), *then also*

$$(Nx_1, Nx_2, Ly_1, Ly_2)$$

is a rational solution of (7).

Moreover, from (10) and § 10 we deduce that

The lines C, D are associate if, and only if, the roots of (10) *separate* $0, \infty$ *harmonically, i.e. if the polynomial M vanishes identically.* We shall see in Section VI that, *in this case, the equation* (7) *may have only trivial solutions. If neither M nor both the forms* (9) *vanish identically, and the resultant of* (8) *is different from zero, we obtain as follows a non-trivial rational parametric solution of* (7).

We can satisfy $N = 0$ identically, by taking

$$\left.\begin{aligned} x_1 &= t_1\sigma_{21}(0, 0, t_1, t_2)+t_2\sigma_{22}(0, 0, t_1, t_2), \\ x_2 &= -t_1\sigma_{11}(0, 0, t_1, t_2)-t_2\sigma_{12}(0, 0, t_1, t_2), \\ y_1 &= t_1, \quad y_2 = t_2, \end{aligned}\right\} \tag{11}$$

where t_1, t_2 are two parameters. Then (10) has the rational solutions $\lambda_1 = 0$ and $\lambda_2 = -M/L$. From the former we only obtain the trivial solution $(0, 0, t_1, t_2)$ of (7). But $\lambda_2 = -M/L$ gives the non-trivial rational

parametric solution $(-x_1M, -x_2M, y_1L, y_2L)$ where x_1, x_2, y_1, y_2 must be expressed from (11) as functions of t_1, t_2. The *degree* of this solution, after suppressing a possible common factor in t_1, t_2, is $11-p$, where p is an integer satisfying the conditions $0 \leqslant p \leqslant 8$, and defined as follows. The equations

$$L = 0, \quad N = 0$$

represent two quartic ruled surfaces in space, each of which has C, D as directrices, with the respective multiplicities 3, 1 and 1, 3. Hence they intersect further in

$$4.4-3.1-1.3 = 10$$

generators. Those among these lines which lie on the surface $M = 0$ are the lines of F incident with both C and D; and p denotes their number.

V. *Cases in which the R-process fails.*

12. In Section III we have seen how the R-process can generally be applied to a rational quartic surface F containing a rational line C. There are, however, exceptional cases, in which the R-process either cannot be applied, or gives only trivial solutions. We shall now proceed to a systematic study of these cases.

We first suppose that C is a *multiple* line of F. We can confine ourselves to the case in which C is a double line, on account of the following remarks. If F has a rational point P of multiplicity 4 or 3, we can introduce non-homogeneous projective coordinates (x, y, z) such that P becomes the point at infinity of the z-axis. Then the equation of F assumes the form

$$f(x, y) = 0 \quad \text{or} \quad f(x, y)+zg(x, y) = 0$$

respectively, where f and g are rational polynomials of the 4th and 3rd degree in x, y. In the first case the problem of solving the Diophantine equation of F reduces to that of finding the rational points of a plane quartic, and is irrelevant to the questions studied in this paper. In the second case the problem is trivial, being completely solved by

$$z = -f(x, y)/g(x, y).$$

Let us therefore suppose that *C is a double line of F*. On taking C as z-axis, the equation of F reduces to

$$x^2\alpha(x, y, z)+xy\beta(x, y, z)+y^2\gamma(x, y, z) = 0,$$

where α, β, γ are non-homogeneous quadratic polynomials with rational coefficients. A general algebraic method for solving this equation cannot be given, since, for instance, the equation

$$x^2\alpha(x, y, z)+y^2\gamma(x, y, z) = 0$$

has only the trivial solution $x = y = 0$, if we suppose α, γ to be both definite and with the same sign. Some solutions, however, can be found, if we can find rational points on the *rational quartic curve*

$$\alpha(0, 0, \xi)+\eta\beta(0, 0, \xi)+\eta^2\gamma(0, 0, \xi) = 0,$$

which is generally *elliptic**. For, if (ξ, η) is a rational solution of this equation, the conic

$$y = \eta x, \quad \alpha(x, \eta x, z)+\eta\beta(x, \eta x, z)+\eta^2\gamma(x, \eta x, z) = 0$$

lies on F, is rational, and contains the rational point $(0, 0, \xi)$, generally as a simple point. One can then immediately deduce all the rational points of this conic, and so obtain a rational parametric solution of (12), different from $x = y = 0$.

13. We now suppose C to be a *simple* line of F, and examine when the first step of the R-process cannot be applied. This occurs if, and only if, for a general choice of P on C, the point Q considered in § 7 is not uniquely defined, *i.e.* if either (i) the cubic Γ of § 7 contains C as a component, or (ii) Γ contains as a component a line distinct from C passing through P, or (iii) Γ has a multiple point at P. We only need consider the first two cases, since a simple geometric argument shows that (iii) implies (i).

In case (i), the plane π touching F at P *touches this surface along the whole line* C. Assuming C to be the z-axis and π the plane $x = 0$, the equation of F takes the form

$$xf(x, y, z)+y^2h(x, y, z) = 0, \tag{12}$$

where f and h are non-homogeneous rational polynomials of degrees 3 and 2 respectively. This equation may have only trivial solutions, as we shall see in § 16. We notice that *the present case cannot arise when* F *is non-singular*, since a point satisfying $x = y = f(0, 0, z) = 0$ is a multiple point of (12).

---

* This reduction of the problem has a simple geometric interpretation in connection with the (2, 2) algebraic correspondence between the points of F lying on C and their tangent planes.

In case (ii), F is a ruled surface, having the line C as a simple directrix. We can suppose F to be irreducible. For otherwise the component of F which contains C could be determined rationally, and the problem of solving the Diophantine equation of F would thus be reduced to two simpler ones. Then F is homaloidal, and, as we shall now prove, *its Diophantine equation can be solved completely, with a parametric solution involving two parameters.*

From M.N., § 6, we see that only two types of surfaces F arise, according as the locus D of multiple points of F is (a) a triple line skew to C, or (b) a double twisted irreducible cubic. In both cases D is obviously rational.

In case (a), we can introduce projective homogeneous coordinates (x_1, x_2, y_1, y_2) rationally, in such a way that C and D have the equations (6). Then the equation of F takes the form

$$x_1\sigma_1(y_1, y_2)+x_2\sigma_2(y_1, y_2) = 0, \tag{13}$$

where σ_1, σ_2 are binary cubic forms with rational coefficients. The complete solution of (13) is

$$x_1 : x_2 : y_1 : y_2 = -\sigma_2(t_1, t_2) : \sigma_1(t_1, t_2) : t_1 : t_2,$$

and contains in fact the two parameters t_1, t_2.

In case (b), we can introduce projective homogeneous coordinates (x_0, x_1, x_2, x_3) rationally, in such a way that D is represented parametrically by the equations*

$$x_0 : x_1 : x_2 : x_3 = 1 : u : u^2 : u^3. \tag{14}$$

A rational quartic passing through D with multiplicity two has the equation

$$\alpha\tau_0^2+\beta\tau_1^2+\gamma\tau_2^2+\lambda\tau_0\tau_1+\mu\tau_0\tau_2+\nu\tau_1\tau_2 = 0, \tag{15}$$

where α, β, γ, λ, μ, ν are rational numbers, and

$$\tau_0 = x_0x_2-x_1^2, \quad \tau_1 = x_0x_3-x_1x_2, \quad \tau_2 = x_1x_3-x_2^2 \tag{16}$$

are quadratic forms, such that $\tau_0 = 0$, $\tau_1 = 0$, $\tau_2 = 0$ represent three independent quadrics containing D.

It is geometrically obvious that the surface (15) is ruled, its generators being chords of D. We shall now establish this result analytically, in a form suitable for further applications. Let us consider two points P_1, P_2

* This can be easily proved by applying the Noether-Hilbert-Hurwitz result (quoted in § 3) to the rational unicursal curve D.

on D corresponding to the values u_1, u_2 of the parameter u, and put

$$t_1 = u_1 + u_2, \quad t_2 = u_1 u_2. \tag{17}$$

Then from (14) we see that the line $P_1 P_2$ has the Plücker coordinates

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12} = 1 : t_1 : t_1^2 - t_2 : t_2^2 : -t_1 t_2 : t_2, \tag{18}$$

so that $P_1 P_2$ is rational if, and only if, both t_1 and t_2 are rational. On the other hand, the line $P_1 P_2$ is the locus of the point

$$x_0 : x_1 : x_2 : x_3 = 1 + \lambda : u_1 + \lambda u_2 : u_1^2 + \lambda u_2^2 : u_1^3 + \lambda u_2^3, \tag{19}$$

when the parameter λ varies. From (16), (17), (19) we deduce

$$\tau_0 : \tau_1 : \tau_2 = 1 : t_1 : t_2,$$

and so (15) becomes

$$\alpha + \beta t_1^2 + \gamma t_2^2 + \lambda t_1 + \mu t_2 + \nu t_1 t_2 = 0. \tag{20}$$

This expresses the necessary and sufficient condition for the chord $P_1 P_2$ of D to lie on F. From a well-known property of twisted cubics, we see that there is one, and only one, of these chords through every point of F not lying on D. Hence this line is rational, if the point is rational.

We add that the line $P_1 P_2$, as join of the two points arising from (19) for $\lambda = \pm 1$, can also be represented parametrically, from (17), by

$$x_0 : x_1 : x_2 : x_3 = 2 : t_1 + t : t_1^2 - 2t_2 + t t_1 : t_1^3 - 3t_1 t_2 + t(t_1^2 - t_2). \tag{21}$$

When $P_1 P_2$ is rational, *i.e.* if t_1, t_2 are rational numbers satisfying (20), the rational points of $P_1 P_2$ are given by (21), and each only once, for rational values of the parameter t. Hence

The solutions of the biquadratic equation (15), *rational in* x_0, x_1, x_2, x_3 *and not included in* (14), *where the* τ*'s are the quadratic forms* (16), *are all given by* (21), *and each only once. Here* t *denotes an arbitrary rational number, and* (t_1, t_2) *is any rational solution of the quadratic equation* (20).

The conic (20) is irreducible, because (15) is. Hence, from a well-known result, (20) has either no rational solution in t_1, t_2, or an infinity of solutions, which then can all be expressed rationally in terms of a parameter. Hence

An irreducible rational quartic surface F, *possessing an irreducible double twisted cubic* D, *either has no rational points apart from those on* D, *or has an infinity of rational points, all expressible rationally as functions of two parameters.*

The second alternative holds in the case (b) previously considered, since F then contains the rational line C, and therefore also an infinity of rational points lying on C. Let

$$a_0x_0+a_1x_1+a_2x_2+a_3x_3=0, \quad b_0x_0+b_1x_1+b_2x_2+b_3x_3=0$$

represent this line, where the a's, b's are rational. The Plücker coordinates of C are

$$q_{01}:q_{02}:q_{03}:q_{23}:q_{31}:q_{12}$$

$$=a_2b_3-a_3b_2:a_3b_1-a_1b_3:a_1b_2-a_2b_1:a_0b_1-a_1b_0:a_0b_2-a_2b_0:a_0b_3-a_3b_0,$$

connected by the relation

$$q_{01}q_{23}+q_{02}q_{31}+q_{03}q_{12}=0, \tag{22}$$

and so the condition for the incidence of C with the line (18) is

$$q_{23}+q_{12}t_1^2+q_{01}t_2^2+q_{31}t_1+(q_{03}-q_{12})t_2-q_{02}t_1t_2=0. \tag{23}$$

Hence, on identifying this equation with (20), we see that, in the present case,

$$\alpha:\beta:\gamma:\lambda:\mu:\nu=q_{23}:q_{12}:q_{01}:q_{31}:q_{03}-q_{12}:-q_{02}. \tag{24}$$

On substituting for α, β, γ, λ, μ, ν in (15) from (24), we obtain the corresponding equation of F, and the equation (20) is now replaced by (23). This equation has in fact an infinity of rational solutions in t_1, t_2. For, if (as it is not restrictive to suppose) $q_{01}q_{12}\neq 0$, we see from (22) that a solution of (23) is given by

$$t_1=-q_{31}/q_{12}, \quad t_2=q_{12}/q_{01}.$$

From this the complete solution of (23), expressing t_1, t_2 rationally in terms of a parameter, can easily be deduced. On substituting this solution in (21), we obtain the complete rational solution of the present equation (15), additional to (14).

14. In §§ 12, 13 we have dealt with all cases in which the R-process cannot be applied. We shall now classify the cases when the R-process, applied to a rational simple line C on a quartic surface F, gives only trivial solutions of the corresponding Diophantine equation. From §§ 5, 7, this occurs if, and only if, the R-transform of C is still a line, which then can only either (i) be skew to C, or (ii) coincide with C. For simplicity, we shall confine ourselves to the general case, in which F has no multiple points on C.

The theorem of § 9 shows that the case (i) arises if, and only if, *the line C has an associate one on F*. The corresponding reduced form for the Diophantine equation of F has been given in § 11, and is the equation (7) for which the condition $M = 0$ is satisfied identically.

The case (ii) occurs when, and only when, for a general position of a point P on C, there is one and only one line G, distinct from C, having a four-point contact with F at P. Using the terminology of M.N., § 2, we shall then say that C is a *line of the 2nd kind* of F. From M.N., § 6, we see that the locus of G, when P describes C, is an irreducible ruled quartic surface Φ, having C as simple directrix. Moreover, the pencil determined by F and Φ contains a surface Ψ consisting of four planes passing through C, so that F *contains at least* 12 (*possibly irrational*) *lines incident with* C, given by the intersection of Φ and Ψ residual to C.

Since F, C are rational, and Φ, Ψ are defined uniquely by them, it follows that Φ and Ψ are also rational. Hence Φ is one of the surfaces considered in § 13 (ii), and so can only present the two cases investigated there as cases (*a*), (*b*). Moreover Ψ is represented by equating to zero a rational quartic form in two independent linear forms, vanishing simultaneously along C. Hence we deduce, from (13) and (15), (24), that the equation of F (which is a surface of the pencil of Φ and Ψ) can be reduced to one or other of the equations

$$x_1\sigma_1(y_1, y_2)+x_2\sigma_2(y_1, y_2) = \psi(x_1, x_2),$$

$$q_{23}\tau_0^2+q_{12}\tau_1^2+q_{01}\tau_2^2+q_{31}\tau_0\tau_1+(q_{03}-q_{12})\tau_0\tau_2-q_{02}\tau_1\tau_2$$
$$= \psi(a_0x_0+a_1x_1+a_2x_2+a_3x_3,\ b_0x_0+b_1x_1+b_2x_2+b_3x_3),$$

where the notation is the same as in § 13, and ψ denotes a binary quartic form with rational coefficients.

The main cases of equations (1) in which our methods do not succeed, are these two equations, the equation (7) when $M = 0$ holds identically, and the equation (12) of § 13. We shall not study them here, since there are difficulties, purely arithmetical in character, which are shown by various examples of quartic equations, having only trivial solutions, given in the next section.

VI. *Some illustrative examples.*

15. An equation of type (7), having $M = 0$, is

$$xy(z^2-w^2) = zw(x^2+y^2). \tag{25}$$

This represents a *ruled* quartic surface, containing the lines

$$x = y = 0, \qquad z = w = 0$$

as directrices of multiplicity two, and passing simply through the other two pairs of opposite edges of the fundamental tetrahedron, viz. $x = z = 0$, $y = w = 0$, and $x = w = 0$, $y = z = 0$. Each of these pairs consists of two *associate* lines (§ 9), since the harmonic homology determined by each pair transforms (25) into itself. The theorem of § 9 does not apply to the two lines of either pair, since the surface has two double points on each of them. However, an argument similar to that of § 9 shows that *each of such two lines is the R-transform of the other.*

We shall now prove that

The equation (25) *has no rational solutions with* $xyzw \neq 0$, *so that its only rational solutions are the trivial ones indicated above.*

If (25) has a rational solution (x, y, z, w) with $xyzw \neq 0$, we can put

$$\xi = x/y, \quad \zeta = z/w,$$

with ξ, ζ rational and different from zero. Then (25) gives*

$$\xi + \frac{1}{\xi} = \zeta - \frac{1}{\zeta},$$

whence, denoting by 2ω the common (rational) value of the two sides,

$$\xi^2 - 2\omega\xi + 1 = 0, \quad \zeta^2 - 2\omega\zeta - 1 = 0.$$

Since each of these quadratic equations in ξ, ζ has rational roots, it follows that both $\omega^2 - 1$ and $\omega^2 + 1$ must be squares of rational numbers, so that $\omega^4 - 1$ is a non-zero square. But this is impossible, since the Diophantine equation $X^4 - Y^4 = Z^2$ has no solutions in positive integers†.

16. Another simple example is given by the quartic surface F represented by

$$x^4 - y^4 = 2(z^2 - w^2)^2. \tag{26}$$

This equation has only the rational solutions

$$x = \pm y, \qquad z = \pm w,$$

because the Diophantine equation $X^4 - Y^4 = 2Z^2$ has no solutions in positive integers†.

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* Hence $\xi = \text{const.}$, $\zeta = \text{const.}$ (connected by this equation) give the generators of (25).

† *Cf., e.g.*, L. E. Dickson, *l.c.* in § 2.

These rational solutions are trivial, and represent geometrically four simple lines of F, which are the sides of the skew quadrilateral having the points

$$(0, 0, 1, 1), \quad (1, 1, 0, 0), \quad (0, 0, 1, -1), \quad (1, -1, 0, 0)$$

as consecutive vertices. The first and the third of these points are double for F, and are the only multiple points of this surface. It follows that *F is non-ruled*, because any ruled surface, not a cone and of order greater than two, has a multiple curve. Moreover, it is easily seen that *F has a fixed tangent plane along each of the four rational lines*, so that (26) can be reduced in many ways to the form (12) of § 13, and the R-transformation cannot be applied to any of the four lines. The T-transformation cannot be applied successfully to either pair of opposite sides of the quadrilateral above, because the plane touching F along either line of the pair meets the other line at the double point of F lying on it.

17. The quartic surface

$$x^4+2y^4=(z^2+yw)^2 \tag{27}$$

has the double point $O(0, 0, 0, 1)$, and no further multiple point, so that *it is non-ruled*. Since the Diophantine equation $X^4+2Y^4=Z^2$ has no solutions in positive integers*, we see that

The only rational solutions of (27) *are*

$$y=0, \quad x=\pm z. \tag{28}$$

These equations represent two lines, meeting at O, and the surface (27) has not a fixed tangent plane along any of them. *Each of the lines* (28) *is its own R-transform.* For the lines

$$x-ty\mp z=2tx-t^2y-w=0$$

have four-point contacts with the surface (27) at the points $(1, 0, \pm 1, 2t)$, which describe the lines (28) when the parameter t varies.

18. We shall now prove that

The equation

$$3(x^2+yz-\tfrac{1}{12}w^2)^2+3z^3w-3z^2w^2+zw^3-\tfrac{1}{12}w^4=0 \tag{29}$$

* *Cf., e.g.,* L. E. Dickson, *l. c.* in § 2.

has only the rational solutions

$$2x \pm w = z = 0 \tag{30}$$

and

$$x : y : z : w = t : t^2 : -1 : 0, \tag{31}$$

where t is a parameter.

In addition, we shall show that

We obtain the solution (31) *by applying the R-process once to either of the trivial solutions* (30) *of* (29). *The reiteration of the R-process gives no new solutions, but leads only to $x = z = w = 0$.*

From (29) we see that (31) is the only solution with $w = 0$. If $w \neq 0$, we define the rational numbers

$$\xi = (x^2+yz)/w^2 - \tfrac{1}{12}, \quad \eta = z/w. \tag{32}$$

Then (29) becomes

$$3\xi^2 + 3\eta^3 - 3\eta^2 + \eta - \tfrac{1}{12} = 0. \tag{33}$$

A simple calculation shows that this cubic reduces to the homogeneous form

$$X^3 + Y^3 + Z^3 = 0, \tag{34}$$

by performing the projective transformation

$$\xi = \frac{Y-Z}{6(Y+Z)}, \quad \eta = \frac{X+Y+Z}{3(Y+Z)} \tag{35}$$

on ξ, η. Since the only rational solutions of (34) are*

$$X = Y+Z = 0, \quad Y = Z+X = 0, \quad Z = X+Y = 0,$$

we see from (35) that the only rational solutions of (33) are

$$\xi = \pm\tfrac{1}{6}, \quad \eta = 0.$$

In correspondence to these, the equations (32) give (30) as the only rational solutions of (29) having $w \neq 0$.

The quartic surface (29) contains an *elliptic pencil of conics*, given by (32), where ξ, η are any two constants connected by (33). This pencil has a single base point at the point O (0, 1, 0, 0), and this is double for

* About this long-established case of Fermat's last theorem, *cf.*, *e.g.*, G. H. Hardy and E. M. Wright, *op. cit.* in § 2, pp. 192–194.

(29), and is the only multiple point of the surface. The two lines (30) and the conic (31) meet at O, and give two particular conics of this pencil.

On joining one, say C, of the lines (30) with the point Q given by (31) for a general t, we obtain the plane

$$2x \pm w + 2tz = 0. \tag{36}$$

This meets (29) in a cubic Γ residual to C, whose equations are (36) and

$$w^2(y + t^2 z \pm tw) + 3z(y + t^2 z \pm tw)^2 + 3z^2 w - 3zw^2 + w^3 = 0.$$

The last equation shows that two of the intersections of Γ and C are fixed at O, the remaining one being the variable point $P(1, 2t \pm 2, 0, \mp 2)$. Moreover the tangential point of P is Q, and the tangential point of Q is the fixed point O. Hence the additional results stated at the beginning of this paragraph follow from § 6.

VII. *Surfaces containing certain sets of lines.*

19. *Every rational quartic surface F containing three rational lines C_1, C_2, C_3, two by two skew, and one at least containing no multiple points, is represented by a Diophantine equation which possesses a non-trivial rational parametric solution, of degree at least* 3 *and at most* 11.

If F has no multiple points on C_1, then, from the final result of § 9, at most one of the pairs C_1, C_2 and C_1, C_3 consists of associate lines of F. There is therefore no restriction in supposing that C_1, C_2 are non-associate. Hence, from § 10, we obtain on F as transform of C_1 by $T_{C_1 C_2}$ a rational unicursal curve C_1', of order $11 - p \geqslant 3$, where $p \geqslant 0$ is the number of rational and irrational lines of F incident with C_1 and C_2. The parametric representation of C_1', which can be obtained rationally as indicated in § 11, gives the solution alluded to above.

20. *Any rational quartic surface F containing four rational lines C_1, C_2, C_3, C_4, which are generators of the same system of a quadric Q, and one of which at least contains no multiple points, is represented by a Diophantine equation which possesses a non-trivial rational parametric solution, of degree at least* 3 *and at most* 7.

This follows from the argument of § 19, since now $p \geqslant 4$. In fact, the intersection of F and Q consists of C_1, C_2, C_3, C_4, and of four generators of Q of the other system, each of which is therefore incident with each of the C's. For, if P is any point of this intersection lying on none of

the C's, then the generator D of the other system of Q containing P has five intersections with F, given by P and the intersections of D with C_1, C_2, C_3, C_4, so that D lies on F.

21. We notice that

The hypotheses of § 19 *imply those of* § 20, *if two of the lines* C_1, C_2, C_3 *are associate.*

For, if C_1, C_2 are associate, then the harmonic homology determined by these two lines transforms C_3 into another rational line C_4, which, from § 9, must also lie on F. The result follows, because C_4 is obviously a generator of the quadric Q containing C_1, C_2, C_3, defined by the condition that, on Q, C_1 and C_2 are harmonically separated by C_3 and C_4.

The following result completes the theorem of § 19, giving a solution which differs from that obtained there.

Let F *be a rational quartic surface containing a rational triplet of lines** C_1, C_2, C_3, *two by two skew, and suppose that the quadric* Q *determined by* C_1, C_2, C_3 *has no other lines lying on* F. *Then the Diophantine equation of* F *has a rational parametric solution of the* 5*th degree.*

Since the triplet C_1, C_2, C_3 is rational, it follows that Q is rational, and contains two systems of rational generators†. Hence Q cuts F, residually to C_1, C_2, C_3, in a rational unicursal curve of the 5th order. This curve contains an infinity of rational points, given by the intersections, not lying on C_1, C_2, C_3, of F with the rational generators of Q incident with C_1, C_2, C_3.

22. The argument employed in § 19 shows that

If a non-singular rational quartic surface F *contains two pairs of associate rational lines, and there are two lines of different pairs which are skew, then the Diophantine equation of* F *possesses a non-trivial rational parametric solution, of degree at least* 3 *and at most* 11.

---

* This does not imply that any of the lines C_1, C_2, C_3 is rational.

† For a general rational plane intersects Q in a rational conic, Γ say, and meets C_1, C_2, C_3 in a rational triplet of points, lying on Γ. Apart from Γ, there is an infinity of rational conics containing this triplet of points, and each of them intersects Γ residually in a rational point. The two generators of Q containing any of these rational points of Γ obviously constitute a rational pair; and each of them is rational, since one of them meets C_1, C_2 and C_3, while the other is skew to C_1, C_2 and C_3.

Hence

If a non-singular rational quartic F contains two pairs of associate rational lines, and all the solutions of its Diophantine equation are trivial, then the two pairs of associate lines are the opposite sides of a skew quadrilateral.

The two surfaces of §§ 15, 16 are singular, but otherwise all the conditions of this theorem hold. It would be important to find out whether non-singular surfaces satisfying this theorem exist. We shall not answer this question, but we shall show that it is tantamount to asking *whether a Diophantine equation*

$$xy(ax^2+by^2+cz^2+dw^2)+zw(a'x^2+b'y^2+c'z^2+d'w^2)=0 \qquad (37)$$

may have only trivial solutions, and represent a non-singular surface.

We first remark that, by a proper choice of the rational and homogeneous coordinates (x, y, z, w), four rational lines which are the sides of a skew quadrilateral can be represented by the equations

$$x=z=0, \quad z=y=0, \quad y=w=0, \quad w=x=0. \qquad (38)$$

Hence, if F is a rational quartic containing these lines, its equation is of the type

$$xy\sigma(x, y, z, w)+zw\tau(x, y, z, w)=0, \qquad (39)$$

where σ and τ are quadratic forms with rational coefficients. The coefficients of x^2, y^2 in σ and those of z^2, w^2 in τ must be different from zero, since none of the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) is multiple for F. Moreover, from § 9, each of the two pairs of skew lines (38) consists of associate lines if, and only if, (39) remains unaltered when we change the signs of x and z, as well as when we change the signs of x and w. From this we easily deduce that the equation (39) must in fact be of the type (37).

We shall now prove that

From any given rational solution (x', y', z', w') of (37), *such that $x'y'z'w' \neq 0$, it is generally possible to deduce another rational solution of* (37), *satisfying a similar condition.*

This follows by applying the final remark of § 6 to the point

$$P'(x', y', z', w'),$$

and either of the rational lines lying on the surface (37). We now give another process, involving these four lines symmetrically. Let us consider the pencil of quadrics

$$xy=\lambda zw,$$

having the skew quadrilateral considered above as base curve. This cuts out on (37), apart from its base quadrilateral, a pencil of elliptic quartic curves, Γ say, represented by the last equation and

$$\lambda(ax^2+by^2+cz^2+dw^2)+(a'x^2+b'y^2+c'z^2+d'w^2)=0.$$

There is one, and only one, curve Γ passing through the rational point P'. This curve is rational, and has at P' a rational osculating plane, π' say, having in general three, and only three, intersections with Γ coinciding at P'. There is consequently one, and only one, further intersection of Γ and π'. This is a rational point of Γ, lying on (37), and therefore giving a new rational solution of this equation*.

The calculations needed for determining this point are rather long, but offer no difficulty, and so we shall only give the final result. On taking $(x/x', y/y', z/z', w/w')$ as new variables, (37) is transformed into a new equation of the same type, having the solution (1, 1, 1, 1). We can therefore suppose without loss of generality that P' is the point (1, 1, 1, 1), and so the coefficients in (37) satisfy the condition

$$a+b+c+d+a'+b'+c'+d'=0.$$

The curve Γ which contains P' is then

$$xy=zw, \quad \alpha x^2+\beta y^2=\gamma z^2+\delta w^2,$$

where

$$\alpha=a+a', \quad \beta=b+b', \quad \gamma=-c-c', \quad \delta=-d-d'$$

are connected by

$$\alpha+\beta=\gamma+\delta.$$

Put for shortness

$$r=\alpha-\gamma, \quad s=\beta-\gamma, \quad t=\alpha+\gamma.$$

Then the solution of (37) derived from (1, 1, 1, 1) as indicated before, is

$$\begin{aligned} x:y:z:w = {} & (r^2-3s^2-2st)(r^2-2rs-2rt-s^2) \\ & :(r^2+s^2+2st)(r^2+2rs+2rt-s^2) \\ & :(r^2-3s^2-2st)(r^2+2rs+2rt-s^2) \\ & :(r^2+s^2+2st)(r^2-2rs-2rt-s^2). \end{aligned}$$

* This process cannot be applied usefully when the point P' lies on a side of the skew quadrilateral. For then, as is easily seen, it only leads to another point of the same line.

The same procedure can now be applied to this solution, thus obtaining in general a new solution of (37); and so on indefinitely.

We add a remark about the non-trivial solutions of (37) satisfying $xyzw = 0$, which, from the geometrical point of view, is a particularly simple application of § 8. The intersection of the plane $w = 0$ with the surface (37) consists of two lines, and of the rational conic

$$ax^2+by^2+cz^2 = 0.$$

When $abc \neq 0$ and well-known *arithmetical conditions* are satisfied, this equation can be solved in the rational domain, thus giving a *rational parametric solution of the second degree* of (37).

23. We obtain immediately other less obvious cases in which the Diophantine equation (37) has non-trivial solutions, by making the surface (37) contain, in addition to the lines (38), any given rational line distinct from $x = y = 0$ and $z = w = 0$. For this line is then skew to at least one of the lines (38), and non-associate with it, so that the result of § 10 can be applied.

The study of the various corresponding types of equations (37) offers no difficulty, and we shall confine ourselves to one of them. If the coefficients of (37) satisfy the conditions

$$a+c+a'+c' = 0, \quad b+d+b'+d' = 0, \tag{40}$$

then the lines $x+z = y+w = 0$, $x-z = y-w = 0$ lie on (37). These two lines are incident with the first and third of the lines (38), and skew with the remaining two. Moreover, the lines

$$x+z = y+w = 0, \quad x-z = y-w = 0, \quad x = w = 0, \quad y = z = 0$$

are four generators of the same system of a quadric, since the first two are interchanged by the harmonic homology defined by the other two. From § 20, it follows that the lines incident with both of

$$x+z = y+w = 0, \quad x = w = 0, \tag{41}$$

and lying on F, are at least four in number. Then a simple argument shows that there are no more than four such lines. Hence, from § 10,

The Diophantine equation (37), *where the coefficients are rational numbers satisfying* (40), *has a rational parametric solution of the seventh degree, obtainable as the transform of either of the lines* (41) *by means of the T-transformation defined by these two lines,*

This extends a classic result, obtained by Euler* (without any geometric interpretation) when

$$a' = b' = c = d = 0, \quad c' = -a, \quad d' = -b.$$

24. *Let F be a rational quartic surface, intersecting a quadric Q in four rational generators C_1, C_2, C_3, C_4 of one system, and four rational generators D_1, D_2, D_3, D_4 of the other system. Then, in general, the Diophantine equation of F has a discontinuous infinity of rational parametric solutions. This result holds even when F is specialized in such a way that the eight lines C and D form four pairs of associate lines.*

We begin by remarking that *the surfaces F considered above depend on 27 parameters, and constitute a single continuous system.* For the eight lines $C_1, C_2, C_3, C_4, D_1, D_2, D_3, D_4$ depend continuously on 17 parameters (9 fixing the position of the quadric Q, and 8 determining the 8 generators on Q). Moreover it is easily seen that the quartics through eight such lines constitute a linear system, of dimension 10.

We can then prove the following theorem, having as an immediate corollary the first statement of this paragraph.

On the general F, possibly specialized as indicated above, the six T-transformations determined by the six pairs of lines C, and the six T-transformations determined by the six pairs of lines D, generate an infinite discontinuous group G of birational transformations. By applying the transformations of G to a line C or D, we obtain on F a discontinuous infinity of unicursal curves, each giving a rational parametric solution of the Diophantine equation of F.

In Q.S. § 17, this theorem was established for a projective transform of a particular surface F_0 (which we shall consider in detail in § 25). The theorem therefore also holds for the general F, since F can be continuously reduced to F_0. For, should the group G be finite, or should the transforms by G of a line C or D be only finite in number, then such results would necessarily still be true for F_0, by reasons of continuity. Moreover, the eight lines C and D of F may be also specialized so as to consist of four pairs of associate lines [in which case four of the twelve T-transformations determined by them are homographic (§ 9)], since this holds for the eight rational lines of F_0.

We add that the theorem can be established without using any argument of continuity, by applying the "fundamental theorem" of

* *Cf., e.g.,* L. E. Dickson, *op. cit.* in § 2, pp. 61–62.

Q.S. § 16 to the eight lines C and D of F, in the same way as it is employed for F_0 in Q.S. § 17. This method has the advantages (i) that *it can be used also for particular surfaces* F, and (ii) that *it gives the orders and the other projective characters of the unicursal curves obtained on* F *by means of* G.

25. The particular surface F_0 considered in § 24, to which all the results stated there for F apply, is the surface represented by *Euler's equation*

$$x^4+y^4=z^4+w^4.$$

From Q.S. § 2 (or from next paragraph), we see that the only rational lines lying on F_0 are the lines

$$x+z=y+w=0,\ x-z=y-w=0;\quad x+w=y-z=0,\ x-w=y+z=0,$$

and

$$x+z=y-w=0,\ x-z=y+w=0;\quad x+w=y+z=0,\ x-w=y-z=0.$$

These eight lines constitute the complete intersection of F_0 with the quadric

$$x^2+y^2=z^2+w^2,$$

on which the first four lines are generators of one system, while the others are generators of the other system. The semicolons divide the eight lines into four pairs, each consisting of two associate lines of F_0.

From Q.S. § 2 (or from next paragraph), we see that F_0 contains only four lines incident with any two skew non-associate rational lines of F_0. Hence, from § 10, we know how to obtain a *rational parametric solution of the seventh degree* of Euler's equation. This was obtained by Euler in a form contained as a particular case in the final result of § 23*. But now, in addition, we have proved that *Euler's equation has a discontinuous infinity of rational parametric solutions*, and we also know how these solutions (and in particular their orders) can be obtained.

VIII. *On Diophantine equations* $\phi(x, y)=\psi(z, w)$.

26. We shall now apply the general theory developed in the previous sections in the study of the equations of the type

$$\phi(x, y)=\psi(z, w), \tag{42}$$

* For Euler's equation reduces to $XY(X^2+Y^2)=ZW(Z^2+W^2)$, on putting

$$x:y:z:w=X+Y:Z-W:Z+W:X-Y.$$

where ϕ and ψ are rational quartic forms, say

$$\left.\begin{aligned}\phi(x, y) &= ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4,\\ \psi(z, w) &= a'z^4+4b'z^3w+6c'z^2w^2+4d'zw^3+e'w^4.\end{aligned}\right\} \tag{43}$$

We begin by studying the surface F represented by (42) *in the complex domain*, so that, in all this paragraph, the coefficients in (43) need not be rational.

It is well known that ϕ has the relative projective invariants

$$I = ae-4bd+3c^2, \quad J = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix},$$

whose vanishing is the condition that the set formed by the roots of the equation

$$\phi(x, y) = 0 \tag{44}$$

should be equianharmonic or harmonic respectively. The ratio I^3/J^2 is an absolute projective invariant, expressible as a function of the cross-ratio of these roots (independent of the order in which the roots are taken). Moreover, the discriminant D of ϕ is given by*

$$D = I^3-27J^2.$$

The invariants I', J', D' can likewise be introduced for ψ, and are similarly related to the equation

$$\psi(z, w) = 0. \tag{45}$$

We suppose the surface F to be *non-singular*. It is easily seen that this is expressed by the condition

$$D\,D' \neq 0. \tag{46}$$

This is tantamount to supposing that F intersects the line p of equations $z = w = 0$ in *four distinct points* P_1, P_2, P_3, P_4, represented by (44), and the line q of equations $x = y = 0$ in *four distinct points* Q_1, Q_2, Q_3, Q_4, represented by (45). The equations (42), (44), (45) then show that

The 16 *lines* $P_i\,Q_j$ $(i, j = 1, 2, 3, 4)$ *lie on* F, *and are the only lines of* F *meeting both* p *and* q.

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* *Cf., e.g.*, G. Salmon, *Lessons introductory to the modern higher algebra*. 3rd ed. (Dublin, 1876), 187–189, or F. Enriques and O. Chisini, *Lezioni*, quoted in § 4, Vol. I (Bologna, Zanichelli, 1915), § 4. From the last relation it follows that $I = 0$ and $J = 0$ cannot hold together if $D \neq 0$.

We shall call the lines of $P_i Q_j$ lines of *type I*, and any other line lying on F a line of *type II*.

A line of type II is skew with at least one of the lines p, q. If, *e.g.*, it is skew with q, then it can be represented by equations of the form

$$z = \alpha x + \beta y, \quad w = \gamma x + \delta y, \tag{47}$$

satisfying (42) identically in x, y. Hence, from (46), it follows that

$$\alpha\delta - \beta\gamma \neq 0, \tag{48}$$

so that *every line of type II is skew with both p and q*. We see consequently that, if (47) is a line of type II lying on F, then the equation

$$z/w = (\alpha x + \beta y)/(\gamma x + \delta y) \tag{49}$$

represents a *projectivity* between the lines p, q, transforming the four points P into the four points Q.

Conversely, a projectivity between p and q satisfying this condition is represented by an equation (49), where α, β, γ, δ are subject to (48) and can still be altered by an arbitrary non-zero common factor. Moreover the projectivity transforms (44) into (45). It follows immediately that the common factor of α, β, γ, δ is determined, but for a quartic root of unity, if we impose the further condition that $\phi(x, y)$ is transformed by (49) into $\psi(z, w)$. Hence the projectivity considered corresponds to *four distinct lines of type II* lying on F^*. We shall say that the former is *related* to each of the latter, and that these lines constitute a *4-set*.

From well-known properties of the projectivities transforming four given points of a line into four given points of another line†, we easily deduce that

The total number of lines lying on the non-singular surface F represented by (42) *is* 16, 32, 48, *or* 64, *according as*

(i) $I^3 J'^2 - I'^3 J^2 \neq 0$, (ii) $I^3 J'^2 - I'^3 J^2 = 0$, *but* $IJI'J' \neq 0$,

(iii) $J = J' = 0$ (*but* $II' \neq 0$), (iv) $I = I' = 0$ (*but* $JJ' \neq 0$)‡.

---

* A simple argument shows that *these four lines constitute a harmonic set of generators of a quadric, which contains p, q and meets F again in four lines of type I.*

† *Cf.*, *e.g.*, F. Enriques and O. Chisini, *loc. ult. cit.*

‡ Case (iii) has been investigated thoroughly in Q.S. From M.N., the surfaces considered in (iv) have the maximum number of lines a non-singular quartic can possess.

We add the obvious remark that *any line P_iQ_j of type I is incident with six, and only six, other lines of type I; three of these are in the plane P_iq and meet at P_i, and the remaining three are in the plane Q_jp and meet at Q_j.* Moreover, it is easily seen that P_iQ_j is incident with a line of type II of F if, and only if, the projectivity related with this line transforms P_i into Q_j. Since the number of projectivities satisfying this condition is 0, 1, 2, 3 in cases (i), (ii), (iii), (iv) respectively, it follows that

In the cases (i), (ii), (iii), *or* (iv), *any line of type I of F is incident with* 0, 4, 8, *or* 12 *respectively of the lines of type II of this surface.*

27. From now on we suppose that the coefficients in (43) are rational, so that the surface F represented by (42) is also rational. We shall say that the two forms (43) are *rationally equivalent*, if they can be transformed one into the other by means of a linear substitution (47) whose coefficients are rational and subject to (48). If two forms ϕ, ψ are rationally equivalent, then their absolute projective invariants are equal. The converse is of course untrue. But necessary and sufficient (arithmetical) conditions for the rational equivalence of ϕ and ψ have been given*.

We shall say, moreover, that *the Diophantine equation* (42) *is rationally equivalent to*

$$\phi_1(x, y) = \psi_1(z, w), \tag{50}$$

and also that (42) *can be reduced to the form* (50), if ϕ_1, ψ_1 are binary quartic forms, rationally equivalent to ϕ, ψ respectively. The two surfaces represented by (42), (50) are then related by a rational homographic transformation.

The following theorem gives *all* the cases in which a rational non-singular surface (42) contains a rational line.

A rational non-singular surface (42) *contains a rational line of type I if, and only if, each of the equations* (44), (45) *has at least one rational solution, i.e. if* (42) *can be reduced to the form*

$$x\sigma(x, y) = z\tau(z, w), \tag{51}$$

where σ, τ are rational cubic binary forms. The surface (42) *contains a rational line of type II if, and only if, ϕ and ψ are rationally equivalent, i.e.*

* *Cf.* B. Segre, "Equivalenza ed automorfismi delle forme binarie in un dato anello o campo numerico", due to appear in the *Revista Univ. Nac. Tucuman*, § 8. This paper will be referred to as E.A.

(42) *can be reduced to the form*

$$\phi(x, y) = \phi(z, w). \tag{52}$$

The 4-set determined by such a line contains one, and only one, further rational line, and the two rational lines of the 4-set constitute a pair of associate lines of the surface.

All these results, except the last one, are immediate consequences of § 26. The last result follows from § 9, by noticing that the surface (52) contains the lines

$$x+z = y+w = 0, \quad x-z = y-w = 0, \tag{53}$$

and is transformed into itself by the harmonic homology determined by them (represented by the simultaneous interchanging of x and z, and of y and w).

28. We now study the equations (51) and (52). We prove first that

The Diophantine equation (51) *has a rational parametric solution of degree* 13 *or* 9, *in the cases* (i) *or* (ii) *respectively* (*with the notation of* § 26). *The existence of non-trivial rational solutions is, however, doubtful in the remaining two cases* (iii) *or* (iv); *then the equation* (51) *can be reduced to the forms*

$$XY(\alpha X^2+\beta Y^2) = ZW(\gamma Z^2+\delta W^2) \tag{54}$$

or

$$X(\alpha X^3+\beta Y^3) = Z(\gamma Z^3+\delta W^3) \tag{55}$$

respectively, where α, β, γ, δ *are non-zero rational numbers.*

Since (51) is non-singular and has the trivial solution $x = z = 0$, we can, by §§ 13, 14, apply to this the R-process and deduce a non-trivial solution, with only two possible exceptions, arising when the line $x = z = 0$ either has an associate one on the surface represented by (51), or is a line of the second kind. Neither exception can occur in the cases (i) or (ii). For then (51) contains in all 6 or 10 lines incident with $x = z = 0$ (§ 26), and the latter is the only line incident with each of the former. Hence the line $x = z = 0$ cannot have an associate one, since then the surface would contain 10 lines incident with both this line and its associate (§ 10). Nor can $x = z = 0$ be a line of the second kind, since then the surface would contain at least 12 lines incident with it (§ 14).

Thus in cases (i), (ii), *we obtain a non-trivial rational parametric solution of* (51) *by applying the R-process to* $x = z = 0$. The degree of this para-

metric solution is 13 or 9 in cases (i) or (ii) respectively, as is easily seen on using the result of § 7 (iii), and recollecting the properties established at the end of § 26.

In case (iii), the equation (51) can be written as (42), where ϕ, ψ are given by (43) and satisfy the conditions (46) and

$$e = e' = J = J' = 0.$$

Then $d \neq 0$, because, if $d = 0$, from $e = J = 0$ we deduce $c = 0$, implying $I = 0$, and therefore $D = 0$, in contradiction to (46). Likewise we see that $d' \neq 0$. Hence we can put

$$x = X, \quad y = (-cX + Y)/2d, \quad z = Z, \quad w = (-c'Z + W)/2d';$$

and a simple calculation shows that (42) is thus reduced to the form (54).

In case (iv), the equation (51) can be written as (42), where ϕ, ψ are given by (43) and satisfy the conditions (46) and

$$e = e' = I = I' = 0.$$

We see again from (46) that $dd' \neq 0$. Hence we can put

$$x = X, \quad y = -cX/2d + Y, \quad z = Z, \quad w = -c'Z/2d' + W;$$

and a simple calculation shows that (42) is now reduced to the form (55).

From §§ 14, 22, we immediately infer that

The surface (54) *contains two pairs of associate rational lines, sides of a skew quadrilateral, represented by* (38). *The rational line* $X = Z = 0$ *is a line of the second kind for the quartic surface* (55). *Hence it is impossible to deduce further solutions of either equation by applying the R-process or the T-process to such lines.*

29. The last result leaves undecided whether (54), (55) have non-trivial solutions when the coefficients α, β, γ, δ are indeterminate. We do not study this important but difficult question (extending one already pointed out in § 22), but confine ourselves to some special equations (54) and (55), having non-trivial solutions.

From § 23 we know that (54) has non-trivial solutions if the surface (54) contains rational lines, in addition to the lines (38). We have then two cases to distinguish, according as such an additional line is of type I or of type II (§ 26). It is easily seen that the second, but generally not the first, case arises if, and only if, (54) can be reduced to an equation of the same type with $\alpha = \gamma$, $\beta = \delta$. Hence, from § 23, a *rational parametric solution of the seventh degree* can then be obtained.

In the remaining case, either $-\alpha/\beta$ or $-\gamma/\delta$ is the square of a rational number, and (54) can be reduced to the form

$$xy(ax^2+by^2)=zw(z^2-w^2). \tag{56}$$

The surface (56) contains the skew non-associate lines

$$x=z=0, \quad y=z+w=0. \tag{57}$$

Moreover, from § 26, we see that (56) contains two lines of type I, and no line of type II, incident with both lines (57). Hence, from § 10, we deduce that

The Diophantine equation (56) *has a rational parametric solution of degree* 9, *obtainable* (*in the way indicated in* § 11) *as the transform of one of the lines* (57), *by the T-transformation defined by these lines.*

30. Another noteworthy type of equation (54) is given by

$$xy(x^2+y^2)=kzw(z^2+w^2), \tag{58}$$

where k is a non-zero rational number. If (58) has a non-trivial solution (x, y, z, w), then

$$xyzw\neq 0, \quad xz+yw\neq 0, \quad xw-yz\neq 0.$$

We can therefore define the non-zero rational numbers

$$\lambda=\frac{xw-yz}{xz+yw}, \quad \mu=\frac{xz+yw}{z^2+w^2}, \quad \nu=\frac{z}{w}.$$

From these equations we deduce

$$x:y:z:w=(\lambda+\nu)\mu:(1-\lambda\nu)\mu:\nu:1, \tag{59}$$

whence, on substituting in (58) and suppressing the non-zero factor $\nu(1+\nu^2)w^4$, we obtain

$$k=\mu^4(\lambda+\nu)(1-\lambda\nu)(1+\lambda^2)/\nu. \tag{60}$$

Hence

If the Diophantine equation (58) *has non-trivial solutions, then* k *can be expressed in the form* (60), *where* λ, μ, ν *are non-zero rational numbers. Conversely, when* $k\neq 0$ *is given by* (60), *the equation* (58) *has the non-trivial solution* (59).

We notice the particular cases arising from the second part of this theorem, on assuming

$$\lambda=1, \mu=\nu, \quad \text{or} \quad \mu=\nu=1.$$

The corresponding values of k given by (60) are then

$$k = 2\mu^3(1-\mu^2) \quad \text{and} \quad k = 1-\lambda^4,$$

while the respective solutions of (58) are

$$x:y:z:w = \mu+\mu^2:\mu-\mu^2:\mu:1 \quad \text{and} \quad x:y:z:w = 1+\lambda:1-\lambda:1:1.$$

We add that, *from the solution* (59) *of the equation* (58), *where* k *is given by* (60), *other rational solutions can generally be deduced, by means of the methods of* §§ 6, 22.

31. If the surface (55) contains a rational line in addition to $X = Z = 0$, this line must be either of type I or of type II (§ 26). In the second case it is easily seen from § 26 that the two rational lines of the surface lie in a plane meeting the surface again in two irrational lines, so that nothing can be deduced about the further rational solutions of (55).

The only interesting case is the remaining one, *i.e.* that of an equation (55) reducible to the form

$$x(x^3+y^3) = kz(z^3+w^3). \tag{61}$$

The surface (61) contains the skew non-associate lines

$$x = z = 0, \quad x+y = z+w = 0.$$

Moreover, from § 26, we see that (61) contains two lines of type I, and four lines of type II, incident with both these lines. Hence, from § 10, we deduce that (61) *has a rational parametric solution of the fifth degree*. A simple calculation gives this solution explicitly, in the form

$$x:y:z:w = kt:t^5:t^4:k,$$

where t is a parameter.

32. We now proceed to study the Diophantine equation (52). We can suppose this not to be of the type (51), already investigated. Then the only rational lines lying on (52) are those of type II, and the determination of these lines is tantamount to obtaining the rational linear substitutions on x, y which transform the binary quartic form $\phi(x, y)$ into itself (§ 26).

From E.A, § 23, we deduce that

The surface (52) *contains four rational lines of type II if, and only if,* (52) *is rationally equivalent to either*

$$\alpha(X^4+Y^4)+\beta X^2Y^2+\gamma XY(X^2+Y^2)$$
$$= \alpha(Z^4+W^4)+\beta Z^2W^2+\gamma ZW(Z^2+W^2) \tag{62}$$

or

$$\alpha(x^4+y^4)+\beta x^2y^2+\gamma xy(x^2-y^2)=\alpha(z^4+w^4)+\beta z^2w^2+\gamma zw(z^2-w^2), \quad (63)$$

where α, β, γ are rational. The surface contains eight rational lines of type II if, and only if, (52) *is rationally equivalent to*

$$\alpha(x^4+y^4)+\beta x^2y^2=\alpha(z^4+w^4)+\beta z^2w^2. \quad (64)$$

In every other case, (52) *contains two rational lines only of type II, given by* (53).

Since the lines (53) are associate (§ 27), they are of no help in studying the general equation (52). We shall therefore restrict ourselves to the equations (62), (63), (64). The necessary and sufficient conditions for the reducibility of (52) to one of these forms can be found in E.A., § 23,

33. The four rational lines lying on (62) are

$$X+Z=Y+W=0, \qquad X+W=Y+Z=0,$$
$$X-Z=Y-W=0, \qquad X-W=Y-Z=0.$$

They are the sides of a skew quadrilateral, having the points

$$(1,-1,-1,1), \quad (1,1,-1,-1), \quad (1,-1,1,-1), \quad (1,1,1,1)$$

as consecutive vertices, and constitute two pairs of associate lines (§ 27). It follows from § 22 that we can *reduce* (62) *to the form* (37) by introducing new variables

$$x:y:z:w$$
$$=X+Y+Z+W:X+Y-Z-W:X-Y+Z-W:X-Y-Z+W.$$

We perform this calculation in two steps, by introducing first the variables

$$x_1:y_1:z_1:w_1=X+Y:X-Y:Z+W:Z-W.$$

Thus (62) becomes

$$\alpha_1x_1^4+\beta_1x_1^2y_1^2+\gamma_1y_1^4=\alpha_1z_1^4+\beta_1z_1^2w_1^2+\gamma_1w_1^4, \quad (65)$$

where

$$\alpha_1=2\alpha+\beta+2\gamma, \quad \beta_1=12\alpha-2\beta, \quad \gamma_1=2\alpha+\beta-2\gamma.$$

Next we put

$$x:y:z:w=x_1+z_1:x_1-z_1:y_1+w_1:y_1-w_1.$$

Thus (65) reduces in fact to the form (37), where

$$a=b=2\alpha_1, \quad a'=b'=c=d=\beta_1, \quad c'=d'=2\gamma_1.$$

From the final remark of § 22 we see that this equation has a rational para metric solution of the second degree if $ac \neq 0$ and, moreover,

$$-c/a = -\beta_1/2\alpha_1$$

is the sum of the squares of two rational numbers. This condition is satisfied if it holds for either $-\alpha_1\beta_1/2$ or $-\alpha_1\beta_1$*. Hence

The Diophantine equation (65) *has a rational parametric solution of the second degree if either* $-\alpha_1\beta_1$ *or* $-\gamma_1\beta_1$ *is different from zero and equal to the sum of the squares of two rational numbers. The Diophantine equation* (62) *has a rational parametric solution of the second degree if either of*

$$(2\alpha+\beta\pm 2\gamma)(\beta-6\alpha)$$

is different from zero and equal to the sum of the squares of two rational numbers†.

Moreover, on applying the result of § 23, we see that

If $\alpha_1+\beta_1+\gamma_1 = 0$ *or* $\alpha = 0$, *the equation* (65) *or* (62) *has a rational parametric solution of the seventh degree.*

We add that, without subjecting the coefficients in (62), (65) to any restriction,

When a non-trivial solution of (62) *or* (65) *is known, we can generally deduce other non-trivial solutions of the same equation by a proper application of the processes given in* §§ 6, 22.

It is obvious that (37) reduces to (58) on assuming

$$a = b = 2, \quad a' = b' = c = d = 0, \quad c' = d' = -2k.$$

Hence, from the above transformation of (65) into an equation of the form (37), we see that

The results established in § 30 *about the equation* (58) *give immediately similar results for the equation*‡

$$x_1^4 - ky_1^4 = z_1^4 - kw_1^4. \tag{66}$$

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* This follows from the identity $(u+v)^2+(u-v)^2 = 2(u^2+v^2)$.

† The second part of this theorem has already been obtained, in a slightly different and incomplete form, by A. Cunningham, *Math. Quest., Educ. Times* (2), 19 (1911), 27.

‡ In particular, it follows that, for $k = 2\mu^3(1-\mu^2)$, the equation (66) has the non-trivial solution $x_1 = 2\mu$, $y_1 = \mu+1$, $z_1 = 2\mu^2$, $w_1 = \mu-1$. This has already been noticed by A. Gérardin, *Sphinx-Oedipe*, 8 (1913), 13.

34. The four rational lines lying on (63) are

$$x+z=y+w=0, \quad x-z=y-w=0,$$

$$x+w=y-z=0, \quad x-w=y+z=0.$$

Since these are four generators of the same system of the quadric

$$x^2+y^2=z^2+w^2,$$

the result of § 20 can be applied to the equation (63). The following more precise statement can be proved, in a way similar to that indicated in § 25 for Euler's equation [included in (63) for $\beta=\gamma=0$].

The equation (63), *where* α, β, γ *are arbitrary rational numbers, has a rational parametric solution of the seventh degree*.*

35. The equation (64) includes Euler's equation for $\beta=0$, and contains the same eight rational lines as this surface (§ 25). Hence

The equation (64), *where* α, β *are arbitrary rational numbers, has a discontinuous infinity of rational parametric solutions, obtainable as indicated in* § 24.

36. In this paper we have chiefly been concerned with the Diophantine equations of quartic surfaces containing rational lines. The research could be extended to quartic surfaces containing other rational unicursal curves of given type, *e.g.* conics, cubics, etc.; but in general the extension would be far from simple. In § 38, as a particularly interesting example, we shall determine all the rational non-singular surfaces

$$ax^4+by^4+cz^4+dw^4=0, \tag{67}$$

which contain rational irreducible conics†.

It is obvious that any equation (67) is of the type (42) *in three different ways.* We shall now prove the stronger converse:

If the equation of a non-singular rational quartic F *can be written in the form* (42) *in at least two non-equivalent ways, which possibly involve two different systems of rational coordinates, then the equation of* F *can be reduced to the form* (67) *by a proper choice of rational coordinates.*

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* Only Euler's equation and another special case (in which $\alpha=1$, $\beta=6$, $\gamma=0$) have been previously considered. About the latter case, *cf.* A. Gérardin, *Sphinx-Oedipe*, 5 (1910), 187.

† The condition for (67) to be non-singular is clearly $abcd \neq 0$.

The Hessian of the quaternary form $\phi(x, y)-\psi(z, w)$ is obviously given by the product of the Hessians of the binary forms $\phi(x, y)$ and $\psi(z, w)$. Hence the Hessian of (42) consists of eight planes, four of which project the Hessian of (44) from the rational line $x = y = 0$, while the other four project the Hessian of (45) from the rational line $z = w = 0$. It follows from the hypotheses made for F that the Hessian H of F consists of eight planes, which can be distributed in two different ways into two quadruplets of planes, passing through two skew rational lines. Thus we obtain two pairs of lines, which obviously must be two pairs of opposite edges of a tetrahedron, having rational vertices, whose faces, taken with proper multiplicities, constitute H. We shall prove that H consists of the faces of this tetrahedron, each counted twice; and that, on taking this tetrahedron as fundamental in a new rational system of coordinates, the equation of F reduces to the type (67).

It is first of all not difficult to show that, if $\phi(x, y)$ is a binary quartic form of non-vanishing discriminant, such that the discriminant of its Hessian vanishes, then $\phi(x, y)$ is harmonic, and the Hessian of $\phi(x, y)$ is proportional to the square of a quadratic form of non-vanishing discriminant. This form can be reduced rationally to kxy, if it has rational roots, and then $\phi(x, y)$ assumes the form ax^4+by^4. The results to be proved now follow at once, since both the quartics ϕ and ψ, appearing in one of the representations of F of the type (42), are reduced to this form when the coordinates are chosen as indicated.

37. If we permute the variables x, y, z, w arbitrarily in (67), and at the same time alter these variables by arbitrary rational non-zero constant factors, we deduce a new equation, again of the form (67). Any two equations of this type, thus deducible one from the other, will be called *equivalent*. From § 36, this definition agrees with that of § 26.

The equation (67) becomes

$$x_1^4+x_2^4+x_3^4+x_4^4 = 0, \tag{68}$$

on putting

$$x_1 : x_2 : x_3 : x_4 = a^{\frac{1}{4}}x : b^{\frac{1}{4}}y : c^{\frac{1}{4}}z : d^{\frac{1}{4}}w, \tag{69}$$

with arbitrary choice of the quartic roots. In the complex domain *every* homographic transformation of (68) into itself can be represented by permuting the x's and multiplying them by arbitrary quartic roots of unity (*cf.* Q.S., § 1). Hence, from (69), we deduce without difficulty *all the rational homographic transformations of* (67) *into itself*. For shortness we

do not state the result explicitly. This would require five different cases, according as the pairs of coefficients a, b, c, d, having as ratio the fourth power of a rational number, are 0, 1, 2, 3, or 6 in number.

From (69) and Q.S., § 2 (or, alternatively, using § 26), we see that

The only rational non-singular surfaces (67) *having rational lines are those equivalent to the surfaces* (66). *The number of rational lines* (*i.e. of distinct trivial solutions*) *is four or eight, according as* $|k|$ *is not or is the fourth power of a rational number.*

38. From Q.S., §§ 11–13, we know that the irreducible conics lying on the surface (68) all belong to the following two categories, which include 192 and 128 conics respectively.

(i) The conic of equations

$$x_3+x_4=\tau_1 x_1+\tau_2 x_2, \quad 2x_3^2+3x_3x_4+2x_4^2=\tau_1\tau_2 x_1 x_2, \tag{70}$$

where τ_1, τ_2 are quartic roots of -1, and its transforms by the homographies changing (68) into itself (§ 37).

(ii) The conic of equations

$$x_3=x_1+x_2, \quad x_4^2=\pm i\sqrt{2}(x_1^2+x_1x_2+x_2^2), \tag{71}$$

and its transforms by the homographies changing (68) into itself.

Let us transform (68) into (67), by means of (69). The first of the equations (70) represents a rational plane in coordinates (x, y, z, w) if, and only if, (67) is equivalent to Euler's equation

$$x^4+y^4=z^4+w^4. \tag{72}$$

From (70) we see that (72) contains in fact *eight rational conics of the first kind*, given by

$$x+y=z+w, \quad x^2+xy+y^2+z^2+zw+w^2=0, \tag{73}$$

and by the equations deducible from these by any change of the signs of the coordinates.

Likewise, the equation (67) must be equivalent to

$$x^4+y^4+z^4+d_1w^4=0,$$

with d_1 rational and different from zero, if the first of the equations (71) is to represent a rational plane in coordinates (x, y, z, w). Then the

second equation represents a rational cone if, and only if, $\sqrt{d_1}/i\sqrt{2}$ is rational, *i.e.* if $d_1 = -2h^2$, with h rational and different from zero. Thus we reduce (67) to the form

$$x^4+y^4+z^4 = 2h^2w^4. \tag{74}$$

This surface contains in fact *eight rational conics of the second kind*, given by

$$z = x+y, \quad hw^2 = \pm(x^2+xy+y^2), \tag{75}$$

and by the conics obtained on altering the signs of x and y arbitrarily*.

Hence *a surface* (67) *contains a rational irreducible conic if, and only if, it is equivalent to either* (72) *or* (74).

We notice now that the second of the equations (73) has no rational solutions. The second of the equations (75) has rational solutions if, and only if,

$$h = \pm(\alpha^2+\alpha\beta+\beta^2),$$

where α, β are rational numbers not both zero, since $h \neq 0$. With such a value for h, an elementary calculation gives the following parametric solution of (75):

$$\begin{aligned} x:y:z:w = &-\alpha t^2+2\beta t+(\alpha+\beta) \\ &:(\alpha+\beta)t^2+2\alpha t-\beta:\beta t^2+2(\alpha+\beta)t+\alpha:t^2+t+1. \end{aligned} \tag{76}$$

And it is well known that any other similar solution can be deduced from (76) on performing a rational projective substitution upon the parameter t. Hence

The only Diophantine equations (67) *having a rational parametric solution of the second degree are those equivalent to*

$$x^4+y^4+z^4 = 2(\alpha^2+\alpha\beta+\beta^2)^2w^4, \tag{77}$$

where α, β are arbitrary rational numbers, $\alpha = \beta = 0$ excepted. The equation (77) *has in fact the parametric solution* (76). *Moreover, all its rational parametric solutions of the second degree can be deduced from* (76) *by per-*

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* A permutation of x, y, z gives nothing new. For $z = x+y$ implies

$$x^2+xy+y^2 = x^2-xz+z^2,$$

and so the interchanging of y and z has the same effect as the altering of x in sign.

forming a rational projective substitution upon the parameter t, and possibly by altering in sign either or both x, y.*

Addendum.

The extension of the property given at the end of § 2, to the non-singular algebraic surfaces of order $n \geqslant 4$, results from the following theorem.

Let F be any algebraic surface of order $n \geqslant 4$, having no multiple points. Then it is impossible to define the coordinates of a point P of F as rational functions of the coordinates of two points M and N of F, in such a way that P describes the whole surface F when either M or N moves on F, the other remaining fixed in a general position.

Assume first that $n = 4$. We suppose that it is possible to define P as a function of M, N, of the type indicated by the theorem. Then, on fixing M in a general position upon F, we obtain a unirational transformation of F into itself, *i.e.* an algebraic $(\nu, 1)$-correspondence (where ν is an integer $\geqslant 1$) associating N and P. Thus the general point P of F corresponds to a set of ν points N; moreover, this set generates on F an involution I_ν, of order ν, when P describes F. Since F has a canonical curve of order zero, and all its genera are equal to one, it follows that I_ν is generated by a group G_ν, of order ν, of birational transformations of F into itself†. We distinguish two cases, according as G_ν varies or remains unaltered, when M now moves on F.

In the first case the transformations of G_ν describe a continuous system of birational transformations of F into itself. But this is impossible, since the existence of such a continuous system would imply that F has the arithmetic genus $p_a \leqslant 0$‡.

In the second case, G_ν, and therefore also I_ν, is independent of M, and we define a birational transformation of F into itself as follows. Let us

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* Rational parametric solutions of the second degree have already been obtained, in different forms and by several authors, for the equation $x^4+y^4+z^4 = 2w^4$. *Cf.* F. Ferrari, *L'interméd. des math.*, 16 (1909), 83; E. Miot, *ibid.*, 17 (1910), 214; A. Gérardin, *Assoc. Franç.*, 39 (1910), I, 44, and *Sphinx-Oedipe*, 5 (1910), 180; 6 (1911), 3; 8 (1913), 119; A. Martin, *Math. Magazine*, 2 (1910), 351; E. N. Barisien, *Nouv. Ann. Math.*, (4), 11 (1911), 280; R. D. Carmichael, *Diophantine analysis* (New York, 1915), 81. All such solutions are, of course, included in our result, for $\alpha = 1$, $\beta = 0$.

† *Cf.* F. Enriques, "Sulle trasformazioni razionali delle superficie di genere uno", *Rend. R. Acc. Bologna* (1909–10), 71–75; L. Godeaux, "Sur les involutions douées d'un nombre fini de points unis, appartenant à une surface algébrique", *Rend. R. Acc. Lincei*, (5) 23 (1914), 408–413.

‡ *Cf.* F. Enriques, "Sulle superficie algebriche che ammettono un gruppo continuo di trasformazioni birazionali in se stesse", *Rend. Circ. Mat. Palermo*, 20 (1905), 61.

fix two general positions M_1 and M_2 for M. Then we associate two positions P_1 and P_2 of P, when P_1 corresponds to M_1 and a set of I_ν and P_2 corresponds to M_2 and the same set of I_ν. P_1 and P_2 are related by a non-identical birational transformation of F into itself, which varies continuously when M_1 moves on F, since otherwise the point P previously defined as a function of M and N would be independent of M. But again this is impossible, since $p_a = 1$.

The contradictions arrived at prove the theorem for $n = 4$.

When the surface F is of order $n > 4$, and therefore of arithmetic genus $p_a > 1$, the theorem is easier to prove. For then every unirational transformation of F into itself is necessarily homographic (and so birational)*. Hence, from Enriques' result last quoted, F can admit only a finite number of such transformations, whence follows the impossibility of representing P as a rational function of M and N of the type indicated by the theorem.

In conclusion, I wish to thank Prof. Mordell for his help in revising my manuscript.

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* *Cf.* F. Severi, "Sulle transformate razionali di un' ipersuperficie algebrica priva di punti multipli", *Math. Annalen*, 109 (1933), 4–6.

ON THE ISOPERIMETRIC RATIO FOR A HARMONIC SURFACE

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1. The object of this note is to establish for the isoperimetric ratio $I = A/L^2$ an upper estimate of the type

$$I = O(1/n) \quad \text{as} \quad n \to \infty, \tag{1.1}$$

in the case of a harmonic surface S of area A whose boundary has finite length L and whose diameter d does not exceed L/n. In other words we show that $A \leqslant KLd$, where K is an absolute constant. The surface S, which may be situated in space of a finite or infinite number of coordinates, is supposed represented on a simple plane domain—a domain bounded by a closed Jordan curve—by means of a vector function $x(u, v)$ whose components are harmonic functions.

We shall establish (1.1) in two stages: first we show that it may be deduced from the weaker result

$$I \leqslant K, \tag{1.2}$$

and secondly we give a proof of (1.2), valid in space of a finite or infinite number of coordinates, with a K somewhat smaller than $\frac{1}{4}$.

Morse and Tompkins (9) established a result of the type (1.2) for space of finite dimensionality r but only with a constant K which depends on r; they observed also that $I \neq O(n^{-2})$, and they raised the question of an estimate for I which tends to 0 as $n \to \infty$, a question of some importance in the Calculus of Variations. It has been pointed out to me by Prof. J. E. Littlewood that (1.1) is the best possible result of this type (*vide* below § 3) apart from the determination of a constant factor.

With regard to (1.2) a number of writers have occupied themselves with this, in some form or other, under assumptions which are usual in elementary analysis and which we are anxious to avoid here. It is probable that the best value of K is $1/(4\pi)$ and a proof of this result in three dimensions

has been promised by Shiffman (**10**). See also Carleman (**6**), Radó (**3**), Beckenbach and Radó (**5**), Lozinski (**8**).

2. We begin with some definitions and preliminary remarks. It will be clear from these that we could, without loss of generality, suppose the space finite-dimensional: we shall not do this, as it makes no serious difference to our arguments†.

We deal with vectors and scalars which are functions of the point (u, v) in the closure, or in the interior, of D. We make free use of the notation (r, θ) for (u, v), where $u+iv = re^{i\theta}$. When any of u, v, r, θ occur as suffixes they denote corresponding partial derivatives, whereas the suffix k denotes a k-th component of a vector.

We use the standard notation for scalar product and modulus. Thus if x and y are vectors, each with a finite or enumerably infinite sequence of components (x_k and y_k), we write

$$xy = \Sigma x_k y_k, \quad x^2 = xx, \quad |x| = \sqrt{(x^2)}.$$

We also combine two vectors by the complex notation: in particular‡

$$|x+iy| = \sqrt{(x^2+y^2)}.$$

Questions of convergence or of term by term integration of such expressions may be relevant: they will, however, be ignored when they can be disposed of by trivial appeals to

$$xy \leqslant |x| . |y| \tag{2.1}$$

or to the inequality $(2.1)_N$ obtained by replacing the left-hand side of (2.1) by the first N terms of $\Sigma x_k y_k$.

In dealing with vectors having only a finite number of components, we shall regard the vectors as unaltered by assigning to them an infinity of further components all of which vanish.

Given, in the closure of D, a curve C and a vector function $x(u, v)$, we write

$$T(C) = \int_C |dx|$$

† A slight simplification could also have been made by supposing that the functions concerned have uniformly continuous partial derivatives: this is permissible, by the familiar function-theory process of considering our functions in a slightly smaller region than the one given.

‡ By Minkowski's inequality with index 2, the modulus satisfies the triangle inequality and all its usual consequences. This, and a great deal of the present paper (originally intended for the *Journal*), has been added to meet the wishes of a referee.

for the total variation of the composite function obtained by expressing $x(u, v)$ in terms of the parameter t along C. We observe that

$$T(C) = \lim_N T_N(C), \tag{2.2}$$

where $T_N(C)$ is obtained by replacing, in the definition of $T(C)$, the vector x by one consisting of the first N components of x. In fact any number $a < T(C)$ is less than some finite sum of the type $V = \Sigma|\Delta x|$ of which $T(C)$ is the upper bound. For large N, the number a is thus also less than an expression V_N similarly formed for the N-dimensional vector function, and we have *a fortiori* $a < T_N(C)$. The relation (2.2) now follows if we make $a \to T(C)$ after observing that $T(C) \geqslant T_N(C)$.

A vector function $x(u, v)$ will be regarded as defining a harmonic surface S, if its components $x_k(u, v)$ are harmonic functions in D and are continuous in the closure of D. In most of the sequel we shall suppose, as we may without loss of generality, that D is the interior of the unit circle, and that the points on S which correspond to the three *cube roots of unity* divide the boundary of S into three parts of equal length. The justification of this reduction of the problem lies partly in the fact that a conformal transformation of D into the unit circle does not disturb the harmonic nature of $x(u, v)$, and partly in the fact that the definitions of area and of length that we shall give satisfy certain simple requirements: invariance of both area and length on S under a conformal transformation of D, and continuity of the length of arc along the boundary. (This last to secure trisection of L.)

In addition we shall suppose that $x(0, 0) = 0$; this is clearly no loss of generality.

If these additional hypotheses are made, then, since $x(0, 0) = 0$, the maximum modulus of $x(u, v)$ cannot exceed the maximum oscillation, *i.e.* the diameter $d \leqslant L/n$ of S; and moreover, from a well known result of Schwarz [(1) Ch. I, Th. 112, 117] it follows that

$$|x_u + ix_v| \leqslant 2d/(1-r). \tag{2.3}$$

In particular x_u and x_v are bounded in any interior concentric circle of the unit circle, and similarly therefore they have bounded derivatives in such a circle and are uniformly continuous in it.

A definition of the length of a curve on S corresponding to a curve C in the closure of D, which has the requisite invariance, is furnished by the total variation $\int_C |dx|$ of $x(u, v)$ on C. In particular the length of arc

$s(t)$ is defined on the boundary of S and is less than or equal to L. We still have to show that $s(t)$ is continuous.

To this effect, we observe that, by elementary considerations,

$$0 \leqslant s(t)+s_N(t) \leqslant L-L_N, \tag{2.4}$$

where $s_N(t)$ and L_N are the quantities corresponding to $s(t)$ and to L for the surface S_N defined by the first N components of $x(u, v)$. Now (2.2) clearly requires $L-L_N$ to tend to zero, so that by (2.4), $s(t)$ is the uniform limit of $s_N(t)$ and is therefore *continuous*. [We remark that this implies that the boundary values of $x(u, v)$, and therefore all its values, are continuous—but we shall not actually use this fact.]

It remains only to give a suitable definition of area. We shall do this in two stages.

Suppose first that S is situated in a finite-dimensional vector space. The Lebesgue-Fréchet area† then has the requisite invariance under conformal representation of D. If we choose D to be the unit circle it follows from semi-continuity of area, and from its increasing character when D is expanded, that the area A of S is the limit of the area of the part of S defined in a concentric smaller circle of the unit circle D. From the remarks following (2.3) the area of such a part of S is given by the classical double integral over the appropriate circle

$$\iint \left(x_\theta^2 x_r^2-(x_\theta x_r)^2\right)^{\frac{1}{2}} dr\, d\theta, \tag{2.5}$$

It follows by passing to the limit that A is given by (2.5) extended over the unit circle.

Let us observe further that, when the square of the integrand of (2.5) is written as a sum of squares of Jacobians, it is clear that it increases with the adjunction of new components to $x(u, v)$.

This being so, we now remove the restriction that S is situated in a finite dimensional space. A definition of the area A of S, invariant under conformal representation of D, is obtained by writing $A = \lim_N A_N$, where A_N is the area of the surface with N components only, these N components coinciding with the first N components of $x(u, v)$. We have just seen that A_N increases with N; in fact its integrand does so in (2.5), so that the limit certainly exists.

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† See the tract by T. Radó, "The Problem of Plateau", *Ergeb. d. Math.*, 2 (1933). The present results, and indeed more general ones, can also be based on new definitions of area and of length which I hope to publish in due course.

Taking limits under the integral sign in the case in which D is the unit circle (this reduction being now permissible) we see that the expression (2.5) by the theorem on integration of a monotone sequence, is a valid expression for the area A in the general case as soon as D has been transformed into the unit circle.

In addition to the facts so far established, we shall use the fact that, for certain elementary combinations of domains and boundaries, A and L have obvious additivity properties.

3. From now, D is the unit circle, $x(0, 0) = 0$ and L is trisected by the points corresponding to the cube roots of unity. By (2.3) the maximum modulus of $x_u + ix_v$ is at most $2Ln^{-\frac{1}{3}}$ in the circle $r \leqslant 1-n^{-\frac{2}{3}}$. In particular

(3.1) *The length of the curve on S which corresponds to a radius of the circle $u^2+v^2 \leqslant (1-n^{-\frac{2}{3}})^2$ is at most $2Ln^{-\frac{1}{3}}$.*

In this connection we shall also require to establish a simple variant for harmonic functions of an inequality for *analytic* functions, due to Carlson (7), in the same order of ideas as the classical theorem of Fejér and F. Riesz.

(3.2) *Let $x(u, v)$ be continuous in $u^2+v^2 \leqslant 1$ and harmonic for $u^2+v^2 < 1$. Let $\alpha(C, w)$ denote the sum of the elementary angles subtended by the elements of a curve C in $u^2+v^2 \leqslant 1$ at the point w of the circumference Γ $(u^2+v^2 = 1)$. Then*

$$\int_C |dx| \leqslant \frac{1}{\pi} \int_\Gamma \alpha(C, w)\,|dx(w)|. \tag{3.3}$$

The exact definition of $\alpha(C, w)$ is, of course, a total variation of an angle, one arm of which is fixed while the other joins w to the moveable point on C. From this definition we see that $\alpha(C, w)$ is the upper bound of $\alpha(P, w)$, where P is an inscribed polygon of C.

I say that if (3.3) were false for C it would be false also for some inscribed polygon P. For we could find a number a less than the left-hand side and greater than the right-hand side of (3.3).

By definition of total variation, there would be a finite sum of the type $\Sigma|\Delta x| > a$, and *a fortiori* we should have $\int_P |dx| > a$ for some inscribed polygon P.

On the other hand, since $\alpha(C, w) \geqslant \alpha(P, w)$, the number a exceeds $\frac{1}{\pi}\int_\Gamma \alpha(P, w)\,|dx(w)|$.

It is thus sufficient to prove (3.3) for a polygon P; we need therefore only treat the case in which C is a single segment, and we may suppose the latter parallel to the axis of u.

Writing $z=u+iv$ and $w-z=|w-z|e^{it}$, where $t=t(w, z)$, we have, by taking the real part in a formula of Carlson (7),

$$\frac{\partial x}{\partial u}=-\frac{1}{\pi}\int_\Gamma \frac{\sin t\, dx(w)}{|w-z|}. \tag{3.4}$$

This formula, valid for a *real* function $x(u, v)$, clearly remains valid for a vector function. Hence, since v is constant along C,

$$\int_C |dx| = \int_C |x_u|\, du \leqslant \frac{1}{\pi}\int_\Gamma a(w)\,|dx(w)|,$$

where
$$a(w)=\int_C \frac{|\sin t|\, du}{|w-z|}=a(C, w)$$

because the element du of C subtends at w the elementary angle $|\sin t|\, du/|w-z|$.

This completes the proof of (3.3).

(3.5) Corollary. *Let C be a radius whose extremity is the midpoint of a unit circle arc δ of length $n^{-\frac{1}{3}}$ such that $\int_\delta |dx| \leqslant Ln^{-\frac{1}{3}}$. Then*

$$\int_C |dx| \leqslant 4Ln^{-\frac{1}{3}}.$$

Denoting by C' the part of C for which $\sqrt{(u^2+v^2)} > 1-n^{-\frac{2}{3}}$, it is sufficient by (3.1) to show that $\int_{C'} |dx| \leqslant 2Ln^{-\frac{1}{3}}$. This, however, follows at once from (3.3) since we have $a(C', w)\leqslant\pi$ on the arc δ and† $a(C', w)\leqslant\pi n^{-\frac{1}{3}}$ on the complementary unit circle arc $\Gamma-\delta$.

4. We are now in a position to derive (1.1) from (1.2). We determine, on each of the three unit circle arcs determined by the cube roots of unity, an arc δ of length $n^{-\frac{1}{3}}$ such that $\int_\delta |dx| \leqslant Ln^{-\frac{1}{3}}$. This is evi-

† If w is in $\Gamma-\delta$, the distance from w to C' is $p \geqslant \sin \frac{1}{2}n^{-\frac{1}{3}}$. The triangle formed by w and C' has area m, where

$$\tfrac{1}{2}n^{-\frac{1}{3}}p \geqslant m \geqslant \tfrac{1}{2}p^2\, a(C', w).$$

Hence $a(C', w) \leqslant n^{-\frac{1}{3}}p^{-1} \leqslant n^{-\frac{1}{3}}\operatorname{cosec}\frac{1}{2}n^{-\frac{1}{3}}\pi n^{-\frac{1}{3}}$.

dently possible. Since the mid-points A, B, C of the three arcs thus obtained are separated by the cube roots of unity, and since the latter correspond to points of division of the boundary of S into three equal parts, it is clear that A, B, C correspond to points of division of the boundary of S into three parts L_1, L_2, L_3, of sum L, none of which can exceed $\frac{2}{3}L$. The greatest of these three parts therefore lies between $\frac{1}{3}L$ and $\frac{2}{3}L$. We denote by A and B the ends of the corresponding unit circle arc.

Then A and B divide the unit circumference into two arcs corresponding to lengths θL and $(1-\theta)L$, where $\frac{1}{3} \leqslant \theta \leqslant \frac{1}{2}$. Moreover, by (3.5), the radii with the extremities A and B correspond to curves on S of lengths at most $4Ln^{-\frac{1}{3}}$. It follows that S can be divided into two parts whose boundaries are of lengths not exceeding

$$(\theta+8n^{-\frac{1}{3}})L \quad \text{and} \quad (1-\theta+8n^{-\frac{1}{3}})L,$$

where $\frac{1}{3} \leqslant \theta \leqslant \frac{1}{2}$; this division being the one corresponding to the division of the unit circle by the pair of radii.

This being so, let us write $\phi(n) = n^{-1}\psi(n)$ for the upper bound of the isoperimetric ratio I for all S whose diameter does not exceed L/n. Clearly $L^2\phi(n)$ cannot exceed the upper bound in θ of the sum of the expression

$$L^2(\theta+8n^{-\frac{1}{3}})^2\phi[(\theta+8n^{-\frac{1}{3}})n]$$

with the corresponding one with $1-\theta$ instead of θ. Hence, writing h for $8n^{-\frac{1}{3}}$, we have

$$(4.1) \quad \psi(n) \leqslant \max_{\frac{1}{3}\leqslant\theta\leqslant\frac{1}{2}} \{(\theta+h)\psi[(\theta+h)n]+(1-\theta+h)\psi[(1-\theta+h)n]\}.$$

This is easily seen to imply that ψ is bounded. In fact, as remarked by a referee, it follows first that the majorant $\Psi(n) = \max_{m\leqslant n}\psi(m)$ of ψ is subject to

$$\Psi(n) \leqslant (1+2h)\Psi[(\tfrac{2}{3}+h)n],$$

and secondly, after an easy calculation, that for all large n a certain majorant Ψ^* of Ψ [*e.g.* $(1+40\sqrt{h})\Psi(n)$ for $h < \frac{1}{40}$, *i.e.* for $n > 320^3$] satisfies

$$\Psi^*(n) \leqslant \Psi^*[(\tfrac{2}{3}+h)n] \leqslant \Psi^*(\tfrac{3}{4}n)$$

and is therefore bounded. Thus (1.1) follows from (1.2) as asserted.

We may add that in (1.1) the O cannot be replaced by o. This follows at once, as was pointed out to me by Prof. J. E. Littlewood, by considering, for integral n, the surface $x = r^n\cos n\theta$, $y = r^n \sin n\theta$ represented on the unit circle $r \leqslant 1$. Similar considerations show that $\psi(nt) \geqslant \psi(t)$ when n is an integer.

5. It only remains now to prove (1.2). We shall prove first what is in effect a more general result, namely (8.1) below.

We shall change our notation, as we now use the Poisson integral. We write r, θ for the polar coordinates of the point (u, v) and

$$(5.1) \quad P(r, \theta) = (1-r^2)/\Delta = 1+\Phi_\theta, \quad Q(r, \theta) = -2r\sin\theta/\Delta = -r\Phi_r$$

for the Poisson kernels, where $1-re^{i\theta} = \Delta^{\frac{1}{2}} e^{\frac{1}{2}i\Phi}$ and $|\Phi| < \pi$ for $r\cos\theta \leqslant 1$.

We observe that P, Q are not of integrable square in the unit circle: for on the one hand the expression

$$P^2+Q^2+1 = 2(1+r^2)/\Delta = 2Pf(r)$$

say, has $L\pi f(r)$ for its integral in θ and is thus not integrable over the area of the unit circle; and on the other hand P^2 has $2\pi f(r)$ for its integral in θ, as we see at once from the useful formula

$$(5.2) \qquad \int P(r, \theta-\phi)\, P(\rho, \theta)\, d\theta = 2\pi P(r\rho, \phi),$$

which is included in the Poisson integral formula—since P is harmonic—or by appealing to the following relation, easily proved by recurrence,

$$(5.3) \qquad \int P^{n+1}(r, \theta) d\theta = 2\pi P_n\left(\frac{1+r^2}{1-r^2}\right),$$

which connects the Poisson kernel P with the Legendre polynomial P_n.

From these remarks about P and Q it is easy to see that, as observed in (9), there is no inequality of the type (1.2) when the area A is replaced by the Dirichlet integral: in fact the finiteness of L does not even imply that the partial derivatives of $x(u, v)$ are of integrable square. It is this fact that renders (1.2) non-trivial†.

6. It is convenient to write θ', θ'' for $\theta-\phi$, $\theta-\psi$ and g', g'' for $g(r, \theta')$, $g(r, \theta'')$, where g is any function of r, θ. We define further T, T^*, $\bar{T}$

† An inequality for the Dirichlet integral is provided by Green's theorem in the form $\iint (x_u^2+x_v^2)\, du\, dv \leqslant Ld^*$, where d^* is the diameter of the conjugate surface $y(u, v)$. For the left-hand side is

$$\iint \{(yx_u)_v - (yx_v)_u\}\, du\, dv = \int y(x_u\, du + x_v\, dv) = \int y\, dx.$$

But here d^*, as is well-known, need not be finite with L. See Zygmund (4).

by the relations

$$4\pi^2 rT = P'Q'' - P''Q' = \frac{2r}{\Delta'\Delta''}(1-r^2)(\sin\theta' - \sin\theta''),$$

$$4\pi^2 rT^* = r\frac{\partial(\Phi', \Phi'')}{\partial(r,\theta)} = (P'-1)Q'' - (P''-1)Q'$$
$$= \frac{8r^2}{\Delta'\Delta''}\sin\left(\frac{\theta'-\theta''}{2}\right)\left[\cos\frac{\theta'-\theta''}{2} - r\cos\frac{\theta'+\theta''}{2}\right],$$

$$4\pi^2 r\bar{T} = P'Q'' - P''Q' + \frac{1-r^2}{1+r^2}(Q'-Q'')$$
$$= \frac{4r^2}{\Delta'\Delta''}\frac{1-r^2}{1+r^2}\sin(\theta'-\theta'') = \frac{4r^2 P'P''}{1-r^4}\sin(\theta'-\theta'').$$

We require the following lemma:

(6.1) *Let* $0 < |\phi-\psi| = 2nH < \pi$. *Then, in the unit circle,*

(*a*) $\iint |\bar{T}|\,dr\,d\theta = \left|\iint T\,dr\,d\theta\right| = \left|\iint T^*\,dr\,d\theta\right| = \frac{1}{2}-H$;

(*b*) $\iint |T^*|\,dr\,d\theta = \frac{1}{2}-H+H^2$; (*c*) $\iint |T|\,dr\,d\theta < \frac{1}{2}$.

Proof. (*a*) The first two equalities follow from the expressions for T, T^* and $\bar{T}$ in terms of P', Q'', P'', Q' and from the fact that $\bar{T}$ has constant sign. Therefore we need only evaluate one of the double integrals. It is simplest to deal with that of $\bar{T}$ using (5.2): if we put $t = r^2$ after integrating in θ, it becomes

$$\int Q(t, \phi-\psi)\,dt/(2\pi t) = -\Phi(1, \phi-\psi)/(2\pi)$$

which has the value stated since $|\Phi(1,\theta)| = \pi - |\theta|$ for $|\theta| < \pi$.

(*b*) Alternatively we may complete the proof of (*a*), and at the same time prove (*b*), by evaluating the area mapped by Φ', Φ'' on the unit circle. This area is found to consist of a pair of right-angled triangles with a common vertex, and their difference is the double integral of the Jacobian $4\pi^2 T^*$, while their sum is the double integral of the modulus.

(*c*) We deduce this from (*a*). We may suppose $\psi = -\phi = \pi H$ and we then note that the sign of T is $+$ in the semicircle (I) $|\theta| < \frac{1}{2}$ and $-$ in the complementary semicircle (II); while T^* has constant sign on each side of the chord joining $e^{i\phi}$, $e^{i\psi}$ and is thus positive in (II). We can therefore write

$$\iint |T|\,dr\,d\theta = \iint T\,dr\,d\theta - 2\iint_{\text{II}} T^*\,dr\,d\theta + 2\iint_{\text{II}} dr\,d\theta\,\frac{Q'-Q''}{4\pi^2 r},$$

and the last integral is easy to evaluate, since the integral in r of $(Q'-Q'')/r$ has the value $(\Phi''-\Phi')_{r=1}$ which is constant and equal to $2\pi H$ on the arc of unit circle in (II). Hence

$$\iint |T|\,dr\,d\theta = \tfrac{1}{2} - H - 2\iint_{\mathrm{II}} T^*\,dr\,d\theta + H < \tfrac{1}{2}.$$

7. For convenience of notation, and in order to obtain a more general result, we introduce two vectors X and Y in place of x_θ and $-rx_r$, and we reduce our assumptions by supposing only that

(i) X is a vector with a finite or infinite number of components X_k each of which is a real harmonic function in the interior of the unit circle;

(ii) Y is the vector with components Y_k vanishing at the origin and conjugate to the X_k;

(iii) the expression $\int_0^{2\pi} |X(r,\theta)|d\theta$ has for $r<1$ a finite upper bound L^*.

In the special case in which X is x_θ, the conditions (i) and (ii) are clearly satisfied, and we have $Y = -rx_r$. Furthermore, applying (3.3) to a circle C of radius r concentric with the unit circle Γ, we see that

$$\int |x_\theta(r,\theta)|d\theta \leqslant 2\int_\Gamma |dx(w)| = 2L$$

and therefore that $L^* \leqslant 2L < \infty$. (Actually $L^* = L$, but we do not use this.) The condition (iii) is thus satisfied also in the special case.

It is easy to extend to our vectors X, Y the familiar Poisson-Stieltjes representation:

(7.1) *There exists a vector function $F(\theta)$, for $0 \leqslant \theta \leqslant 2\pi$, such that*

(*a*) $$F(\theta) \text{ has the components } F_k(\theta) = \lim_{r\to 1}\int_0^\theta X_k(r,\theta)\,d\theta;$$

(*b*) $$2\pi X = \int P(r,\theta-\phi)\,dF(\phi), \quad 2\pi Y = \int Q(r,\theta-\phi)\,dF(\phi);$$

(*c*) $$\int_0^{2\pi} |dF(\theta)| \leqslant L^*.$$

Actually there is equality in (*c*)—the opposite inequality being deducible from (*b*)—but we do not require this.

Proof of (7.1). (*a*) and (*b*) simply state that each component of X, and each component of Y, has the corresponding representation: this is

the case [(2) 187-189; note the last lines of §158] because, for each k, the expression $\int |X_k(r, \theta)|\, d\theta$ is certainly bounded by L^* in the circle $r < 1$.

To prove (c), we observe that, as r describes any sequence tending to 1, we have for any interval (a, b)

$$|F(b)-F(a)|^2 = \sum_k \left(\lim_r \int_a^b X_k(r, \theta)\, d\theta\right)^2 = \sum_k \lim_r \left(\int_a^b X_k\, d\theta\right)^2,$$

which by Fatou's inequality for sums [(1) Intr. §4, Th. 11] is

$$\leqslant \liminf_r \sum_k \left(\int_a^b X_k\, d\theta\right)^2 = \liminf_r \left(\int_a^b X\, d\theta\right)^2,$$

because the vector $\int X\, d\theta$ has the components $\int X_k\, d\theta$. Hence, remembering that $\left(\int X\, d\theta\right)^2 \leqslant \left(\int |X|\, d\theta\right)^2$ by the usual application of (2.1), we deduce that

$$|F(b)-F(a)| \leqslant \liminf_r \int_a^b |X|\, d\theta,$$

and from this it follows, by addition and by forming an upper bound, that

$$\int |dF| \leqslant \liminf_r \int_0^{2\pi} |X|\, d\theta \leqslant L^*.$$

This completes the proof.

We shall use the following variant of (7.1):

(7.2) *There exists for* $0 \leqslant \theta \leqslant 2\pi$ *an increasing* $s(\theta)$, *where* $0 \leqslant s \leqslant L^*$, *and a vector* $f(\theta)$, *where* $|f| \leqslant 1$, *such that*

$$2\pi X = \int_0^{2\pi} P(r, \theta-\phi)\, f(\phi)\, ds(\phi), \quad 2\pi Y = \int_0^{2\pi} Q(r, \theta-\phi)\, f(\phi)\, ds(\phi).$$

Proof. Let $s(\theta)$ be the total variation of $F(\theta)$ in $(0, \theta)$. Since the components $F_k(\theta)$ satisfy a Lipschitz condition in $s(\theta)$, they have, outside an $s(\theta)$-length zero, bounded network derivatives $f_k(\theta)$ with respect to $s(\theta)$, and are the integrals of these derivatives with respect to $s(\theta)$. This is still the case if we replace *all* the $f_k(\theta)$ by 0 at a point where *at least one* does not exist, for such alterations affect only an $s(\theta)$-length zero.

If (a, b) is the typical interval of the network tending to any θ outside this negligible set, we obtain

$$\Sigma f_k^2 = \Sigma \lim \left(\frac{F_k(b)-F_k(a)}{s(b)-s(a)}\right)^2$$

$$\leqslant \liminf \Sigma \left(\frac{F_k(b)-F_k(a)}{s(b)-s(a)}\right)^2 = \liminf \left(\frac{|F(b)-F(a)|}{s(b)-s(a)}\right)^2.$$

Since this limit does not exceed unity, we have $|f| \leqslant 1$, where f is the vector with the components f_k. Moreover we have, for every interval (a, b), $F(b)-F(a) = \int_a^b f\,ds$, and so, by addition,

$$\int g\,dF = \int gf\,ds,$$

where g is any real *step function*. This last formula now clearly extends to a *continuous* g, and we may therefore replace, in (7.1) (b), dF by $f\,ds$. This proves (7.2).

8. We are now in a position to establish the following theorem, which is an extension of (1.2) to the more general vector functions X, Y:

(8.1) *If* $0 < L^* < \infty$, *then*

$$\iint \frac{dr\,d\theta}{r} \{X^2\,Y^2-(XY)^2\}^{\frac{1}{2}} < \tfrac{1}{4}L^{*2}.$$

Proof. We write

$$rD_{mn} = X_m\,Y_n-X_n\,Y_m; \quad J_{mn}(\phi, \psi) = f_m(\phi)\,f_n(\psi)-f_n(\phi)f_m(\psi),$$

so that

$$(8.2) \quad r^2 \sum_{m>n} (D_{mn})^2 = X^2\,Y^2-(XY)^2; \quad \sum_{m>n} (J_{mn})^2 \leqslant f^2(\phi)\,f^2(\psi) \leqslant 1;$$

and the series concerned are convergent. Also, by (7.2), remembering that T changes sign when ϕ, ψ are interchanged,

$$(8.3) \qquad D_{mn} = \tfrac{1}{2}\iint T(r, \theta, \phi, \psi)\,J_{mn}(\phi, \psi)\,ds(\phi)\,ds(\psi).$$

We square the right-hand side, write the result in the form of a quadruple integral, and increase it by taking absolute values inside the integral. We may then sum with respect to m, n for $m > n$ under the integral sign, to obtain

$$\sum_{m>n} (D_{mn})^2 \leqslant \tfrac{1}{4}\iiiint |T(r, \theta, \phi, \psi)\,T(r, \theta, \phi', \psi')|\,R\,ds(\phi)\,ds(\psi)\,ds(\phi')\,ds(\psi'),$$

where $R = \Sigma |J_{mn}(\phi, \psi)J_{mn}(\phi', \psi')| \leqslant 1$ by Hölder's inequality with index 2 on account of (8.2). Thus

$$\sum_{m>n} (D_{mn})^2 \leqslant \left\{\tfrac{1}{2}\iint |T(r, \theta, \phi, \psi)|\,ds(\phi)\,ds(\psi)\right\}^2,$$

and after passing to the square root we obtain the desired result by integrating in r, θ and by using (6.1) (c) and (7.2) in conjunction with Fubini's theorem.

9. In the special case $X = x_\theta$, we have seen that $L^* \leqslant 2L$, but actually (7.1)(c) shows that $L^* \leqslant L$, since, by (7.1)(a), $F(\theta) = x(1, \theta) - x(1, 0)$. On the other hand, the double integral in (8.1) reduces to the classical expression, given by (2.5), for the area A of S. Thus (8.1) states that

$$A < KL^2,$$

where $K = \frac{1}{4}$. It is, however, easy to reduce this value of K; for we can arrange, by the argument of the beginning of § 4, that the unit circumference contains two arcs AB and DE, separated by angles $\frac{1}{3}\pi$, such that the portions of the boundary of S corresponding to AB and DE have lengths not less than $\frac{1}{3}L$ and equal to $\frac{1}{3}L$ respectively. Replacing, as we may, T by $\bar{T}$ in the proof of (8.1) since X now has mean value 0, and using at the end (6.1) (a) in place of (6.1) (c), we get a smaller value for K because $H \geqslant \frac{1}{6}$ in a substantial part of the lengths of arc $s(\phi)$, $s(\psi)$.

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ON SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

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1. There is a theorem of Khintchine's* which establishes a correspondence between the simultaneous approximations by rationals to a *set* of real numbers $(\theta_1, \ldots, \theta_n)$ and the approximations to zero by a homogeneous linear *form*

$$f = x_1\theta_1 + \ldots + x_n\theta_n - \xi. \tag{1}$$

The theorem is stated and discussed in Koksma's monograph†, and we restate it here as

THEOREM 1 (Khintchine). *Let any set of real numbers $(\theta_1, \ldots, \theta_n)$ be given, and let $r = \sqrt{(x_1^2 + \ldots + x_n^2)}$. Let β_1 be the upper bound of all indices β such that, for every positive ϵ, the inequality*

$$|f| \leqslant \epsilon r^{-n-\beta} \tag{2}$$

is soluble in integers $x_1, \ldots, x_n, \xi$, with $r \neq 0$. Let β_2 be the upper bound of all indices β such that, for every positive ϵ, the inequalities

$$|y\theta_i - \eta_i| \leqslant \epsilon y^{-(1+\beta)/n} \quad (i = 1, \ldots, n) \tag{3}$$

are soluble in integers $y, \eta_1, \ldots, \eta_n$, with $y \neq 0$. Then

$$\beta_1 \geqslant \beta_2 \geqslant \frac{\beta_1}{n^2 + n\beta_1 - \beta_1}; \tag{4}$$

and if $\beta_1 = \beta_2$ and β_2 is an attained upper bound, then β_1 is also attained; and if $\beta_2 = \beta_1/(n^2 + n\beta_1 - \beta_1)$ and β_1 is attained, then β_2 is also attained.

This theorem points to a certain type of symmetry between "set-approximations" and "form-approximations". The first aim of the

* A. Khintchine, *Rend. Circ. Mat. Palermo*, 50 (1926), 170–195, Theorem 6.

† J. F. Koksma, "Diophantische Approximationen", Vol. 4 of *Ergebnisse der Mathematik*, p. 66, Satz 1.

present paper is to generalize Theorem 1 in such a way that this symmetry becomes explicit, both in the statement of the results and in the proof; thus, in particular, the two inequalities (4) will no longer have to be proved separately, but will appear as special cases of the same general formula. The second aim of the paper is to transfer the results of Theorem 1 from simultaneous approximations in which the errors are to be made uniformly small to simultaneous approximations in which only the product of the errors is to be made small. The first aim leads us to Theorems 3 and 4, and the second to Theorem 5. Finally, in Theorem 2, the main result of the paper, we have combined the two methods of generalization, and thus obtained a theorem which reduces to a minimum the complications of proof, and which includes all the other results as corollaries.

2. We introduce the following notations and definitions. θ will denote any $n \times m$ matrix whose elements are real numbers θ_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, m)$:

$$f_i = \sum_{j=1}^{m} \theta_{ij} x_j - \xi_i \quad (i = 1, \ldots, n): \tag{5}$$

$$g_j = \sum_{i=1}^{n} \theta_{ij} y_i - \eta_j \quad (j = 1, \ldots, m): \tag{6}$$

$(\alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_m)$ will denote any set of $m+n$ non-negative indices. We shall say that θ *admits the indices* (α_i, β_j) *horizontally* if, for all positive values of A and B, the inequalities

$$|Af_i|^{\beta_j} |Bx_j|^{\alpha_i} \leqslant 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, m) \tag{7}$$

are soluble in integers $(x_1, \ldots, x_m;\ \xi_1, \ldots, \xi_n)$ with not all the x_j zero. We make the convention that if an expression such as the left-hand side of (7) contains an indeterminate factor 0^0, then that factor is to be interpreted as unity. We say that θ *admits the indices* (α_i, β_j) *vertically* if, for all positive values of A and B, the inequalities

$$|Ay_i|^{\beta_j} |Bg_j|^{\alpha_i} \leqslant 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, m) \tag{8}$$

are soluble in integers $(y_1, \ldots, y_n;\ \eta_1, \ldots, \eta_m)$ with not all the y_i zero.

The lemma which follows is a straightforward generalization of a well-known result in Diophantine approximation*.

* The "well-known result" is the special case in which all the α_i are equal, and $\beta_j = 1$. See Koksma, *loc. cit.*, p. 5, Satz 1.

LEMMA. *If* $\sum_{i=1}^{n} \alpha_i < \sum_{j=1}^{m} \beta_j$, *then every* $n \times m$ *matrix* θ *admits the indices* (α_i, β_j) *horizontally.*

Let A and B be any two positive numbers. Since $\Sigma\alpha_i < \Sigma\beta_j$ and the α_i are non-negative, we can choose a number R so large that

$$B^{-m} R^{\Sigma\beta_j} > \prod_{i=1}^{n} (2 + A R^{\alpha_i}). \tag{9}$$

We consider all the sets of integers $(x_1, \ldots, x_m)$ such that $0 \leqslant x_j < B^{-1} R^{\beta_j}$ $(j = 1, \ldots, m)$. The number of such sets is at least $B^{-m} R^{\Sigma\beta_j}$. With each of these sets we associate the set of integers $(\xi_1, \ldots, \xi_n)$ such that $0 \leqslant f_i < 1$ $(i = 1, \ldots, n)$. We thus obtain a distribution of points, in number at least $B^{-m} R^{\Sigma\beta_j}$, with coordinates $(f_1, \ldots, f_n)$; and the points are all contained in a unit cell of n-dimensional space. We divide up this unit cell, by sections parallel to the coordinate primes, into congruent cuboids whose edge parallel to the i-th coordinate axis is $A^{-1} R^{-\alpha_i}$ $(i = 1, \ldots, n)$. The number of these cuboids will be at most $\prod_{i=1}^{n} (2 + A R^{\alpha_i})$. Hence, by the condition (9), there will be at least one cuboid of the subdivision which contains two points $(f_1, \ldots, f_n)$ corresponding to two different sets of integers $(x_1, \ldots, x_m)$. We subtract term by term the two sets $(x_1, \ldots, x_m)$ and their associated sets $(\xi_1, \ldots, \xi_n)$. We thus obtain m integers $x_1, \ldots, x_m$, not all zero, such that $|x_j| < B^{-1} R^{\beta_j}$ $(j = 1, \ldots, m)$, and n further integers $\xi_1, \ldots, \xi_n$, such that $|f_i| \leqslant A^{-1} R^{-\alpha_i}$ $(i = 1, \ldots, n)$. With these values of x_j and ξ_i,

$$|A f_i|^{\beta_j} |B x_j|^{\alpha_i} \leqslant 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, m),$$

and since A and B are arbitrary the lemma is proved.

3. In this section we state and prove our main theorem.

THEOREM 2. *Let* $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m)$ *be any set of non-negative indices not all zero; and let* ζ *satisfy the equation*

$$\sum_{\alpha_i > \zeta} (\alpha_i - \zeta) - \sum_{\beta_j > -\zeta} (\beta_j + \zeta) + \zeta = 0. \tag{10}$$

There will always be one value of ζ *satisfying* (10), *and, with trivial exceptions, not more than one. Let indices* (ρ_i, σ_j) *be defined as follows*:

$$\rho_i = \max(0, \alpha_i - \zeta)\ (i = 1, \ldots, n); \quad \sigma_j = \max(0, \beta_j + \zeta)\ (j = 1, \ldots, m). \tag{11}$$

Then if any $n \times m$ *matrix* θ *admits the indices* (α_i, β_j) *horizontally, it will admit the indices* (ρ_i, σ_j) *vertically; and if it admits the indices* (α_i, β_j) *vertically, it will admit the indices* (ρ_i, σ_j) *horizontally.*

The left-hand side of (10) is a continuous function of ζ; it takes the value $\Sigma a_i+(n-1)(-\zeta)$ when ζ is equal to minus the largest β_j, and the value $-(m-1)\zeta-\Sigma\beta_j$ when ζ is equal to the largest a_i; and it decreases strictly in the interval between these values. Hence it will take the value zero once, and only once, in this interval. Further, it is a non-increasing function of ζ outside the interval, and so will take the value zero only once altogether, except in the trivial cases when either all a_i or all β_j are zero.

The statement that θ admits the indices (a_i, β_j) horizontally is equivalent to the statement that the transposed matrix θ' admits the indices (β_j, a_i) vertically, and *vice versa*. Suppose now that we have proved the first half of the conclusion of Theorem 2 for θ'. Then we shall have proved the following statement. If θ admits the indices (a_i, β_j) vertically, then it will admit the indices (ρ_i', σ_j') horizontally, where ρ_i' and σ_j' are given by

$$(11')\quad \rho_i' = \max(0, a_i+\zeta')\ (i=1, \ldots, n);\quad \sigma_j' = \max(\beta_j-\zeta')\ (j=1, \ldots, m);$$

and ζ' satisfies the equation

$$(10')\qquad \sum_{\beta_j>\zeta'}(\beta_j-\zeta')-\sum_{a_i>-\zeta'}(a_i+\zeta')+\zeta'=0.$$

But (10′) will be satisfied by $\zeta'=-\zeta$, and then (11′) will give $\rho_i'=\rho_i$, $\sigma_j'=\sigma_j$. Hence we shall have proved the second half of the conclusion of Theorem 2 for θ. It is therefore sufficient, for a proof of Theorem 2, to prove the first half of the conclusion for a general matrix θ. This we do in the following paragraphs.

Let any two numbers $A\geqslant 1$, $B\geqslant 1$, be given. Then, by the hypothesis of the first half of our theorem, there exists a solution of the inequalities (7) in integers x_j not all zero. Suppose first that $\Sigma a_i<\Sigma\beta_j$. Then the left-hand side of (10) is negative for $\zeta=0$; and, since it is a non-increasing function of ζ, this implies that $\zeta<0$. By (10) and (11),

$$\Sigma\rho_i-\Sigma\sigma_j=-\zeta>0,$$

that is to say, $\Sigma\rho_i>\Sigma\sigma_j$. Hence we may apply the lemma to the transposed matrix θ', and conclude that θ' admits the indices (σ_j, ρ_i) horizontally. This, as explained earlier, is equivalent to the conclusion of the first half of our theorem. We may therefore suppose henceforth that

$$(12)\qquad \Sigma a_i\geqslant\Sigma\beta_j,\quad \Sigma\rho_i\leqslant\Sigma\sigma_j.$$

Since not all a_i and β_j are zero, there will be at least one a_i, a_1 say, which is strictly positive. Then, by (7), if j is such that $\beta_j=0$, $|Bx_j|^{a_1}\leqslant 1$

(here we use the convention regarding the interpretation of 0^0). In other words, for $j = 1, \ldots, m$ we have either $\beta_j > 0$ or $|Bx_j| \leqslant 1$. There will therefore exist a number $S \geqslant 1$ such that

$$B|x_j| \leqslant S^{\beta_j} \quad (j = 1, \ldots, m), \tag{14}$$

and since not all x_j vanish, we can choose S so that equality holds in (14) for some j. Without loss of generality, we suppose

$$B|x_m| = S^{\beta_m} \geqslant 1. \tag{15}$$

If $B > 1$, it follows from (15) that $\beta_m > 0$. In the proof of Theorem 2 we shall eventually take $B > 1$, but for application to Theorem 5 we need to include the case $B = 1$. We therefore make the new hypothesis:

$$\text{if} \quad B = 1 \quad \text{then} \quad \beta_m > 0, \tag{16}$$

so that $\beta_m > 0$ in any case. The new hypothesis is satisfied vacuously as far as Theorem 2 is concerned.

By (10) and (12), $\zeta = \Sigma\sigma_j - \Sigma\rho_i \geqslant 0$. Hence, by (10) again and (11),

$$\Sigma\rho_i = \Sigma\sigma_j - \zeta = \Sigma\beta_j + (m-1)\zeta \geqslant \beta_m > 0,$$

and so there is at least one i such that $\rho_i > 0$. We write $Y_i = (1+3^{m-1})S^{\rho_i}$ if $\rho_i > 0$, and $Y_i = 1$ if $\rho_i = 0$, and consider all the sets of integers $(y_1, \ldots, y_n)$ satisfying the conditions

$$0 \leqslant y_i < Y_i \quad (i = 1, \ldots, n). \tag{17}$$

The number of such sets is at least $\prod_{i=1}^{n} Y_i$, and since some $\rho_i > 0$ this number is not less than $(1+3^{m-1})S^{\Sigma\rho_i}$. With each of the sets we associate the set of integers $(\eta_1, \ldots, \eta_{m-1})$ such that $0 \leqslant g_j < 1$ $(j = 1, \ldots, m-1)$. We thus obtain a distribution of points in $(m-1)$-dimensional space with coordinates $(g_1, \ldots, g_{m-1})$; the points are in number at least $(1+3^{m-1})S^{\Sigma\rho_i}$, and all lie in a unit cell of the space. Now let $\delta = B^{-1/(m-1)}$ if $m > 1$, and $\delta = 1$ if $m = 1$; in any case $\delta \leqslant 1$. We divide up the unit cell, by sections parallel to the coordinate primes, into congruent cuboids whose edge parallel to the j-th coordinate axis is $\delta S^{-\sigma_j}$ $(j = 1, \ldots, m-1)$. The number of cuboids is at most $\prod_{j=1}^{m-1}(2+\delta^{-1}S^{\sigma_j})$. The average number of points in a cuboid is thus at least $a = (1+3^{m-1})S^{\Sigma\rho_i}\prod_{j=1}^{m-1}(2+\delta^{-1}S^{\sigma_j})^{-1}$. Now $\delta^{-1}S^{\sigma_i} \geqslant 1$, and so

$$a \geqslant (1+3^{m-1})S^{\Sigma\rho_i}\prod_{j=1}^{m-1}(3\delta^{-1}S^{\sigma_j})^{-1} > S^{\sum_1^n \rho_i - \sum_1^{m-1}\sigma_j}\delta^{m-1}.$$

Since $\zeta \geqslant 0$, we have, by (10),

$$\sum_1^n \rho_i - \sum_1^{m-1} \sigma_j = \sigma_m - \zeta = \beta_m,$$

and so finally

$$a > S^{\beta_m} \delta^{m-1} \geqslant S^{\beta_m} B^{-1} = |x_m|,$$

by (15). Therefore at least one cuboid will contain more than $|x_m|$ points, and at least two of these points will correspond to values of the y_i and η_j giving the same residue for

$$x_1\eta_1 + \ldots + x_{m-1}\eta_{m-1} - \xi_1 y_1 - \ldots - \xi_n y_n \pmod{x_m}.$$

We subtract term-by-term the two sets of integers $(y_1, \ldots, y_n)$ corresponding to the two points, and also the two associated sets $(\eta_1, \ldots, \eta_{m-1})$. We thus obtain n integers $(y_1, \ldots, y_n)$ not all zero, and $m-1$ further integers $(\eta_1, \ldots, \eta_{m-1})$, such that

$$|y_i| < Y_i \qquad (i = 1, \ldots, n), \tag{18}$$

$$|g_j| < \delta S^{-\sigma_j} \quad (j = 1, \ldots, m-1), \tag{19}$$

$$x_1\eta_1 + \ldots + x_{m-1}\eta_{m-1} - \xi_1 y_1 - \ldots - \xi_n y_n + x_m \eta_m = 0 \tag{20}$$

for some integer η_m.

Now we have identically

$$\sum_{j=1}^m x_j g_j - \sum_{i=1}^n y_i f_i = \sum_{i=1}^n y_i \xi_i - \sum_{j=1}^m x_j \eta_j.$$

Hence, by (20),

$$x_m g_m = \sum_{i=1}^n y_i f_i - \sum_{j=1}^{m-1} x_j g_j.$$

Now, by (7) and (15), since $\beta_m > 0$,

$$|f_i| \leqslant A^{-1} (Bx_m)^{-\alpha_i/\beta_m} = A^{-1} S^{-\alpha_i} \quad (i = 1, \ldots, n).$$

Hence, using (14) and (19),

$$|x_m g_m| \leqslant \sum_{i=1}^n A^{-1} |y_i| S^{-\alpha_i} + \sum_{j=1}^{m-1} \delta S^{\beta_j - \sigma_j} B^{-1}. \tag{21}$$

When $\rho_i > 0$, then

$$Y_i = (1 + 3^{m-1}) S^{\rho_i} = (1 + 3^{m-1}) S^{\alpha_i - \zeta}.$$

When $\rho_i = 0$, then $Y_i = 1$, and so $|y_i| = 0$. Therefore we may replace (18) by

$$|y_i| < (1+3^{m-1})\, S^{\alpha_i - \zeta} \quad (i = 1, \ldots, n). \tag{22}$$

Also
$$\beta_j - \sigma_j \leqslant -\zeta \quad (j = 1, \ldots, m-1).$$

Therefore (21) leads to the inequality

$$|x_m g_m| \leqslant n(1+3^{m-1})\, A^{-1} S^{-\zeta} + (m-1)\, \delta B^{-1} S^{-\zeta}.$$

Hence, since $\zeta \geqslant 0$ and $\sigma_m = \beta_m + \zeta$, we have, substituting for $|x_m|$ from (15),

$$|g_m| \leqslant \{n(1+3^{m-1})\, A^{-1} B + (m-1)\, \delta\}\, S^{-\sigma_m}. \tag{23}$$

Now A and B are arbitrary, and so we may take $A = n(1+3^{m-1})\, B^2$. We may then combine (19) and (23) into the inequalities

$$|g_j| \leqslant \{B^{-1} + (m-1)\, \delta\}\, S^{-\sigma_j} \quad (j = 1, \ldots, m). \tag{24}$$

Let H and K be any positive numbers. When i is such that $\rho_i > 0$, then by (18) and (24),

$$|Hy_i|^{\sigma_j} |Kg_j|^{\rho_i} \leqslant \{(1+3^{m-1})\, H\}^{\sigma_j} \{(B^{-1} + (m-1)\, \delta)\, K\}^{\rho_i},$$

and the right-hand side of this inequality is less than unity if B is chosen sufficiently great. When on the other hand $\rho_i = 0$, then, by (18), $|y_i| = 0$, and so

$$|Hy_i|^{\sigma_j} |Kg_j|^{\rho_i} = 0^{\sigma_j} |Kg_j|^0 \leqslant 1.$$

Therefore θ admits the indices (ρ_i, σ_j) vertically; and, by what was said earlier, this is sufficient to complete the proof of Theorem 2.

4. In the special case $m = 1$, the statement of Theorem 2 can be considerably simplified. Although the simplifications are obvious and trivial, it may be worth while to state the theorem explicitly in the simplified form, because the case $m = 1$ is in fact the most interesting case.

We say that a *set* of real numbers $(\theta_1, \ldots, \theta_n)$ is *approximable with indices* $(\alpha_1, \ldots, \alpha_n)$ if for every positive ϵ the inequalities

$$|q\theta_i - p_i| \leqslant \epsilon q^{-\alpha_i} \quad (i = 1, \ldots, n) \tag{25}$$

are soluble in integers $p_1, \ldots, p_n, q$, with $q \neq 0$. We say that a *form* $f = x_1\theta_1 + \ldots + x_n\theta_n - \xi$ is *approximable with indices* $(\alpha_1, \ldots, \alpha_n)$, if for every positive ϵ the inequalities

$$|x_i| \leqslant \epsilon |f|^{-\alpha_i} \quad (i = 1, \ldots, n) \tag{26}$$

are soluble in integers $x_1, \ldots, x_n, \xi$, with not all of $x_1, \ldots, x_n$ zero. With these definitions, the simplified form of Theorem 2 becomes the following

THEOREM 3. *Let $(a_1, \ldots, a_n)$ be any set of non-negative indices not all zero, and let α be defined by the equation*

$$\sum_{a_i > \alpha} (a_i - \alpha) = 1. \tag{27}$$

Then α is uniquely determined, and $\alpha + 1 > 0$. Let

$$\omega_i = \max\left(0, \frac{a_i - \alpha}{1+\alpha}\right) \quad (i = 1, \ldots, n). \tag{28}$$

Then, if the set $(\theta_1, \ldots, \theta_n)$ is approximable with indices $(a_1, \ldots, a_n)$, the form f is approximable with indices $(\omega_1, \ldots, \omega_n)$. And if the form f is approximable with indices $(a_1, \ldots, a_n)$, the set $(\theta_1, \ldots, \theta_n)$ is approximable with indices $(\omega_1, \ldots, \omega_n)$.

We leave to the reader the verification that this theorem is actually the same as Theorem 2 when $m = 1$. The only doubtful point is the precise connection between the values of ϵ in (25) and (26) and the values of A and B in (7) and (8); but it is easy to see that if both A and B are large then ϵ can be made small, and conversely.

5. We could at this stage deduce Theorem 1 without difficulty from Theorem 3. What we actually do is to make the deduction from Theorem 2 instead of Theorem 3, and we thus arrive at Theorem 4 instead of Theorem 1. We thereby preserve the symmetry of Theorem 2 between m and n, so that the length of the deduction is reduced by one-half. And Theorem 1 finally appears as the special case $m = 1$ of Theorem 4.

THEOREM 4. *Let any $n \times m$ matrix θ of real numbers be given, and let f_i and g_j be defined by (5) and (6). Let*

$$r = \sqrt{(x_1^2 + \ldots + x_m^2)}, \quad s = \sqrt{(y_1^2 + \ldots + y_n^2)}.$$

Let β_1 be the upper bound of all indices β such that, for every positive ϵ, the inequalities

$$|f_i| \leqslant \epsilon r^{-(m+\beta)/n} \quad (i = 1, \ldots, n) \tag{27}$$

are soluble in integers $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n$, with $r \neq 0$. Let β_2 be the upper bound of all indices β such that, for every positive ϵ, the inequalities

$$|g_j| \leqslant \epsilon s^{-(n+\beta)/m} \quad (j = 1, \ldots, m) \tag{28}$$

are soluble in integers $y_1, \ldots, y_n, \eta_1, \ldots, \eta_m$, *with* $s \neq 0$. *Then*

$$(29) \quad \beta_1 \geqslant \frac{m\beta_2}{n(m+n-1)+(n-1)\beta_2}; \quad \beta_2 \geqslant \frac{n\beta_1}{m(m+n-1)+(m-1)\beta_1};$$

and if equality holds in the former condition (29) *and* β_2 *is an attained upper bound, then* β_1 *is also attained; and if equality holds in the latter condition* (29) *and* β_1 *is attained, then* β_2 *is also attained.*

We suppose that β' is any index not exceeding β_2, equality being admitted only if β_2 is attained. Then the inequalities (28) will have a solution with $\beta = \beta'$, for every positive number ϵ. In the terminology of Theorem 2, this means that θ admits the indices (α_i, β_j) vertically, where

$$\alpha_i = 1 \ (i = 1, \ldots, n); \quad \beta_j = (n+\beta')/m \ (j = 1, \ldots, m).$$

Equation (10) is then satisfied by $\zeta = -\beta'/(m+n-1)$, and we find from (11)

$$\rho_i = 1 + \frac{\beta'}{m+n-1} \qquad (i = 1, \ldots, n),$$

$$\sigma_j = \frac{n}{m} + \frac{(n-1)\beta'}{m(m+n-1)} \quad (j = 1, \ldots, m).$$

By Theorem 2, θ admits the indices (ρ_i, σ_j) horizontally, and from this it follows that for every positive ϵ the inequalities

$$|f_i|^{(n/m)+(n-1)\beta'/\{m(m+n-1)\}} \leqslant \epsilon r^{-[1+\{\beta'/(m+n-1)\}]} \quad (i = 1, \ldots, n)$$

are soluble under the usual conditions. Therefore, for every positive ϵ, the inequalities (27) are soluble with

$$\frac{m+\beta}{n} = \frac{m(m+n-1+\beta')}{n(m+n-1)+(n-1)\beta'},$$

$$\beta = \frac{m\beta'}{n(m+n-1)+(n-1)\beta'}.$$

It follows from the definition of β_1 that

$$\beta_1 \geqslant \frac{m\beta'}{n(m+n-1)+(n-1)\beta'},$$

for all β' having the postulated relation to β_2. Hence the first inequality (29) is proved; and if equality holds in it and β_2 is attained, then β_1 is attained.

The other clauses of the theorem can now be proved by exactly the same argument, interchanging β_1 with β_2 and m with n. We have thus fully established Theorem 4 and, as a special case, Theorem 1.

6. THEOREM 5. *Let any $n \times m$ matrix θ with real elements be given, and let f_i and g_j be defined by* (5) *and* (6). *Let $\overline{x_j} = \max(1, |x_j|)$ $(j = 1, \ldots, m)$, and $\overline{y_i} = \max(1, |y_i|)$ $(i = 1, \ldots, n)$. Let*

$$F = \left(\prod_{i=1}^{n} |f_i| \right)^{1/n}, \quad G = \left(\prod_{j=1}^{m} |g_j| \right)^{1/m}, \quad S = \left(\prod_{j=1}^{m} \overline{x_j} \right)^{1/m}, \quad T = \left(\prod_{i=1}^{n} \overline{y_i} \right)^{1/n}.$$

Let π_1 be the upper bound of all indices π such that, for every positive ϵ, the inequality

$$F \leqslant \epsilon S^{-(m+\pi)/n} \tag{30}$$

is soluble in integers $x_1, \ldots, x_m, \xi_1, \ldots, \xi_n$, with not all x_j zero. Let π_2 be the upper bound of all indices π such that, for every positive ϵ, the inequality

$$G \leqslant \epsilon T^{-(n+\pi)/m} \tag{31}$$

is soluble in integers $y_1, \ldots, y_n, \eta_1, \ldots, \eta_m$, with not all y_i zero. Then

$$\pi_1 \geqslant \frac{m\pi_2}{n(m+n-1)+(n-1)\pi_2}; \quad \pi_2 \geqslant \frac{n\pi_1}{m(m+n-1)+(m-1)\pi_1}; \tag{32}$$

and if equality holds in the former condition (32) *and π_2 is attained, then π_1 is attained; and if equality holds in the latter condition* (32) *and π_1 is attained, then π_2 is attained.*

We suppose that π' is any index not exceeding π_1, equality being admitted only if π_1 is attained. Then the inequality (30) will have solutions with $\pi = \pi'$, for every positive ϵ. Given ϵ, we can find a solution in which $|f_i| < 1$ $(i = 1, \ldots, n)$ and $S > 1$. (We make $S > 1$ if we derive our solution from a general solution with a smaller value of ϵ by doubling all the x_j and ξ_i; and we can make $|f_i| < 1$, by suitable choice of ξ_i, without disturbing the validity of (30).) We suppose henceforth that a value of ϵ is fixed in the range $0 < \epsilon \leqslant 1$, and that corresponding values of x_j and ξ_i are fixed so that (30) is satisfied with $\pi = \pi'$, $|f_i| < 1$, and $S > 1$. We can therefore find a set of non-negative indices $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ such that

$$\overline{x}_j = S^{\beta_j} \quad (j = 1, \ldots, m), \tag{33}$$

$$|f_i| \leqslant S^{-\alpha_i} \quad (i = 1, \ldots, n), \tag{34}$$

$$S^{-\Sigma\alpha_i} = \epsilon^n S^{-m-\pi'} = \epsilon^n S^{-\Sigma\beta_j - \pi'}. \tag{35}$$

Since $\Sigma\beta_j = m$, we may suppose without loss of generality that $\beta_m > 0$, $\overline{x}_m = |x_m|$.

We can now appeal to the argument used in the proof of Theorem 2, taking the special case in which $A = B = \delta = 1$. The conditions (7) are satisfied, and so are (14), (15) and (16). Further, since $S > 1$ and $\epsilon \leqslant 1$, we have from (35)

$$\Sigma \alpha_i - \Sigma \beta_j \geqslant \pi'. \tag{36}$$

We assume for the present that $\pi' \geqslant 0$. We define ζ, ρ_i, and σ_j by (10) and (11), and then condition (12) will be satisfied in virtue of (36). The whole argument of the proof of Theorem 2 can now be applied. We conclude that there exist integers $y_1, \ldots, y_n$, not all zero, and integers $\eta_1, \ldots, \eta_m$, such that (18), (19) and (23) are satisfied. Substituting the values $A = B = \delta = 1$ in these inequalities, and forming the products G and T, we find

$$T < \left(\prod_{i=1}^{n} Y_i \right)^{1/n} \leqslant (1+3^{m-1})\, S^{\Sigma \rho_i / n}, \tag{37}$$

$$G < (n+m-1+n3^{m-1})^{1/m}\, S^{-\Sigma \sigma_j / m}. \tag{38}$$

Thus finally

$$GT^{(n+\pi)/m} \leqslant D S^{\{(\Sigma \rho_i - \Sigma \sigma_j)/m\} + \pi \Sigma \rho_i / mn}, \tag{39}$$

where D is a number depending only on m, n and π.

By (10) and (12), $\Sigma \sigma_j - \Sigma \rho_i = \zeta \geqslant 0$; and so, by (11), $\sigma_j = \beta_j + \zeta$ $(j = 1, \ldots, m)$. Hence

$$\Sigma \sigma_j = \Sigma \beta_j + m\zeta = m + m\zeta, \tag{40}$$

$$\Sigma \rho_i = m + (m-1)\,\zeta, \quad \Sigma \alpha_i \leqslant \Sigma(\rho_i + \zeta) = m + (m+n-1)\,\zeta. \tag{41}$$

We take in the inequality (39)

$$\pi = \frac{n\pi'}{m(m+n-1) + (m-1)\,\pi'}. \tag{42}$$

Then by (39), (40) and (41),

$$GT^{(n+\pi)/m} \leqslant D S^{\{\pi' - (m+n-1)\zeta\}/\{m(m+n-1)+(m-1)\pi'\}} \leqslant D S^{(\pi' + m - \Sigma \alpha_i)/\{m(m+n-1)+(m-1)\pi'\}}.$$

Now by (35), $S^{\pi' + m - \Sigma \alpha_i} = \epsilon^n$. Hence

$$G \leqslant D \epsilon^{n/\{m(m+n-1)+(m-1)\pi'\}}\, T^{-(n+\pi)/m}, \tag{43}$$

and so (31) can be solved, for every positive ϵ, if π is given by (42). There remains the case in which $\pi' < 0$. In this case $\pi < 0$, and it is a simple corollary of the lemma we proved earlier that (31) will have solutions for

all positive ϵ. Therefore, in every case, we have by the definition of π_2

$$\pi_2 \geqslant \frac{n\pi'}{m(m+n-1)+(m-1)\pi'},$$

for every π' having the postulated relation to π_1. Hence the second inequality (32) is proved; and if equality holds in it and π_1 is attained, then π_2 is attained. The remaining clauses of Theorem 5 now follow by symmetry, as in Theorem 4, and the proof is thereby completed.

7. In the special case $m = 1$, the results of Theorem 5 are strictly analogous to those of Theorem 1, and thus we have achieved what was described in § 1 as the second aim of this paper. The particular interest of Theorem 5 is due to the fact, that when $m = 1$ and $n > 1$ it is not known whether it is ever the case that $\pi_1 = 0$ and is not attained. Theorem 5 shows that this will be the case if and only if $\pi_2 = 0$ and is not attained. Whether this fact simplifies the problem is, however, doubtful.

Khintchine left open the question whether Theorem 1 is a "best possible" result, but expressed the opinion that it probably is. It seems equally probable that the other theorems of this paper are best possible, in all cases when the results do not follow trivially from the lemma of § 2. For example, the results of Theorem 2 will be trivial one way and non-trivial the other if $\Sigma\alpha_i > \Sigma\beta_j$, and *vice versa* if $\Sigma\alpha_i < \Sigma\beta_j$; when $\Sigma\alpha_i = \Sigma\beta_j$ the results are non-trivial both ways. In the case $\Sigma\alpha_i = \Sigma\beta_j$ it is easy to see that the results must be best possible, because we then have $\rho_i = \alpha_i$ and $\sigma_j = \beta_j$; the two halves of the theorem thus become equivalent to the following statement. *If* $\Sigma\alpha_i = \Sigma\beta_j$, *then any matrix* θ *admitting the indices* (α_i, β_j) *horizontally will also admit them vertically, and conversely.*

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SOME IDENTITIES IN COMBINATORY ANALYSIS

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1. If we use the abbreviations

$$x_n = 1-x^n; \quad \bar{x}_n = 1+x^n; \quad x_n! = x_1 x_2 \dots x_n; \quad \bar{x}_n! = \bar{x}_1 \bar{x}_2 \dots \bar{x}_n;$$

$$x_n{}^2 = x_{2n}; \quad x_n{}^2! = x_2 x_4 \dots x_{2n}; \quad x! = x_\infty!; \quad \bar{x}! = \bar{x}_\infty!; \quad x^2! = x_\infty{}^2!,$$

the well-known Rogers-Ramanujan identities can be written in the forms

$$(1.1) \qquad \prod_{n=0}^{\infty} \frac{1}{x_{5n+1} x_{5n+4}} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{x_n!},$$

$$(1.2) \qquad \prod_{n=0}^{\infty} \frac{1}{x_{5n+2} x_{5n+3}} = \sum_{n=0}^{\infty} \frac{x^{n(n+1)}}{x_n!}.$$

Three further identities given by Rogers are

$$(1.3) \qquad \prod_{n=0}^{\infty} \frac{1}{x_{7n+1} x_{7n+2} x_{7n+5} x_{7n+6}} = \bar{x}! \sum_{n=0}^{\infty} \frac{x^{2n^2}}{x_n{}^2! \, \bar{x}_{2n}!},$$

$$(1.4) \qquad \prod_{n=0}^{\infty} \frac{1}{x_{7n+1} x_{7n+3} x_{7n+4} x_{7n+6}} = \bar{x}! \sum_{n=0}^{\infty} \frac{x^{2n(n+1)}}{x_n{}^2! \, \bar{x}_{2n}!},$$

$$(1.5) \qquad \prod_{n=0}^{\infty} \frac{1}{x_{7n+2} x_{7n+3} x_{7n+4} x_{7n+5}} = \bar{x}! \sum_{n=0}^{\infty} \frac{x^{2n(n+1)}}{x_n{}^2! \, \bar{x}_{2n+1}!}.$$

The history of the Rogers-Ramanujan identities is given in Hardy, **2**, and Hardy gives a proof* of them which combines Rogers' method, given in **5**, with formulae which are limiting cases of Jackson's identity† giving the sum of a certain basic hypergeometric series. Recently Dyson‡ used a similar method to prove (1.3)–(1.5). Although he did not use Jackson's identity in the course of the proof, his formulae for α_n, β_n are actually derivable from that identity. To obtain his formula for γ_n by a direct

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* Hardy, **2**, 110–111.

† See Bailey, **1**, § 8.3 (1). I shall refer to this as "Tract".

‡ Dyson, **3**.

limiting process, it appears necessary to use Watson's transformation* of a basic series ${}_8\Phi_7$.

My first object in this paper is to combine Rogers' method freely with known formulae for the sums of basic hypergeometric series. In this way we obtain formulae from which the above identities can easily be derived, but perhaps the most interesting results obtained† are three which give series for products in which the powers of x advance by 9 instead of 5 as in (1.1) and (1.2) or 7 as in (1.3)–(1.5). These three formulae are

$$(1.6)\qquad \prod_{n=0}^{\infty}\frac{1}{x_{9n+1}x_{9n+2}x_{9n+7}x_{9n+8}}=\frac{(x^3!)^2}{x!\,x^9!}\sum_{n=0}^{\infty}\frac{x_{3n}!\,x^{3n^2}}{x_n^{3}!\,x_{2n}^3!},$$

$$(1.7)\qquad \prod_{n=0}^{\infty}\frac{1}{x_{9n+2}x_{9n+4}x_{9n+5}x_{9n+7}}=\frac{(x^3!)^2}{x!\,x^9!}\sum_{n=0}^{\infty}\frac{x_{3n+1}!\,x^{3n(n+1)}}{x_n^{3}!\,x_{2n+1}^3!},$$

$$(1.8)\qquad \prod_{n=0}^{\infty}\frac{1}{x_{9n+1}x_{9n+4}x_{9n+5}x_{9n+8}}=\frac{(x^3!)^2}{x!\,x^9!}\sum_{n=0}^{\infty}\frac{x_{3n}!\,x_{3n+2}x^{3n(n+1)}}{x_n^{3}!\,x_{2n+1}^3!},$$

where $x_n^{3}!=x_3x_6\dots x_{3n}$, and $x^9!$ is defined in a similar manner.

Alternative forms of (1.6)–(1.8) are

$$(1.6a)\qquad 1+\frac{x_1x_2}{x_3x_6}x^{3.1^2}+\frac{x_1x_2x_4x_5}{x_3x_6x_9x_{12}}x^{3.2^2}+\dots=\prod_{n=0}^{\infty}\frac{x_{9n+4}x_{9n+5}}{x_{9n+3}x_{9n+6}},$$

$$(1.7a)\qquad 1+\frac{x_2x_4}{x_6x_9}x^{3(1.2)}+\frac{x_2x_4x_5x_7}{x_6x_9x_{12}x_{15}}x^{3(2.3)}+\dots=\prod_{n=0}^{\infty}\frac{x_{9n+8}x_{9n+10}}{x_{9n+6}x_{9n+12}},$$

$$(1.8a)\qquad 1+\frac{x_1x_5}{x_6x_9}x^{3(1.2)}+\frac{x_1x_4x_5x_8}{x_6x_9x_{12}x_{15}}x^{3(2.3)}+\dots=\prod_{n=0}^{\infty}\frac{x_{9n+7}x_{9n+11}}{x_{9n+6}x_{9n+12}},$$

where, on the left of (1.8a), the x's in the numerators are the same as in (1.7a), except that x_2 is replaced by x_1, and the last x has its suffix increased by 1.

Later in the paper I show that the method can be extended to obtain all the most general known transformations of basic hypergeometric series.

2. Rogers' method of obtaining the Rogers-Ramanujan identities consists in using the classical identities

$$(2.1)\qquad \prod_{n=1}^{\infty}(1+ax^n)=\sum_{n=0}^{\infty}\frac{x^{\frac{1}{2}n(n+1)}}{x_n!}a^n,$$

$$(2.2)\qquad x!\prod_{n=1}^{\infty}(1+2x^{n-\frac{1}{2}}\cos\theta+x^{2n-1})=1+\sum_{n=1}^{\infty}2x^{\frac{1}{2}n^2}\cos n\theta.$$

* Tract, § 8.5 (2).

† See, however, §§ 10 and 11, which were added later.

Evidently

$$\prod_{n=1}^{\infty}(1+2ax^n\cos\theta+a^2x^{2n}) = \prod_{n=1}^{\infty}(1+ae^{i\theta}x^n)(1+ae^{-i\theta}x^n)$$

$$= \sum_{r=0}^{\infty}\frac{x^{\frac{1}{2}r(r+1)}}{x_r!}(ae^{i\theta})^r \sum_{s=0}^{\infty}\frac{x^{\frac{1}{2}s(s+1)}}{x_s!}(ae^{-i\theta})^s$$

$$= 1+\sum_{n=1}^{\infty}\frac{B_n(\theta)}{x_n!}a^n,$$

where

$$(2.3)\quad \frac{B_{2n}(\theta)}{x_{2n}!} = \frac{x^{n(n+1)}}{(x_n!)^2}\left(1+\frac{x_n}{x_{n+1}}2x\cos 2\theta+\frac{x_nx_{n-1}}{x_{n+1}x_{n+2}}2x^4\cos 4\theta+\ldots\right),$$

$$(2.4)\quad \frac{B_{2n+1}(\theta)}{x_{2n+1}!}$$

$$= \frac{x^{(n+1)^2}}{x_n!\,x_{n+1}!}\left(2\cos\theta+\frac{x_n}{x_{n+2}}2x^2\cos 3\theta+\frac{x_nx_{n-1}}{x_{n+2}x_{n+3}}2x^6\cos 5\theta+\ldots\right).$$

Replacing a by $x^{-\frac{1}{2}}$, and using (2.2), we obtain

$$(2.5)\qquad 1+\sum_{n=1}^{\infty}2x^{\frac{1}{2}n^2}\cos n\theta = x!\left[1+\sum_{n=1}^{\infty}\frac{B_n(\theta)}{x_n!}x^{-\frac{1}{2}n}\right].$$

This is a trigonometrical identity, and we can therefore replace the cosines by any numbers for which the series are convergent.

3. In the following work we replace the cosines by numbers so chosen that the series corresponding to $B_n(\theta)$ become basic hypergeometric series which we can sum. It is convenient to use the abbreviation

$$(a)_n = (1-a)(1-ax)\ldots(1-ax^{n-1}).$$

Then, with the usual notation for basic hypergeometric series*, it is known that

$$(3.1)\quad {}_6\Phi_5\left[\begin{matrix} a, & x\sqrt{a}, & -x\sqrt{a}, & b, & c, & x^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & ax/b, & ax/c, & ax^{n+1}\end{matrix}\;;\; ax^{n+1}/bc\right]$$

$$= \frac{(ax)_n\,(ax/bc)_n}{(ax/b)_n\,(ax/c)_n}.$$

* I have written x for the more usual q to keep the work more in line with the sources quoted. The formula (3.1) is a limiting case of Jackson's identity.

If we let $a \to 1$ in this, it becomes

$$(3.2) \qquad 1+\sum_{r=1}^{n}(1+x^r)\frac{(b)_r(c)_r(x^{-n})_r}{(x/b)_r(x/c)_r(x^{n+1})_r}\left(\frac{x^{n+1}}{bc}\right)^r=\frac{(x)_n(x/bc)_n}{(x/b)_n(x/c)_n}.$$

If, however, we put $a=x$, we get

$$(3.3) \qquad \sum_{r=0}^{n}(1-x^{2r+1})\frac{(b)_r(c)_r(x^{-n})_r}{(x^2/b)_r(x^2/c)_r(x^{n+2})_r}\left(\frac{x^{n+2}}{bc}\right)^r=\frac{(x)_{n+1}(x^2/bc)_n}{(x^2/b)_n(x^2/c)_n}.$$

When we put $b=-x^{-n}$ and let $c\to\infty$, these formulae give Dyson's formulae derived from the values of α_n and β_n. His formula derived from the value of γ_n, as already remarked, appears to need Watson's transformation. If in that formula* we make $a\to 1$ and put $cd=x^2$, the series ${}_4\Phi_3$ reduces to two terms. Thus†, with $c=-\omega x$, $d=-\omega^2 x$, $e=x^{-n}$, where ω is an imaginary cube root of unity, we find that

$$(3.4) \quad 1+\sum_{r=1}^{n}(1+x^{3r})\frac{(f)_r(g)_r(x^{-n})_r}{(x/f)_r(x/g)_r(x^{n+1})_r}\left(\frac{x^n}{fg}\right)^r$$
$$=\frac{(x)_n(x/fg)_n}{(x/f)_n(x/g)_n}\left[1-\frac{(1-f)(1-g)(1-x^{-n})}{1-fgx^{-n}}\right],$$

which gives Dyson's result for γ_n when we put $f=-x^{-n}$ and let $g\to\infty$.

The formulae (3.2) and (3.4) are suitable for use with (2.3), and (3.3) is suitable for use with (2.4).

4. We now put 0 for $2\cos(2r+1)\theta$ and

$$(-1)^r(1+x^r)\frac{(\alpha)_r(\beta)_r}{(x/\alpha)_r(x/\beta)_r}\frac{x^{-\frac{1}{2}r(r-1)}}{(\alpha\beta)^r}$$

for $2\cos 2r\theta$ in (2.3), and use (3.2). Then (2.5) gives

$$(4.1) \quad 1+\sum_{n=1}^{\infty}(-1)^n(1+x^n)x^{\frac{1}{2}n(3n+1)}\frac{(\alpha)_n(\beta)_n}{(x/\alpha)_n(x/\beta)_n}\frac{1}{(\alpha\beta)^n}$$
$$=x!\sum_{n=0}^{\infty}\frac{(x/\alpha\beta)_n}{(x)_n(x/\alpha)_n(x/\beta)_n}x^{n^2}.$$

If $\alpha\to\infty$, $\beta\to\infty$, this leads to (1.1). Actually (4.1) is a limiting case of Watson's transformation, obtained by letting three of the numerator parameters tend to infinity. Now in (2.3) and (2.5) change x into x^2

* Tract, § 8.5 (2).

† A more general value of c simply gives a combination of (3.1) and (3.4).

and replace the odd cosines by zero. Then

$$\frac{B_{2n}(\theta)}{x_{2n}^2!} = \frac{x^{2n(n+1)}}{(x_n^2!)^2}\left(1+\frac{x_n\overline{x}_n}{x_{n+1}\overline{x}_{n+1}}2x\cos 2\theta+\frac{x_nx_{n-1}\overline{x}_n\overline{x}_{n-1}}{x_{n+1}x_{n+2}\overline{x}_{n+1}\overline{x}_{n+2}}2x^4\cos 4\theta+\dots\right)$$

and
$$1+\sum_{n=1}^{\infty}2x^{4n^2}\cos 2n\theta = x^2!\left[1+\sum_{n=1}^{\infty}\frac{B_{2n}(\theta)}{x_{2n}^2!}x^{-2n}\right].$$

Putting $(1+x^r)\frac{(a)_r}{(x/a)_r}x^{-r^2}\left(\frac{1}{a}\right)^r$ for $2\cos 2r\theta$, and using (3.2), we get

$$1+\sum_{n=1}^{\infty}(1+x^n)x^{3n^2}\frac{(a)_n}{(x/a)_n}\frac{1}{a^n} = x^2!\sum_{n=0}^{\infty}\frac{x^{2n^2}(-x^{n+1}/a)_n}{\overline{x}_{2n}!\,x_n^2!\,(x/a)_n}. \tag{4.2}$$

When $a\to\infty$ this leads to (1.3), and when $a\to 0$ it gives

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(5n-1)} = x^2!\sum_{n=0}^{\infty}\frac{x^{3n^2}}{\overline{x}_{2n}!\,x_n^2!}$$

from which*

$$\prod_{n=0}^{\infty}\frac{1}{x_{5n+1}x_{5n+4}} = \overline{x}!\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n^2}}{\overline{x}_{2n}!\,x_n^2!}. \tag{4.3}$$

This expresses the product occurring in (1.1) as a different (and less elegant) series.

We now change x into x^3 in (2.3) and (2.5), put 0 for $2\cos(2r+1)\theta$, and $(-1)^r(1+x^r)x^{-\frac{1}{2}r(3r+1)}$ for $2\cos 2r\theta$, and use (3.2) again. We then obtain the formula

$$1+\sum_{n=1}^{\infty}(-1)^n(1+x^n)x^{\frac{1}{2}n(9n-1)} = x^3!\sum_{n=0}^{\infty}\frac{x_{3n}!\,x^{3n^2}}{x_n^3!\,x_{2n}^3!}. \tag{4.4}$$

The left-hand side can be expressed as a product by Jacobi's classical formula, and we obtain (1.6) or equivalently (1.6a).

5. We now turn to the case when we put 0 for the even cosines, and combine (2.4) with (3.3). The work is exactly similar, and so I shall only state the results. The straightforward application of the method gives

$$\sum_{n=0}^{\infty}(-1)^n(1-x^{2n+1})\frac{(a)_n(\beta)_n}{(x^2/a)_n(x^2/\beta)_n}\frac{x^{\frac{1}{2}n(3n+5)}}{(a\beta)^n} = x!\sum_{n=0}^{\infty}\frac{(x^2/a\beta)_n}{(x)_n(x^2/a)_n(x^2/\beta)_n}x^{n(n+1)}, \tag{5.1}$$

* Rogers, 5, p. 330 (5). The formula was rediscovered by Jackson, 6, first formula on p. 176, but the product $(1+q)(1+q^3)\dots$ on the right should be $(1+q)(1+q^2)\dots$.

which leads to (1.2) when $a\to\infty$, $\beta\to\infty$. Changing x into x^2, and proceeding as before, we get

$$(5.2)\quad \sum_{n=0}^{\infty}(1-x^{2n+1})\frac{(a)_n}{(x^2/a)_n}\frac{x^{3n(n+1)}}{a^n}=x^2!\sum_{n=0}^{\infty}\frac{(-x^{n+2}/a)_n}{x_n^{\,2}!\,\bar{x}_{2n+1}!\,(x^2/a)_n}x^{2n(n+1)},$$

which leads to (1.5) when $a\to\infty$. When $a\to 0$ it gives*

$$(5.3)\qquad \prod_{n=0}^{\infty}\frac{1}{x_{5n+2}x_{5n+3}}=\bar{x}!\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n^2+2n}}{x_n^{\,2}!\,\bar{x}_{2n+1}!},$$

an alternative series for the product on the left of (1.2).

Finally, replacing x by x^3, we get

$$(5.4)\quad \sum_{n=0}^{\infty}(-1)^n(1-x^{2n+1})x^{\frac{1}{2}n(9n+7)}=x^3!\sum_{n=0}^{\infty}\frac{x_{3n+1}^{\,!}}{x_n^{\,3}!\,x_{2n+1}^{3\,!}}x^{3n(n+1)},$$

which leads to (1.7) or equivalently (1.7a).

6. A direct application of (3.4) gives

$$(6.1)\quad 1+\sum_{n=1}^{\infty}(-1)^n(1+x^{3n})\frac{(f)_n(g)_n}{(x/f)_n(x/g)_n}\frac{x^{\frac{1}{2}n(3n-1)}}{(fg)^n}$$

$$=x!\left[1+\sum_{n=1}^{\infty}\frac{(x/fg)_n x^{n^2}}{(x)_n(x/f)_n(x/g)_n}\left\{1-\frac{(1-f)(1-g)(1-x^{-n})}{1-fgx^{-n}}\right\}\right],$$

which gives (1.2) when $f\to\infty$, $g\to\infty$.

Changing x into x^2 in the method, we get

$$(6.2)\quad 1+\sum_{n=1}^{\infty}(-1)^n(1+x^{3n})\frac{(a)_n}{(x/a)_n}\frac{x^{3n^2-n}}{a^n}$$

$$=x^2!\left[1+\sum_{n=1}^{\infty}\frac{\{x^{2n}+(1/a)\}(-x^{n+1}/a)_{n-1}}{x_n^{\,2}!\,\bar{x}_{2n}!\,(x/a)_n}x^{2n^2}\right].$$

When $a\to\infty$ this gives (1.4), and when $a\to 0$ we get†

$$(6.3)\qquad \prod_{n=0}^{\infty}\frac{1}{x_{5n+2}x_{5n+3}}=\bar{x}!\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n^2-2n}}{x_n^{\,2}!\,\bar{x}_{2n}!},$$

which is similar to, but different from, (5.3).

The formula obtained by changing x into x^3 is comparatively trivial.

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* Rogers, **5**, p. 330 (5). See also Jackson, **6**, last formula on p. 175.

† (6.3) is stated by Jackson, **6**, but the sign of q should be changed on one side of his formula. The formula would be correct, for example, if the left-hand side were altered to $1-\frac{q}{(\bar{1})(4)}+\frac{q^3}{(\bar{1})(\bar{3})(4)(8)}-\ldots$, where $(\bar{n})=1+q^n$.

7. So far we have obtained more general identities by merely substituting more general expressions for the cosines. Now we examine the method itself more closely.

The essential part of Rogers' argument lies in the identity

$$x! \sum_{r=0}^{\infty} \frac{x^{\frac{1}{2}r^2}}{x_r!} e^{ri\theta} \sum_{s=0}^{\infty} \frac{x^{\frac{1}{2}s^2}}{x_s!} e^{-si\theta} = 1 + \sum_{n=1}^{\infty} 2x^{\frac{1}{2}n^2} \cos n\theta,$$

and, equating the coefficients of $2\cos n\theta$ on the two sides, this implies that

$$\sum_{s=0}^{\infty} \frac{x^{s^2+ns}}{(x)_s (x^{n+1})_s} = \frac{x_n!}{x!}. \tag{7.1}$$

Thus, taking the case when all the odd cosines are replaced by zero, Rogers' method states that, if

$$\beta_n = \sum_{r=0}^{n} \frac{x_n x_{n-1} \cdots x_{n-r+1}}{(x^{n+1})_r} a_r,$$

then

$$\sum_{n=0}^{\infty} a_n x^{n^2} = x! \sum_{n=0}^{\infty} \frac{\beta_n}{(x_n!)^2} x^{n^2}.$$

This is easily deduced from (7.1), for, substituting the value of β_n, we get

$$x! \sum_{n=0}^{\infty} \frac{\beta_n}{(x_n!)^2} x^{n^2} = x! \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{x^{n^2} a_r}{x_{n-r}! x_{n+r}!},$$

and, putting $n = r+s$, we get

$$x! \sum_{r=0}^{\infty} \frac{a_r x^{r^2}}{x_{2r}!} \sum_{s=0}^{\infty} \frac{x^{s^2+2rs}}{(x)_s (x^{2r+1})_s} = \sum_{r=0}^{\infty} a_r x^{r^2}.$$

Now (7.1) is a particular case of much more general known sums of basic hypergeometric series, and so it is suggested that we might use such sums instead of (7.1).

8. The simplest generalization of (7.1) is

$$\prod_{n=1}^{\infty} (1-ax^n) \sum_{s=0}^{\infty} \frac{a^s x^{s^2+ns}}{(x)_s (ax^{n+1})_s} = (ax)_n, \tag{8.1}$$

which can easily be obtained from the basic analogue of Gauss's theorem*. From this we find, by an argument exactly similar to that of § 7, that, if

$$\beta_n = \sum_{r=0}^{n} \frac{x_n x_{n-1} \cdots x_{n-r+1}}{(ax^{n+1})_r} a_r,$$

* Tract, § 8.4 (3).

then

$$\prod_{n=1}^{\infty}(1-ax^n)\sum_{n=0}^{\infty}\frac{a^n x^{n^2}}{x_n!\,(ax)_n}\beta_n=\sum_{n=0}^{\infty}a^n x^{n^2}\alpha_n.$$

We can now proceed as in §4, choosing α_r so that we can use (3.1) to find β_n instead of using (3.2). Thus, corresponding to the limiting case of (4.1) when $\alpha, \beta\to\infty$, we obtain the formula

$$(8.2)\quad 1+\sum_{n=1}^{\infty}(-1)^n(1-ax^{2n})(ax)_{n-1}\frac{a^{2n}}{x_n!}x^{\frac{1}{2}n(5n-1)}=\prod_{n=1}^{\infty}(1-ax^n)\sum_{n=0}^{\infty}\frac{a^n x^{n^2}}{x_n!},$$

from which the Rogers-Ramanujan identities are usually deduced by putting $a=1$ and $a=x$.

Corresponding to (4.2) we get

$$(8.3)\quad 1+\sum_{n=1}^{\infty}(1-ax^{2n})\frac{(ax)_{n-1}(\alpha)_n}{(x)_n(ax/\alpha)_n}\frac{a^{3n}x^{3n^2}}{\alpha^n}$$
$$=\prod_{n=1}^{\infty}(1-a^2x^{2n})\sum_{n=0}^{\infty}\frac{(-ax^{n+1}/\alpha)_n a^{2n}x^{2n^2}}{x_n^2!\,(-ax)_{2n}(ax/\alpha)_n},$$

from which we derive (4.2) and (5.2) when $a=1$ and $a=x$. If we let $\alpha\to 0$, we get

$$(8.4)\quad 1+\sum_{n=1}^{\infty}(-1)^n(1-ax^{2n})\frac{(ax)_{n-1}}{(x)_n}a^{2n}x^{\frac{1}{2}n(5n-1)}$$
$$=\prod_{n=1}^{\infty}(1-a^2x^{2n})\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}x^{3n^2}}{x_n^2!\,(-ax)_{2n}},$$

which gives another transformation of the series on the left of (8.2).

If in (8.3) we let $\alpha\to\infty$, we get

$$(8.5)\quad 1+\sum_{n=1}^{\infty}(-1)^n(1-ax^{2n})\frac{(ax)_{n-1}}{(x)_n}a^{3n}x^{\frac{1}{2}n(7n-1)}$$
$$=\prod_{n=1}^{\infty}(1-a^2x^{2n})\sum_{n=0}^{\infty}\frac{a^{2n}x^{2n^2}}{x_n^2!\,(-ax)_{2n}},$$

from which (1.3) and (1.5) follow by taking $a=1$ and $a=x$.

Corresponding to (4.4) we get

$$(8.6)\quad \prod_{n=1}^{\infty}(1-a^3x^{3n})\sum_{n=0}^{\infty}\frac{(ax)_{3n}a^{3n}x^{3n^2}}{x_n^3!\,(1-a^3x^3)(1-a^3x^6)\dots(1-a^3x^{6n})}$$
$$=1+\sum_{n=1}^{\infty}(-1)^n(1-ax^{2n})\frac{(ax)_{n-1}}{(x)_n}a^{4n}x^{\frac{1}{2}n(9n-1)},$$

from which we are led to (1.6) and (1.7) by taking $a=1$ and $a=x$.

Now in Watson's transformation take $g = x^{-n}$, $d = ax^2/c$, and the ${}_4\Phi_3$ reduces to two terms. When multiplied by $(1-ax/c)(1-c/x)$, the identity becomes one involving terms in c, $1/c$, and terms independent of c. The terms in c or $1/c$ simply give the identity (3.1). The terms independent of c give*

$$\sum_{r=0}^{n} (1-a^2x^{4r}) \frac{(ax)_{r-1}}{(x)_r} \frac{(e)_r (f)_r (x^{-n})_r}{(ax/e)_r (ax/f)_r (ax^{n+1})_r} \left(\frac{ax^n}{ef}\right)^r$$

$$= \frac{(ax)_n (ax/ef)_n}{(ax/e)_n (ax/f)_n} \left[1+a-\frac{(1-e)(1-f)(1-x^{-n})}{1-efx^{-n}/a}\right].$$

If we take $e = \omega x^{-n}$, $f = \omega^2 x^{-n}$, where ω is an imaginary cube root of unity, and use this with the formulae of this paragraph with a^3, x^3 replacing a, x, we get

$$(8.7) \quad \prod_{n=1}^{\infty} (1-a^3x^{3n}) \sum_{n=0}^{\infty} \frac{(ax)_{3n} (1-a^2x^{3n}) a^{3n} x^{3n^2}}{x_n^{3}! (1-a^3x^3)(1-a^3x^6)\dots(1-a^3x^{6n})(1-ax^{3n})}$$

$$= \sum_{n=0}^{\infty} (-1)^n a^{4n} x^{\frac{1}{2}n(9n-3)} (1-a^2x^{4n}) \frac{(ax)_{n-1}}{(x)_n}.$$

Taking $a = x$ in this formula, we get (1.8) or equivalently (1.8a).

9. More generally, it is easily shown that, if

$$(9.1) \quad \beta_n = \sum_{r=0}^{n} \frac{x_n x_{n-1} \dots x_{n-r+1}}{(ax^{n+1})_r} \alpha_r,$$

where α_r is independent of n, then

$$(9.2) \quad \sum_{n=0}^{\infty} \frac{(\mu_1)_n (\mu_2)_n (\mu_3)_n}{(x)_n (ax)_n (\sigma_1)_n} \beta_n t^n$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r (\mu_3)_r}{(ax)_{2r} (\sigma_1)_r} t^r \alpha_r \Phi \left[\begin{matrix} \mu_1 x^r, & \mu_2 x^r, & \mu_3 x^r; \\ ax^{2r+1}, & \sigma_1 x^r \end{matrix} \; t\right].$$

In this formula there may be any number of μ and σ.

Taking

$$\alpha_r = \frac{(a)_r (x\sqrt{a})_r (-x\sqrt{a})_r (d)_r (e)_r}{(x)_r (\sqrt{a})_r (-\sqrt{a})_r (ax/d)_r (ax/e)_r} (-1)^r x^{\frac{1}{2}r(r+1)} \left(\frac{a}{de}\right)^r,$$

* When we put $a = x$, $e = -x^{-n}$ and let $f \to \infty$, this formula becomes the last formula given by Dyson.

we can use (3.1) to find β_n. Then, choosing

$$\mu_1 = b, \quad \mu_2 = c, \quad \mu_3 = x^{-N}, \quad \sigma_1 = bc/ax^N, \quad t = x,$$

we can sum the series Φ on the right of (9.2) by the analogue of Saalschütz's theorem*, and we are led to Watson's transformation of a well-poised ${}_8\Phi_7$ into a Saalschützian ${}_4\Phi_3$.

We could, of course, change x into x^2 or into x^3 in (9.1) and (9.2) and obtain further transformations, but the series obtained, although well poised, are not of general type.

Again, a similar argument shows that, if

$$\beta_n = \sum_{r=0}^{n} \frac{(x^{-n})_r (kx^n)_r}{(ax^{n+1})_r (ax^{1-n}/k)_r} \alpha_r, \tag{9.3}$$

where α_r is independent of n, then

$$\sum_{n=0}^{\infty} \frac{(k)_n (k/a)_n (\mu_1)_n (\mu_2)_n}{(x)_n (ax)_n (\sigma_1)_n (\sigma_2)_n} t^n \beta_n = \sum_{r=0}^{\infty} \frac{(k)_{2r} (\mu_1)_r (\mu_2)_r}{(ax)_{2r} (\sigma_1)_r (\sigma_2)_r} \left(\frac{kt}{ax}\right)^r \alpha_r \Phi\left[\begin{matrix} kx^{2r}, & k/a, & \mu_1 x^r, & \mu_2 x^r; \\ & ax^{2r+1}, & \sigma_1 x^r, & \sigma_2 x^r \end{matrix} \; t\right], \tag{9.4}$$

where there may be any number of the numbers μ, σ.

For example, if we take

$$\alpha_r = \frac{(a)_r (x\sqrt{a})_r (-x\sqrt{a})_r (b)_r (c)_r (d)_r x^r}{(x)_r (\sqrt{a})_r (-\sqrt{a})_r (ax/b)_r (ax/c)_r (ax/d)_r},$$

where $k = a^2 x/bcd$, we can use Jackson's identity to find β_n. In fact

$$\beta_n = \frac{(ax)_n (ax/cd)_n (ax/bd)_n (ax/bc)_n}{(ax/b)_n (ax/c)_n (ax/d)_n (ax/bcd)_n}.$$

Now in (9.4) take $\mu_1 = x\sqrt{k}$, $\mu_2 = -x\sqrt{k}$, $\mu_3 = e$, $\mu_4 = f$, $\mu_5 = g$, $\mu_6 = x^{-N}$, $\sigma_1 = \sqrt{k}$, $\sigma_2 = -\sqrt{k}$, $\sigma_3 = kx/e$, $\sigma_4 = kx/f$, $\sigma_5 = kx/g$, $\sigma_6 = kx^{N+1}$, and $t = x$; we can sum the series Φ on the right of (9.4) by Jackson's identity, provided that $efg = akx^{N+1}$. We are then led to a known transformation of a well-poised ${}_{10}\Phi_9$ into another series of the same type†.

The extension of the method has thus given us the most general known transformations of well-poised basic hypergeometric series. One might expect also to be able to obtain results for nearly-poised basic series. In fact, if we take α_r as above in (9.3) and choose μ_1, σ_1 in (9.4) so that we

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* Tract, § 8.4 (1).

† Tract, § 8.5 (1).

can sum by the basic analogue of Saalschütz's theorem*, we obtain the formula

$$(9.5)\quad {}_5\Phi_4\left[\begin{matrix} a,\ b,\ c,\ d,\ x^{-N}; \\ ax/b,\ ax/c,\ ax/d,\ a^2x^{-N}/k^2 \end{matrix}\ x\right]$$

$$= \frac{(kx/a)_N\,(k^2x/a)_N}{(kx)_N\,(k^2x/a^2)_N}$$

$$\times {}_{12}\Phi_{11}\left[\begin{matrix} k,\ x\sqrt{k},\ -x\sqrt{k},\ kb/a,\ kc/a,\ kd/a,\ \sqrt{a},\ -\sqrt{a}, \\ \sqrt{k},\ -\sqrt{k},\ ax/b,\ ax/c,\ ax/d,\ kx/\sqrt{a},\ -kx/\sqrt{a}, \\ \sqrt{(ax)},\ -\sqrt{(ax)},\ k^2x^{N+1}/a,\ x^{-N} \\ k\sqrt{(x/a)},\ -k\sqrt{(x/a)},\ ax^{-N}/k,\ kx^{N+1} \end{matrix}\ ;\ x\right],$$

where $k = a^2x/bcd$, which transforms a Saalschützian nearly-poised series into a well-poised series. The formula is exactly analogous to the formula §4.5 (3) of my tract. In other cases where we might expect to find a transformation of a nearly-poised series, the method breaks down, as we have to introduce a power of x into a_r, which is not independent of n, in order to sum the series for β_n. Thus, for the majority of nearly-poised series, the analogy between basic series and series of the ordinary type appears to break down.

10. Further results† can be obtained by replacing some of the numbers a_r by zero. Thus, in §8, if we take $a_r = 0$ when r is odd‡, and

$$a_{2r} = \frac{[ax^2]_{r-1}\,(1-ax^{4r})[f]_r}{x_r^{\,2}\,[ax^2/f]_r}\left(\frac{a}{f}\right)^r x^{2r^2},$$

where $$[a]_r = (1-a)\,(1-ax^2)\ldots(1-ax^{2r-2}),$$

we find from Jackson's identity that

$$\beta_n = \frac{(ax)_n\,[ax/f]_n}{[ax]_n\,(ax/f)_n}.$$

We therefore have the identity

$$(10.1)\quad \prod_{n=1}^{\infty}(1-ax^n)\sum_{n=0}^{\infty}\frac{a^n x^{n^2}\,[ax/f]_n}{x_n!\,[ax]_n\,(ax/f)_n}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{a^{2n}x^{4n^2}(1-ax^{4n})[ax^2]_{n-1}[f]_n}{x_n^{\,2}\,[ax^2/f]_n}\left(\frac{a}{f}\right)^n x^{2n^2}.$$

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* Tract, §8.4 (1).

† §§ 10 and 11 were added on 4 January, 1944.

‡ Equivalent to replacing alternate cosines by zero in Rogers' method.

When $f \to \infty$ this gives

$$\prod_{n=1}^{\infty}(1-ax^n)\sum_{n=0}^{\infty}\frac{a^n x^{n^2}}{x_n!\,[ax]_n} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n a^{3n}(1-ax^{4n})[ax^2]_{n-1}x^{7n^2-n}}{x_n{}^2!},$$

and for $a = x^2$ and $a = 1$ we derive the formulae

$$(10.2)\qquad \sum_{n=0}^{\infty}\frac{x^{n^2+2n}}{x_n!\,x_{n+1}!!} = \prod_{n=1}^{\infty}\frac{(1-x^{14n-2})(1-x^{14n-12})(1-x^{14n})}{1-x^n},$$

$$(10.3)\qquad \sum_{n=0}^{\infty}\frac{x^{n^2}}{x_n!\,x_n!!} = \prod_{n=1}^{\infty}\frac{(1-x^{14n-6})(1-x^{14n-8})(1-x^{14n})}{1-x^n},$$

where $x_n!! = (1-x)(1-x^3)\dots(1-x^{2n-1})$.

If $f \to 0$ in (10.1) we derive in a similar way

$$(10.4)\qquad \sum_{n=0}^{\infty}\frac{x^{\frac{1}{2}n(3n-1)}}{x_n!\,x_n!!} = \prod_{n=1}^{\infty}\frac{(1-x^{10n-4})(1-x^{10n-6})(1-x^{10n})}{1-x^n},$$

$$(10.5)\qquad \sum_{n=0}^{\infty}\frac{x^{\frac{3}{2}n(n+1)}}{x_n!\,x_{n+1}!!} = \prod_{n=1}^{\infty}\frac{(1-x^{10n-2})(1-x^{10n-8})(1-x^{10n})}{1-x^n}.$$

These formulae were all given by Rogers*.

Similarly, if we take $\alpha_r = 0$, unless r is a multiple of 3, and let

$$\alpha_{3r} = \frac{(-1)^r\{ax^3\}_{r-1}(1-ax^{6r})\,a^r x^{\frac{3}{2}r(r-1)}}{\{x^3\}_r},$$

where $$\{a\}_r = (1-a)(1-ax^3)\dots(1-ax^{3r-3}),$$

we find that $$\beta_n = \frac{\{ax^3\}_{n-1}}{(ax^{n+1})_{n-1}}.$$

Thus we have the identity

$$(10.6)\qquad \prod_{n=1}^{\infty}(1-ax^n)\left[1+\sum_{n=1}^{\infty}\frac{a^n x^{n^2}\{ax^3\}_{n-1}}{x_n!\,(ax)_{2n-1}}\right]$$
$$= 1+\sum_{n=1}^{\infty}(-1)^n\frac{\{ax^3\}_{n-1}(1-ax^{6n})}{\{x^3\}_n}\,a^{4n}x^{\frac{1}{2}(27n^2-3n)}.$$

* Rogers, **5**, § 7, (1) and (2).

For $a=1$ and $a=x^3$ we obtain the formulae*

$$(10.7)\qquad 1+\sum_{n=1}^{\infty}\frac{x^{n^2}x_{n-1}^{3}!}{x_n!\,x_{2n-1}!}=\prod_{n=1}^{\infty}\frac{(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})}{1-x^n},$$

$$(10.8)\qquad \sum_{n=0}^{\infty}\frac{x^{n^2+3n}x_n^{3}!}{x_n!\,x_{2n+2}!}=\prod_{n=1}^{\infty}\frac{(1-x^{27n-24})(1-x^{27n-3})(1-x^{27n})}{1-x^n}.$$

Evidently identities more general than (10.1) and (10.6) could be obtained by using §9 instead of §8. For example, if we take $\mu_1=b$, $\mu_2=c$, $t=ax/bc$ in (9.2), and sum the series Φ by the analogue of Gauss's theorem, we get

$$\sum_{n=0}^{\infty}\frac{(b)_n\,(c)_n}{(ax/b)_n\,(ax/c)_n}\left(\frac{ax}{bc}\right)^n\alpha_n$$
$$=\prod_{n=1}^{\infty}\left[\frac{(1-ax^n)(1-ax^n/bc)}{(1-ax^n/b)(1-ax^n/c)}\right]\sum_{n=0}^{\infty}\frac{(b)_n\,(c)_n}{(x)_n\,(ax)_n}\left(\frac{ax}{bc}\right)^n\beta_n,$$

where β_n is given by (9.1) and the process of taking some of the numbers α_r equal to zero will lead to more general results.

11. Mr. F. J. Dyson has called my attention to the fact that Rogers gave expansions for products with powers of x advancing by 5, 7, 10, 14, 15, 20 and 21, though in the cases of 15 and 21 the results are not so simple. The formulae (1.6)–(1.8) above can be written in the forms

$$(A\,1)\qquad \sum_{0}^{\infty}\frac{x^{3n^2}x_{3n}!}{x_n^{3}!\,x_{2n}^{3}!}=\prod_{1}^{\infty}\frac{(1-x^{9n-5})(1-x^{9n-4})(1-x^{9n})}{1-x^{3n}},$$

$$(A\,2)\qquad \sum_{0}^{\infty}\frac{x^{3n(n+1)}x_{3n+1}!}{x_n^{3}!\,x_{2n+1}^{3}!}=\prod_{1}^{\infty}\frac{(1-x^{9n-8})(1-x^{9n-1})(1-x^{9n})}{1-x^{3n}},$$

$$(A\,3)\qquad \sum_{0}^{\infty}\frac{x^{3n(n+1)}x_{3n}!\,x_{3n+2}}{x_n^{3}!\,x_{2n+1}^{3}!}=\prod_{1}^{\infty}\frac{(1-x^{9n-7})(1-x^{9n-2})(1-x^{9n})}{1-x^{3n}}.$$

Mr. Dyson has used the methods of this paper to derive the identities

$$(B\,1)\qquad \sum_{0}^{\infty}\frac{x^{n^2+3n}x_n^{3}!}{x_n!\,x_{2n+2}!}=\prod_{1}^{\infty}\frac{(1-x^{27n-24})(1-x^{27n-3})(1-x^{27n})}{1-x^n},$$

$$(B\,2)\qquad \sum_{0}^{\infty}\frac{x^{n^2+2n}x_n^{3}!}{x_n!\,x_{2n+2}!}=\prod_{1}^{\infty}\frac{(1-x^{27n-21})(1-x^{27n-6})(1-x^{27n})}{1-x^n},$$

$$(B\,3)\qquad \sum_{0}^{\infty}\frac{x^{n^2+n}x_n^{3}!}{x_n!\,x_{2n+1}!}=\prod_{1}^{\infty}\left(\frac{1-x^{9n}}{1-x^n}\right),$$

$$(B\,4)\qquad 1+\sum_{1}^{\infty}\frac{x^{n^2}x_{n-1}^{3}!}{x_n!\,x_{2n-1}!}=\prod_{1}^{\infty}\frac{(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})}{1-x^n}.$$

* (10.7) and (10.8) are due to Dyson. See the next paragraph.

Of these, (B 1) and (B 4) were obtained in § 10. (B 2) can easily be deduced from these by using the identity

$$\prod_1^\infty (1-x^n) = \prod_1^\infty [(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})]$$
$$-x^2 \prod_1^\infty [(1-x^{27n-24})(1-x^{27n-3})(1-x^{27n})]$$
$$-x \prod_1^\infty [(1-x^{27n-21})(1-x^{27n-6})(1-x^{27n})],$$

which is itself easily proved by expanding both sides in powers of x. The methods of § 10 can be used to express the product in (B 3) as a series, but it is rather tedious to obtain the actual series on the left of (B 3).

Mr. Dyson has also used Rogers' methods to deduce the following formulae:

$$\text{(C 1)} \qquad \sum_0^\infty \frac{x^{n^2+4n} x_n{}^6!}{x_{2n+1}! x_{n+1}^4!} = \prod_1^\infty \frac{(1-x^{36n-33})(1-x^{36n-3})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})},$$

$$\text{(C 2)} \qquad \sum_0^\infty \frac{x^{n^2+2n} x_n{}^6!}{x_{2n+1}! x_{n+1}^4!} = \prod_1^\infty \frac{(1-x^{18n-9})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})},$$

$$\text{(C 3)} \qquad 1+\sum_1^\infty \frac{x^{n^2} x_{n-1}^6!}{x_{2n}! x_{n-1}^4!} = \prod_1^\infty \frac{(1-x^{36n-21})(1-x^{36n-15})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})},$$

$$\text{(D 1)} \qquad \sum_0^\infty \frac{x^{\frac{1}{2}(n^2+3n)} x_n{}^3!}{x_n! x_{n+1}! x_{n+1}!!} = \prod_1^\infty \frac{(1-x^{18n-15})(1-x^{18n-3})(1-x^{18n})}{(1-x^n)(1-x^{2n-1})},$$

$$\text{(D 2)} \qquad \sum_0^\infty \frac{x^{\frac{1}{2}(n^2+n)} x_n{}^3!}{(x_n!)^2 x_{n+1}!!} = \prod_1^\infty \frac{1-x^{6n}}{(1-x^n)(1-x^{2n-1})},$$

$$\text{(D 3)} \qquad 1+2\sum_1^\infty \frac{x^{\frac{1}{2}(n^2+n)} x_{n-1}^3!}{x_{n-1}! x_n! x_n!!} = \prod_1^\infty \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})},$$

$$\text{(E 1)} \qquad \sum_0^\infty (-1)^n \frac{x^{\frac{1}{2}(n^2+3n)} x_n{}^3!}{(1+x^{n+1}) x_{2n+1}!} = \sum_0^\infty x^{9n^2+6n} - \sum_1^\infty x^{9n^2-6n},$$

$$\text{(E 2)} \qquad \sum_0^\infty (-1)^n \frac{x^{\frac{1}{2}(n^2+n)} x_n{}^3!}{x_{2n+1}!} = \sum_0^\infty x^{9n^2+3n} - \sum_1^\infty x^{9n^2-3n}.$$

Of these, he remarks that (D 3) is a trivial consequence of (D 1), and the series on the right of (E 1) and (E 2) are of the type which Rogers calls "false ϑ-series". Mr. Dyson tells me that the method of proof is a standard type of argument due to Rogers. In this paper I have considered only one aspect of Rogers' work, and so I shall not attempt to

provide proofs of the formulae. It is not suggested that these are all the formulae which can be obtained from the identities (A) and (B). In fact Mr. Dyson calls them "a rather haphazard selection", but I feel that they add considerably to the interest of the paper.

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ASYMPTOTIC INTEGRALS OF A FOURTH ORDER DIFFERENTIAL EQUATION CONTAINING A LARGE PARAMETER

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1. *Statement of the problem.*

The equation considered in this paper is of very great importance in the problem of turbulent motion; it is of the form

$$\frac{d^4\psi}{dy^4}+[r+\lambda p(y)]\frac{d^2\psi}{dy^2}+[s+\lambda q(y)]\psi=0, \tag{1.1}$$

where $p(y)$ and $q(y)$ are analytic slowly varying functions of a real variable y; r and s are small parameters or bounded functions of y, and λ is a large parameter; the range of y is the segment of the real axis given by

$$y_1 \leqslant y \leqslant y_2. \tag{1.2}$$

The function $p(y)$ has a single simple zero y_0 within this range; it is an ordinary point of the equation, but it is a critical point of the asymptotic expansions; $q(y)$ is assumed not to vanish at the critical point, this case being more complicated.

An equation of this type has two slowly varying integrals and two rapidly varying integrals; the object of this paper is to find the asymptotic expansions of the four integrals. The main difficulty of this problem is to find the continuation of the integrals through the critical point.

For the purposes of the hydrodynamical problem a particular case of the above equation will be considered.

The equation, in the form given below, was discussed in connection with the problem of stability of laminar motion by Tietjens (3), Heisenberg (1) and Tollmien (4).

Tietjens considered the case when $p(y)$ represents a profile consisting of broken straight lines, so that in his work the critical point in the slowly

varying integrals does not occur; the transformation of the rapidly varying integrals was done numerically.

Heisenberg and Tollmien tried to find the continuation of the slowly varying integrals; their results are, however, incomplete. No attempts have been made in the above-mentioned papers to give a theoretical discussion of continuation of the two rapidly varying integrals.

Jeffreys (2) developed a valuable method of finding the transformations of asymptotic integrals for a linear equation of the second order.

Use is made of Jeffreys' idea in the transformation of the rapidly varying integrals; since, however, our equation is of the fourth order, it is necessary to supplement Jeffreys' method by somewhat involved considerations.

The equation which will be discussed in detail is

$$\frac{d^4\psi}{dy^4}+[-2\alpha^2+i\lambda(w-c)]\frac{d^2\psi}{dy^2}+[\alpha^4-i\lambda\alpha^2(w-c)-i\lambda w'']\psi=0 \quad (1.3)$$

where α^2 is a small parameter of the order of unity and λ is a large positive parameter at least equal to 1000 and, usually very much larger; w is an even function of a real variable y which lies within the range

$$-1\leqslant y\leqslant 1. \quad (1.4)$$

Since the coefficients of (1.3) are even functions of y the equation has two even and two odd integrals; the range of y can, therefore, be limited to

$$0\leqslant y\leqslant 1, \quad (1.5)$$

and within this range w satisfies the conditions

$$\left.\begin{aligned} w-c=0 \quad \text{at} \quad y=y_0, \\ w'<0, \quad w''\neq 0, \end{aligned}\right\} \quad (1.6)$$

where the simple root y_0 lies between zero and unity, and α^2, λ, y_0, w and c are all real.

In the physical problem λ is given, and α^2 and c have to be found from four boundary conditions; this problem is, however, beyond the scope of this paper.

We shall only sketch the transformation of the slowly varying integrals; the even, rapidly varying integral will be fully worked out.

A. Slowly varying integrals.

Asymptotic approximations of the two slowly varying integrals are found by making $\lambda=\infty$ in the differential equation (§2).

We discuss in § 3 the transformation through the critical point $\eta = 0$ of the term $\eta \log \eta$, where $\eta = \epsilon(y - y_0)$ and η is positive, and it is shown that it becomes transformed into a large rapidly varying integral for large negative η.

The transformation of terms of the type $\eta^n \log \eta$ $(n > 1)$ is briefly sketched in § 4.

2. *Integrals between $y = 0$ and the critical point y_0.*

Rewrite (1.3) in the form

$$(w-c)(\psi''-\alpha^2\psi)-w''\psi = \frac{i}{\lambda}(\psi^{iv}-2\alpha^2\psi''+\alpha^4\psi) \qquad (2.1)$$

The terms on the right can be omitted in a first approximation since $w-c$ does not vanish within the range under consideration, namely

$$0 \leqslant y < y_0.$$

The equation

$$(w-c)(\psi''-\alpha^2\psi)-w''\psi = 0 \qquad (2.2)$$

can be solved by expansion in series about the critical point in the well known way.

One of the integrals is regular, while the other is of the form

$$\left.\begin{array}{c} P_1(z)+P_2(z)\log(-z), \\ z = y-y_0 \end{array}\right\} \qquad (2.3)$$

(since w'' does not vanish at $y = y_0$), where $P_1(z)$ and $P_2(z)$ are two regular functions; P_2 begins with a term of the first degree in z.

The regular integral and the regular part of the singular integral pass through the critical point unchanged, but the singular part in (2.3) is not valid near the critical point, and it does not preserve its form in passing through that point.

We have, therefore, to consider the term

$$P_2(z)\log(-z)$$

as an asymptotic approximation of the required integral, valid only for negative values of z, starting from a certain distance from $z = 0$; and our main problem is to find the expression of the singular term round the zero point; *i.e.* we have to find the transformations through $z = 0$ of the functions

$$z\log(-z), \quad z^2\log(-z), \quad \ldots.$$

It is worth while to notice that the two integrals of (2.2) can be expanded in ascending powers of a^2 (**1**, 586).

3. *Transformation of* $z \log(-z)$.

It will be shown that the term $z \log(-z)$, where $z < 0$, becomes transformed round $z = 0$ into a very complicated expression, which for positive z tends to a rapidly varying, exponentially increasing, integral of (1.3).

To find the transformation of

$$z \log(-z), \quad z = y - y_0 < 0, \tag{3.1}$$

we have to bear in mind that, since $w - c$ vanishes at $z = 0$, and is very small round that point, the terms on the right in (2.1) cannot be neglected; we have, therefore, to consider the complete equation (1.3). We transform it by taking a new variable η, where

$$\left.\begin{aligned} z = y - y_0 &= \epsilon\eta, \\ \epsilon &= (\lambda w_0')^{-\frac{1}{3}}; \end{aligned}\right\} \tag{3.2}$$

since w' is negative, ϵ is a negative real number.

Expanding $w - c$ and w'' in powers of η, we obtain

$$\frac{d^4\psi}{dy^4} + i\eta \frac{d^2\psi}{dy^2} = \left[2a^2\epsilon^2 - i\frac{w_0''}{w_0'}\frac{\epsilon\eta^2}{2!} - i\frac{w_0'''}{w_0'}\frac{\epsilon^2\eta^3}{3!} - \dots\right]\frac{d^2\psi}{dy^2}$$

$$+\left[i\frac{w_0''}{w_0'}\epsilon + i\left(a^2 + \frac{w_0'''}{w_0'}\right)\epsilon^2\eta + \dots\right]\psi, \tag{3.3}$$

where the derivatives w_0', w_0'', ... are taken with respect to y at the critical point $\eta = 0$; the expansion of $w - c$ in powers of η is permissible since it is an analytic function.

The equation (3.3) can be solved by successive approximations, by expanding the solution in powers of the small parameter ϵ, thus

$$\psi = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{3.4}$$

Combining (3.3) and (3.4) we obtain a set of linear equations, namely

$$\left.\begin{aligned} &\frac{d^4 f_0}{d\eta^4} + i\eta\frac{d^2 f_0}{d\eta^2} = 0, \\ &\frac{d^4 f_1}{d\eta^4} + i\eta\frac{d^2 f_1}{d\eta^2} = i\frac{w_0''}{w_0'}f_0, \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}\right\} \tag{3.5}$$

and $f_0, f_1, \dots$ can be successively determined.

It was shown by Tollmien (4, 28) that the solution which leads to the logarithmic term for positive η is obtained from the equation (3.5)

$$f_1^{iv}+i\eta f_1''=i\frac{w_0''}{w_0'}f_0, \tag{3.6}$$

where, on the right-hand side, f_0 is put equal to a constant.

The solutions which include the slowly varying integrals are obtained by putting f_0 equal to a constant or to $\epsilon\eta$, both of which are integrals of the first equation (3.5).

Since $$\epsilon<0, \tag{3.7}$$

it follows from (3.2) that

$$z=y-y_0\gtrless 0, \quad \eta\gtrless 0, \tag{3.8}$$

and the singular term (3.1), when expressed in terms of the new variable η (apart from a linear term in η), becomes

$$\epsilon\eta\log\eta, \quad \eta>0. \tag{3.9}$$

We have, accordingly, to prove that a certain solution of (3.6) tends asymptotically to (3.9) for large positive η.

Now, although z is usually small within the asymptotic range, η can be made large by taking ϵ sufficiently small, since $z=\epsilon\eta$; it will thus be possible to make use of the asymptotic expansions of the integrals of (3.6).

(*a*) *Transformation of* $1/\eta$.

We have to solve the equation (3.6) for f_1, where f_0 is put equal to a constant, since that leads us to a solution for f_1'' which tends asymptotically to a constant multiple of $1/\eta$ for large positive η.

By choosing a suitable constant value for f_0, the equation (3.6) can be reduced to

$$v^{iv}+i\eta v''=i, \tag{3.10}$$

and it will, in fact, be shown below that a particular integral of this equation behaves like $1/\eta$ for large positive η. (v is used instead of f_1 to avoid confusion.)

To solve (3.10), assume

$$v''=A\int_0^\infty \exp(-t^3-\sigma\eta t)\,dt, \tag{3.11}$$

where A and σ are constants, and the path of integration is not necessarily the real axis.

Substituting (3.11) in (3.10), we find that

$$v'' = 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} \int_0^\infty \exp\left(-t^3 - 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} \eta t\right) dt \tag{3.12}$$

the second derivative of the singular term (3.9) is, except for the factor ϵ, equal to $1/\eta$, for positive η; we have, therefore, to prove that (3.12) tends to $1/\eta$ for large positive η; we accordingly consider (3.12) for large values of η, and for two special cases.

(i) *η is large and positive.*

Only contributions to (3.12) near $t = 0$ will be effective; accordingly t^3 can be neglected, and we get

$$v'' \sim 1/\eta, \tag{3.13}$$

i.e. (3.12) is the required solution.

(ii) *η is large and negative.*

In this case the only significant contribution in (3.12) will be from t satisfying the condition

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left(t^3 + 3^{\frac{1}{3}} e^{\frac{1}{3}\pi i} \eta t\right) = 0, \\ &t = t_0 = |\eta|^{\frac{1}{2}} 3^{-\frac{1}{3}} e^{\frac{1}{2}\pi i}. \end{aligned}\right\} \tag{3.14}$$

Assume

$$t = t_0 + \delta. \tag{3.15}$$

Substituting (3.15) in (3.12), retaining in the exponential only the term proportional to δ^2 and extending the two limits of integration to negative and positive infinity respectively, we get

$$v'' = \frac{\pi^{\frac{1}{2}} e^{\frac{1}{6}\pi i}}{(-\eta)^{\frac{1}{4}}} \exp\left(\tfrac{2}{3} e^{\frac{1}{4}\pi i} (-\eta)^{\frac{3}{2}}\right), \tag{3.16}$$

which is a large solution of the first equation (3.5).

We accordingly obtain the result that the term which behaves like $1/\eta$ between $y = 0$ and the critical point tends to a rapidly varying integral in passing through the critical point.

This result differs essentially from Tollmien's (4, 29) conclusion that the slowly varying integral becomes transformed into a linear combination of two slowly varying integrals in passing through the critical point.

(*b*) *Transformation of* $\eta \log \eta$.

This section deals with the evaluation of two very important constants of integration. An example is given to make the subsequent work clearer.

If, to fix ideas, a certain function $T''(\eta)$ tends asymptotically to $1/\eta$ for large values of η, then the function

$$\int_0^\eta T''(\eta)\, d\eta$$

tends asymptotically to $C+\log\eta$, where C is a constant to be evaluated.

We have found that the integral (3.12) tends asymptotically to $1/\eta$ for large positive η; it is still necessary to find the expression which tends to $\eta \log\eta$; for that purpose we integrate (3.12) with respect to η between 0 and η and, by splitting up the integral into two parts, we get

$$v'+C = \int_0^\infty \frac{\exp(-t^3)-\exp(-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t)}{t}\, dt + \int_0^\infty \frac{\exp(-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t)-\exp(-t^3-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i}\eta t)}{t}\, dt, \quad (3.17)$$

where C is a constant to be found.

The first term in (3.17) is independent of η, and it can be taken to be C; the second term tends to $\log\eta$ for large positive η, as can be shown by neglecting t^3 in the exponential in (3.21).

Accordingly, denoting the second integral on the right-hand side of (3.17) by $T'(\eta)$, instead of by v', we obtain

$$T'(\eta) = \int_0^\infty \frac{\exp(-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t)-\exp(-t^3-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i}\eta t)}{t}\, dt \quad (3.18)$$

as the function which behaves like $\log\eta$ for large positive η.

Integrating (3.18) once more with respect to η from 0 to η, and splitting up the integral into two parts, we get

$$T(\eta)+C = -\int_0^\infty \frac{\exp(-t^3)-1}{3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t^2}\, dt + \int_0^\infty \left(\frac{\eta \exp(-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t)}{t} + \frac{\exp(-t^3-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i}\eta t)-1}{3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t^2} \right) dt, \quad (3.19)$$

where C is a constant to be determined.

Now, the first term on the right-hand side of (3.19) is a constant which can be put equal to C; and we obtain

$$T(\eta) = \int_0^\infty \left(\frac{\eta \exp(-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t)}{t} + \frac{\exp(-t^3-3^{\frac{1}{3}} e^{\frac{1}{6}\pi i}\eta t)-1}{3^{\frac{1}{3}} e^{\frac{1}{6}\pi i} t^2} \right) dt. \quad (3.20)$$

To find the asymptotic behaviour of $T(\eta)$, integrate the expression

$$\log\eta = \int_0^\infty \frac{e^{-at}-e^{-a\eta t}}{t}\,dt, \quad \eta>0, \quad R(a)>0 \tag{3.21}$$

with respect to η from 0 to η,

$$\eta\log\eta-\eta = \int_0^\infty \left(\frac{\eta e^{-at}}{t}+\frac{e^{-a\eta t}-1}{at^2}\right)dt. \tag{3.22}$$

For large positive values of η, the term t^3 can be omitted from the exponential in (3.20), and, by comparing (3.20) with (3.22), we get

$$T(\eta)\sim\eta\log\eta-\eta; \quad \eta>0. \tag{3.23}$$

To find the power series for $T(\eta)$, expand $\exp(-3^{\frac{1}{3}}e^{\frac{1}{6}\pi i}\eta t)$ in powers of η, and integrating each term of the infinite series, beginning with η^2 we get, after a slight change in notation,

$$\left.\begin{aligned} T(\eta) &= E+D\eta-\sum_{n=1}^{\infty}\tfrac{1}{3}\Gamma\left(\frac{n}{3}\right)\frac{(-3^{\frac{1}{3}}e^{\frac{1}{6}\pi i})^n\eta^{n+1}}{(n+1)!},\\ E &= 3^{-\frac{1}{3}}e^{-\frac{1}{6}\pi i}\int_0^\infty\frac{\exp(-t^3)-1}{t^2}\,dt = -0{\cdot}813+0{\cdot}469i,\\ D &= \int_0^\infty\frac{\exp(-3^{\frac{1}{3}}e^{\frac{1}{6}\pi i}t)-\exp(-t^3)}{t}\,dt = -0{\cdot}751-\tfrac{1}{6}\pi i. \end{aligned}\right\} \tag{3.24}$$

The infinite series is convergent for finite values of η; E and D are the constants mentioned at the beginning of this section.

4. *Transformation of* $z^n\log(-z)$.

We now prove that the transformation of the term $\eta^n\log\eta$ is obtained by integrating $T(\eta)$ (3.20) and (3.22) and comparing them for positive large η, and we explain how to effect the transformation of $\eta^n\log\eta$ $(n>2)$.

We discuss briefly the continuation through the critical point of the remaining singular terms

$$z^n\log(-z), \quad n>1, \quad z<0. \tag{4.1}$$

In the η variable, the terms to be considered are

$$\eta^n\log\eta, \quad n>1, \quad \eta>0. \tag{4.2}$$

We shall first sketch the transformation of $\eta^2\log\eta$.

It might appear at first sight that, since $T(\eta)+\eta$ in (3.20) tends to $\eta\log\eta$ by (3.22), the expression

$$\eta T(\eta)+\eta^2 \tag{4.3}$$

is the required function which tends to $\eta^2 \log \eta$; this, however, is not the case, because (4.3) does not appear as a solution of (3.5) in successive approximations.

In the general case no successive integral of (3.5) will tend exactly to $\eta^2 \log \eta$; it always also includes regular terms because these two kinds of terms usually occur together, as can be seen from (2.3); since, however, the regular terms do not change in passing through the critical point, they can be easily accounted for. For instance, we have found that $T(\eta)$ tends to $\eta \log \eta - \eta$; obviously $T(\eta)+\eta$ tends to $\eta \log \eta$.

We have, therefore, to show that an integral of (3.5) in a suitable approximation (in the present case, in the second one) tends to $\eta^2 \log \eta$, apart from some regular terms and a constant factor.

Consider, accordingly, the two equations (3.5)

$$f_1^{\text{iv}} + i\eta f_1'' = k f_0, \quad f_2^{\text{iv}} + i\eta f_2'' = k f_1, \tag{4.4}$$

where k is written for brevity; the remaining terms on the right of (4.4) lead to no new results and need not be considered; the set of non-homogeneous equations being linear, such procedure can be allowed.

Put in (4.4)

$$f_0 = i/k, \quad f_1 = T(\eta); \tag{4.5}$$

we shall prove that, if

$$f_2'(\eta) = c f_1(\eta) + h\eta, \tag{4.6}$$

then $f_2(\eta)$ is a solution of (4.4) for suitable values of the constants c and h; the value of the constant f_0 (4.4) is irrelevant to the proof, and it was merely selected so as to make $f_1(\eta) = T(\eta)$.

Combining (4.4), (4.5) and (4.6), we obtain

$$h = ik, \quad c = -ik. \tag{4.7}$$

It is clear from (3.22), (4.5) and (4.6) that $f_2(\eta)$ tends asymptotically to $\eta^2 \log \eta$, apart from a constant factor and a few regular terms.

In order to find the explicit expression of the function which tends asymptotically to $\eta^2 \log \eta$ for large positive η, we proceed exactly as in the corresponding case of $T(\eta)$.

We integrate $T(\eta)$ (3.20) and (3.22) and, by comparing the two integrals for large positive η, we can find the expression which tends to $\eta^2 \log \eta$.

It is easy to generalize this procedure; in fact, in order to find the expression which tends to $\eta^n \log \eta$, we have to integrate successively (3.20) and (3.22) and compare them for large positive η.

B. Rapidly varying integrals.

Asymptotic exponential integrals are found round the point $y = 0$, and an even solution is then derived (§ 5).

Asymptotic integrals round the critical point $\eta = 0$ are evaluated (§ 6), and the even solution is expressed as a linear combination of the two integrals near $\eta = 0$ (§ 7).

The continuation of the asymptotic solution near η through $\eta = 0$ is then found (§ 8).

5. *Integrals between $y = 0$ and the critical point y_0.*

To find the rapidly varying solution, assume

$$\psi = A \exp(i\xi) \tag{5.1}$$

where A is a slowly varying function of y, and the function ξ includes the large parameter λ.

Differentiating (5.1), substituting in the equation (1.3), and equating to zero the two terms of higher order, we obtain, by some rather tedious calculations,

$$\xi'^2 = i\lambda(w-c), \quad \xi = \int_{y_0}^{y} \xi' \, dy, \quad A = \frac{\text{const}}{\xi'^{\frac{5}{2}}}, \tag{5.2}$$

and the dominant terms of the two large asymptotic integrals are

$$\phi_3 = \frac{1}{\xi'^{\frac{5}{2}}} \sin \xi, \quad \phi_4 = \frac{1}{\xi'^{\frac{5}{2}}} \cos \xi, \tag{5.3}$$

where ϕ is written instead of ψ, the latter being reserved for the odd and even solutions.

The above method can, of course, be immediately applied to the more general equation (1.1).

We now find the even asymptotic solution of (1.3); the odd solution is constructed in a similar manner, and will not be given.

Let ψ_4 be an even solution different from the slowly varying even solution ψ_2; since $y = 0$ is an ordinary point of the equation (1.3), ψ_4 will be completely determined if the values of it together with its derivatives up to the third inclusive are specified at $y = 0$.

Since ψ_4 is an even function, it satisfies the conditions

$$\left.\begin{aligned} &\psi_4, \ \psi_4'' \ \text{finite at } y = 0, \\ &\psi_4', \ \psi_4''' \ \text{zero at } y = 0. \end{aligned}\right\} \tag{5.4}$$

Assuming (5.3), we have

$$\psi_4 = a\psi_1 + b\psi_2 + c\phi_3 + d\phi_4 \tag{5.5}$$

where ψ_1 and ψ_2 are the odd and even slowly varying integrals respectively, and a, b, c, d are constants; we see from (5.3), (5.4) and (5.5) that the dominant term of the asymptotic expansion of ψ_4 is evidently

$$\psi_4 = \frac{1}{\xi'^{\frac{5}{2}}} \cos(\xi - \xi_0), \quad \xi_0 = \int_{y_0}^{0} \xi' \, dy. \tag{5.6}$$

For, if this is the case, the dominant term of ψ_4' is $-\sin(\xi - \xi_0)/\xi'^{\frac{3}{2}}$ which vanishes at $y = 0$, and similarly for ψ_4'''.

The asymptotic integrals (5.3) and (5.6) are valid between $y = 0$ and $y = y_0$, though not too near the latter, since at that point the solution becomes infinite ($\xi' = 0$); they also change their form in passing through the critical point.

The next step, therefore, is to find the continuation of the solution (5.6) through the critical point; this has to be done in the following way.

1. We have to determine the argument of ξ in the asymptotic expansions round $y = 0$.

2. The equation (1.3) is transformed to the critical point $z = 0$, and the asymptotic expansions are found near that point corresponding to the argument of ξ.

3. The two sets of integrals are linked up; *i.e.* the integral round $y = 0$ is expressed as a linear combination of the integrals near the critical point, and the coefficients found.

4. The power expansion of the integral is found round the critical point.

The need for this step is due to the solution at the critical point being expressed as a double integral of Bessel functions; this introduces four constants, the evaluation of which is rather complicated.

The argument of ξ. In order to find the continuation of (5.6) through the critical point it is necessary to fix the argument of ξ.

Assume that

$$0 \leqslant \arg(y - y_0) < 2\pi, \tag{5.7}$$

so that

$$\arg(y-y_0)=0, \quad \text{if} \quad y>y_0, \tag{5.8}$$

where y_0 is the root of $w-c=0$.

Put also $$\arg(-i)=\tfrac{3}{2}\pi. \tag{5.9}$$

We have assumed that $w-c$ is a decreasing function; it starts from some positive value and remains positive up to y_0 (the simplest example is $w-c=y_0^2-y^2$).

Accordingly

$$\xi'^2=i\lambda(w-c)\sim -i\lambda(y-y_0)\sim e^{3\pi i}\lambda(y-y_0), \tag{5.10}$$

λ being positive; whence

$$\left.\begin{aligned} &\arg\xi'=\tfrac{3}{4}\pi, \quad \arg\xi=\tfrac{3}{4}\pi, \quad y>y_0,\\ &\arg\xi'=\tfrac{5}{4}\pi, \quad \arg\xi=\tfrac{9}{4}\pi, \quad y<y_0. \end{aligned}\right\} \tag{5.11}$$

We apply the above results to ξ for small values of z; since

$$w-c=w_0'z, \quad z=y-y_0=\epsilon\eta, \quad \epsilon=(\lambda w_0')^{-\frac{1}{3}}, \tag{5.12}$$

we obtain from (5.2) the following approximate relations:

$$\left.\begin{aligned} &\xi'=e^{5\pi i/4}|\epsilon|^{-\frac{3}{2}}|z|^{\frac{1}{2}},\\ &\xi=e^{9\pi i/4}|\epsilon|^{-\frac{3}{2}}\tfrac{2}{3}|z|^{\frac{3}{2}},\\ &\xi_0=\int_{y_0}^{0}\xi'\,dy=\tfrac{1}{4}\pi y_0^{\,2}\lambda^{\frac{1}{2}}e^{9\pi i/4}. \end{aligned}\right\} \tag{5.13}$$

Although z is small, ξ may be large because ϵ is very small. In (5.13) ξ_0 was rigorously evaluated for a particular expression of $w-c$, namely

$$w-c=y_0^{\,2}-y^2$$

since we need only the argument and the order of magnitude of ξ_0.

6. *Expansion round the critical point.*

The equation (1.3) is transformed to the new variable η of (3.2), and only terms of higher order are retained in (3.3); whence

$$\psi^{\mathrm{iv}}+i\eta\psi''=0 \tag{6.1}$$

and the two rapidly varying integrals can be immediately found; namely

$$v_{\frac{1}{3}}''=\eta^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}e^{9\pi i/4}\eta^{\frac{3}{2}}), \quad v_{-\frac{1}{3}}''=\eta^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}e^{9\pi i/4}\eta^{\frac{3}{2}}), \tag{6.2}$$

where $v_{\frac{1}{3}}''$ and $v_{-\frac{1}{3}}''$ are used instead of ψ'' to avoid confusion, and J is the well-known Bessel function; in the above it was assumed that

$$\arg i = \tfrac{9}{2}\pi i \tag{6.3}$$

in order to make the argument of $i^{\frac{3}{2}}\eta^{\frac{3}{2}}$ equal to that of ξ in (5.13).

It is now necessary to find the asymptotic expansions of the v's for large positive η.

Since $$\arg\,(e^{9\pi i/4}\eta^{\frac{3}{2}}) = \tfrac{9}{4}\pi$$

for positive η, it is contained between π and 3π, which corresponds to the argument of ξ for negative values of z.

Making use of (6.2) and of the asymptotic expansions of the functions $J_{\pm\frac{1}{3}}$ for the above value of the argument (5, 201) we find, after a few transformations and a double integration with respect to η,

$$\left.\begin{aligned} v_{\frac{1}{3}} &= -\left(\frac{3}{\pi}\right)^{\frac{1}{2}} e^{\pi i/24}\eta^{-\frac{5}{4}}\cos\left(\tfrac{2}{3}e^{9\pi i/4}\eta^{\frac{3}{2}} - \tfrac{5}{12}\pi\right), \\ v_{-\frac{1}{3}} &= \left(\frac{3}{\pi}\right)^{\frac{1}{2}} e^{-7\pi i/24}\eta^{-\frac{5}{4}}\cos\left(\tfrac{2}{3}e^{9\pi i/4}\eta^{\frac{3}{2}} - \tfrac{1}{12}\pi\right), \\ &\eta > 0. \end{aligned}\right\} \tag{6.4}$$

The important question of the constant limits of integration will be considered in § 8.

7. *Linking-up the two sets of solutions.*

We now compare the integrals $v_{\frac{1}{3}}$ and $v_{-\frac{1}{3}}$ given by (6.4) with ψ_4 given by (5.6). Expressing ξ (5.13) in terms of the new variable η of (3.2), we get

$$\xi = \tfrac{2}{3}e^{9\pi i/4}\eta^{\frac{3}{2}}. \tag{7.1}$$

We shall have to make use of the asymptotic expressions of the $v_{\pm\frac{1}{3}}$'s (6.4) which, of course, are only valid for large values of ξ, *i.e.* of η in view of (7.1).

On the other hand, in evaluating ξ (5.13) we assumed that z is small; now, since $z = \epsilon\eta$, and ϵ can be made as small as we please, η can assume large values even though z remains small.

Let (5.6), (6.4)

$$\psi_4 = \frac{1}{\xi'^{\frac{5}{2}}}\cos(\xi - \xi_0) = av_{\frac{1}{3}} + bv_{-\frac{1}{3}}, \tag{7.2}$$

where a and b are constants.

Comparing the terms in $\exp(i\xi)$ and $\exp(-i\xi)$ on both sides of (7.2), we find that

$$a = \tfrac{1}{3}\pi^{\frac{1}{2}}|\epsilon|^{\frac{5}{2}} e^{5\pi i/12} \exp(-i\xi_0), \quad b = ae^{2\pi i/3}. \tag{7.3}$$

In the above transformation the term $\exp(i\xi_0)$ was neglected as very small (5.13); ξ_0 in (7.3) refers, of course, to the actual expression of $w-c$ and not to the particular expression assumed in the evaluation of (5.13).

8. *Continuation through the critical point. Evaluation of constants of integration.*

General. The functions $v_{\frac{1}{3}}$ and $v_{-\frac{1}{3}}$ (6.4) were found by a double integration with respect to η of expressions containing Bessel functions (6.2).

For large η the latter can be expressed as a combination of two exponentials; and these must preserve their exponential character after the double integration in order to fit in with the functions (5.3) and (5.6).

A double integration, however, introduces terms linear in η, unless the constant limits of integration are properly chosen.

An example will make clearer the procedure to be followed.

Consider the equation

$$y^{\mathrm{iv}} - y'' = 0,$$

the solution being

$$y_1'' = e^{-x}, \quad y_2'' = e^{x}.$$

An exponential integral can immediately be got from y_1'', namely

$$y_1 = \int_{\infty}^{x}\int_{\infty}^{x} e^{-x}\,dx^2,$$

and similarly from y_2''.

If the expansion of y_1 is required round zero, we transform y_1 as follows:

$$y_1 = \int_{\infty}^{0} dx \int_{\infty}^{x} e^{-x}\,dx + \int_{0}^{x} dx \int_{\infty}^{0} e^{-x}\,dx + \int_{0}^{x}\int_{0}^{x} e^{-x}\,dx^2 = D + Cx + \int_{0}^{x}\int_{0}^{x} e^{-x}\,dx^2,$$

i.e. such double integration introduces two constants for each solution.

The analysis that follows aims at evaluating the four constants which arise from the double integration of the two Bessel functions; the final result, *i.e.* the expansion of ψ_4 round $\eta = 0$ is given in § 9.

It will be necessary to integrate expressions, containing Bessel functions, from infinity to η; for that purpose we have to make use of Bessel functions expressed by means of a definite integral.

Consider the integral (**6**, 340)

$$W_{k,n}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2}-k+n)} \int_0^\infty t^{-k-\frac{1}{2}+n} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+n} e^{-t} dt \qquad (8.1)$$

(this z should not be confused with the previous one); the asymptotic expansion of $W_{k,n}$ is valid when

$$|\arg(z)| < \pi.$$

The Bessel function J_n can be expressed by means of functions $W_{k,n}$ thus (**6**, 368):

$$J_n(z) = \frac{1}{(2\pi z)^{\frac{1}{2}}} \{e^{\frac{1}{2}(n+\frac{1}{2})\pi i} W_{0,n}(2iz) + e^{-\frac{1}{2}(n+\frac{1}{2})\pi i} W_{0,n}(-2iz)\}. \qquad (8.2)$$

It is important to notice that each term on the right of (8.2) separately satisfies Bessel's equation; in fact, they correspond to the Bessel functions $H_n^{(1)}$ and $H_n^{(2)}$.

The solution of (6.1) is (6.2),

$$v_n'' = \eta^{\frac{1}{2}} J_n(\tfrac{2}{3} e^{9\pi i/4} \eta^{\frac{3}{2}}) = e^{2n\pi i} \eta^{\frac{1}{2}} J_n(\tfrac{2}{3} e^{\pi i/4} \eta^{\frac{3}{2}}), \quad n = \pm\tfrac{1}{3}. \qquad (8.3)$$

We shall now consider the function u_n'', where

$$u_n'' = \eta^{\frac{1}{2}} J_n(\tfrac{2}{3} e^{\pi i/4} \eta^{\frac{3}{2}})$$

$$= \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} e^{-\pi i/8} \eta^{-\frac{1}{4}} \{e^{\frac{1}{2}(n+\frac{1}{2})\pi i} W_{0,n}(\tfrac{4}{3} i e^{\pi i/4} \eta^{\frac{3}{2}}) + e^{-\frac{1}{2}(n+\frac{1}{2})\pi i} W_{0,n}(-\tfrac{4}{3} i e^{\pi i/4} \eta^{\frac{3}{2}})\}. \qquad (8.4)$$

As was explained at the beginning of this section, it is necessary to integrate the function u_n'' twice, and to evaluate four integrals. In the analysis that follows we have to change the order of integration of improper integrals with an infinite limit of integration. The proofs that such transformations are permissible will be omitted for the sake of brevity.

We consider separately the two terms in (8.4), beginning with the second term.

$C_{2,n}$; *second term, first constant.*

Put

$$H''_{2,n}(\eta) = \eta^{-\frac{1}{2}} W_{0,n}(\tfrac{4}{3}e^{-\pi i/4}\eta^{\frac{3}{2}})$$

$$= \frac{\exp(-\frac{2}{3}e^{-\pi i/4}\eta^{\frac{3}{2}})}{\Gamma(n+\frac{1}{2})\eta^{\frac{1}{2}}} \int_0^\infty t^{n-\frac{1}{2}} \left(1+\tfrac{3}{4}e^{\pi i/4}\frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{2}} e^{-t}\,dt, \qquad (8.5)$$

where (8.4) has been used; the variable z is given by

$$z = \tfrac{4}{3}e^{-\pi i/4}\eta^{\frac{3}{2}}, \quad \eta > 0. \qquad (8.6)$$

Let

$$\frac{t}{\eta^{\frac{3}{2}}} = \xi, \qquad (8.7)$$

whence, transforming (8.5) to the new variable ξ (not to be confused with ξ previously used), we obtain

$$H''_{2,n} = \frac{\exp(-\frac{2}{3}e^{-\pi i/4}\eta^{\frac{3}{2}})}{\Gamma(n+\frac{1}{2})}\eta^{\frac{3}{2}n+\frac{1}{2}} \int_0^\infty \xi^{n-\frac{1}{2}}(1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}} \exp(-\xi\eta^{\frac{3}{2}})\,d\xi. \qquad (8.8)$$

Since $H''_{2,n}$ vanishes at $\eta = \infty$, we take infinity as the lower limit of integration; hence, after splitting up the integral (8.8) into two parts,

$$H'_{2,n}(\eta) = \int_\infty^0 H''_{2,n}(\eta)\,d\eta + \int_0^\eta H'_{2,n}(\eta)\,d\eta, \qquad (8.9)$$

we obtain the first constant of integration (8.8), (8.9),

$$C_{2,n} = \int_\infty^0 H''_{2,n}(\eta)\,d\eta$$

$$= \frac{1}{\Gamma(n+\frac{1}{2})} \int_0^\infty \xi^{n-\frac{1}{2}}(1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}}\,d\xi \int_\infty^0 \exp(-\tfrac{2}{3}e^{-\pi i/4}\eta^{\frac{3}{2}} - \xi\eta^{\frac{3}{2}})\,\eta^{\frac{3}{2}n+\frac{1}{2}}\,d\eta. \qquad (8.10)$$

where the order of the integrations has been changed.

Transforming to a new variable x, instead of η, given by

$$\eta^{\frac{3}{2}}(\tfrac{3}{2}e^{-\pi i/4}+\xi) = x, \qquad (8.11)$$

where the path of integration can be transformed into the real axis from 0 to ∞, and integrating with respect to x, we get

$$C_{2,n} = -\tfrac{2}{3}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \int_0^\infty \xi^{n-\frac{1}{2}}(1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}}(\tfrac{2}{3}e^{-\pi i/4}+\xi)^{-n-1}\,d\xi. \qquad (8.12)$$

$D_{2,n}$; *second term, second constant.*

Integrating (8.9) once more with respect to η, we have

$$H_{2,n} = \int_{\infty}^{\eta} H'_{2,n}(\eta)\, d\eta = \int_{\infty}^{0} H'_{2,n}(\eta)\, d\eta + \int_{0}^{\eta} H'_{2,n}(\eta)\, d\eta, \qquad (8.13)$$

whence the second constant of integration is

$$D_{2,n} = \int_{\infty}^{0} H'_{2,n}(\eta)\, d\eta$$

$$= \frac{1}{\Gamma(n+\frac{1}{2})} \int_{\infty}^{0} d\eta \int_{\infty}^{\eta} \exp\left(-\tfrac{2}{3}e^{-\pi i/4}\eta^{\frac{3}{2}}\right) \eta^{\frac{3}{2}n+\frac{1}{4}}$$

$$\times \int_{0}^{\infty} \xi^{n-\frac{1}{2}} (1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}} \exp\left(-\xi\eta^{\frac{3}{2}}\right) d\xi. \qquad (8.14)$$

Changing the order of integration and the notation (the letter η under the sign of the integral is changed into x) we get

$$D_{2,n} = \frac{1}{\Gamma(n+\frac{1}{2})} \int_{0}^{\infty} \xi^{n-\frac{1}{2}} (1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}}\, d\xi$$

$$\times \int_{\infty}^{0} d\eta \int_{\infty}^{\eta} \exp\left(-[\tfrac{2}{3}e^{-\pi i/4}+\xi]\, x^{\frac{3}{2}}\right) x^{\frac{3}{2}n+\frac{1}{4}} dx. \qquad (8.15)$$

Changing the variable of integration x, let

$$x = \eta y, \qquad (8.16)$$

where the limits of y are ∞ and 1; it follows that

$$D_{2,n} = \frac{1}{\Gamma(n+\frac{1}{2})} \int_{0}^{\infty} \xi^{n-\frac{1}{2}} (1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}}\, d\xi$$

$$\times \int_{\infty}^{0} \eta^{\frac{3}{2}n+\frac{1}{4}} d\eta \int_{\infty}^{1} \exp\left(-[\tfrac{2}{3}e^{-\pi i/4}+\xi]\, \eta^{\frac{3}{2}} y^{\frac{3}{2}}\right) y^{\frac{3}{2}n+\frac{1}{4}} dy. \qquad (8.17)$$

We change the order of integration in the last two integrals; we also change the variable of integration η by putting

$$z = (\tfrac{2}{3}e^{-\pi i/4}+\xi)\, \eta^{\frac{3}{2}} y^{\frac{3}{2}}, \qquad (8.18)$$

so that the limits of integration of z are ∞ and 0; integrating first with respect to z (which yields a Gamma function), and then with respect to y we get

$$D_{2,n} = \tfrac{2}{3} \frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{1}{2})} \int_{0}^{\infty} \xi^{n-\frac{1}{2}} (1+\tfrac{3}{4}e^{\pi i/4}\xi)^{n-\frac{1}{2}} (\tfrac{2}{3}e^{-\pi i/4}+\xi)^{-n-\frac{5}{2}} d\xi. \qquad (8.19)$$

Since $n = \pm\frac{1}{3}$, the two integrals $C_{2,n}$ and $D_{2,n}$ remain finite when $\xi = 0$ and $\xi = \infty$.

Evaluation of $C_{2,n}$ and $D_{2,n}$.

To evaluate $C_{2,n}$ and $D_{2,n}$ consider the integral

$$I = \int_0^\infty \xi^r (1+\tfrac{3}{4}e^{\pi i/4}\xi)^r (\tfrac{2}{3}e^{-\pi i/4}+\xi)^s d\xi, \tag{8.20}$$

where r and s have values corresponding to $C_{2,n}$ and $D_{2,n}$; the path of integration is the positive real axis.

By changing the variable of integration (8.20) can be reduced to

$$I = (\tfrac{2}{3}e^{-\pi i/4})^{r+s+1} \int_0^\infty \xi^r (1+\tfrac{1}{2}\xi)^r (1+\xi)^s d\xi. \tag{8.21}$$

Changing again the variable of integration, let

$$\xi = \frac{t}{1-t}, \tag{8.22}$$

whence

$$I = (\tfrac{2}{3}e^{-\pi i/4})^{r+s+1} \int_0^1 t^r (1-\tfrac{1}{2}t)^r (1-t)^{-2r-s-2} dt. \tag{8.23}$$

the integral can be easily evaluated in terms of Gamma functions. This was pointed out to me by a Referee, thus

$$\int_0^1 t^r (1-\tfrac{1}{2}t)^r (1-t)^q dt = \frac{1}{2^r}\int_0^1 [1-(1-t)^2]^r (1-t)^q dt$$

$$= \frac{1}{2^{r+1}} \frac{\Gamma(r+1)\,\Gamma(\frac{1}{2}q+\frac{1}{2})}{\Gamma(r+\frac{1}{2}q+\frac{3}{2})}. \tag{8.24}$$

Hence, by (8.12), (8.20), (8.23) and (8.24),

$$C_{2,n} = -(\tfrac{1}{3}\pi)^{\frac{1}{2}} e^{\pi i/4} \sec \tfrac{1}{2}n\pi \tag{8.25}$$

and, by (8.19), (8.20), (8.23) and (8.24),

$$D_{2,n} = \frac{3^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} e^{7\pi i/24} \Gamma(\tfrac{5}{6}+\tfrac{1}{2}n)\,\Gamma(\tfrac{5}{6}-\tfrac{1}{2}n); \tag{8.26}$$

in obtaining these results the duplication formula for the Gamma Function, namely

$$2^{2x-1}\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) = \pi^{\frac{1}{2}}\Gamma(2x),$$

has been used.

$C_{1,n}$; *first term, first constant.*

Consider the first term in (8.4).

$$H''_{1,n} = \eta^{-\frac{1}{2}} W_{0,n}\left(\tfrac{4}{3} i e^{\pi i/4} \eta^{\frac{3}{2}}\right) = \eta^{-\frac{1}{2}} W_{0,n}\left(\tfrac{4}{3} e^{3\pi i/4} \eta^{\frac{3}{2}}\right)$$

$$= \frac{\exp\left(-\frac{2}{3} e^{3\pi i/4} \eta^{\frac{3}{2}}\right)}{\Gamma(n+\frac{1}{2})} \eta^{-\frac{1}{2}} \int_0^\infty t^{n-\frac{1}{2}} \left(1 + \tfrac{3}{4} e^{-3\pi i/4} \frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{2}} e^{-t} dt. \quad (8.27)$$

Since the real part of $-e^{3\pi i/4} \eta^{\frac{3}{2}}$ is positive for positive η, the lower limit of integration has to be chosen so as to make the integral at that limit vanish.

It is accordingly assumed that

$$H'_{1,n} = \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\eta} H''_{1,n}(\eta)\, d\eta = \int_{\infty \exp(-\frac{2}{3}\pi i)}^{0} H''_{1,n}(\eta)\, d\eta + \int_0^\eta H''_{1,n}(\eta)\, d\eta \quad (8.28)$$

i.e. we integrate along a straight line inclined at an angle $-2\pi/3$ to the real axis from ∞ to 0, and then along the positive real axis from 0 to η; this path of integration does not cross any critical point, and it leads to very simple results.

The corresponding constant of integration is

$$C_{1,n} = \int_{\infty \exp(-\frac{2}{3}\pi i)}^{0} H''_{1,n}(\eta)\, d\eta$$

$$= \frac{1}{\Gamma(n+\frac{1}{2})} \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\eta} \exp\left(-\tfrac{2}{3} e^{3\pi i/4} \eta^{\frac{3}{2}}\right) \eta^{-\frac{1}{2}} d\eta$$

$$\times \int_0^\infty t^{n-\frac{1}{2}} \left(1 + \tfrac{3}{4} e^{-3\pi i/4} \frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{2}} e^{-t} dt. \quad (8.29)$$

Put
$$\eta = \eta_1 e^{-\frac{2}{3}\pi i}, \quad (8.30)$$

so that changing to the new variable η_1, and then dropping the suffix in η_1, we have

$$C_{1,n} = \frac{e^{-\frac{1}{2}\pi i}}{\Gamma(n+\frac{1}{2})} \int_\infty^0 \exp\left(-\tfrac{2}{3} e^{-\pi i/4} \eta^{\frac{3}{2}}\right) \eta^{-\frac{1}{2}} d\eta \int_0^\infty t^{n-\frac{1}{2}} \left(1 + \tfrac{3}{4} e^{\pi i/4} \frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{2}} e^{-t} dt. \quad (8.31)$$

Comparing (8.31) and (8.10) before the order of integration was changed, we get

$$C_{1,n} = C_{2,n} e^{-\pi i/2}. \quad (8.32)$$

$D_{1,n}$; *first term, second constant.*

A second integration along the same path leads to

$$H_{1,n} = \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\eta} H'_{1,n}(\eta)\,d\eta = \int_{\infty \exp(-\frac{2}{3}\pi i)}^{0} H'_{1,n}(\eta)\,d\eta + \int_{0}^{\eta} H'_{1,n}(\eta)\,d\eta. \quad (8.33)$$

whence the constant of integration is

$$\begin{aligned} D_{1,n} &= \int_{\infty \exp(-\frac{2}{3}\pi i)}^{0} H'_{1,n}(\eta)\,d\eta \\ &= \frac{1}{\Gamma(n+\frac{1}{2})} \int_{\infty \exp(-\frac{2}{3}\pi i)}^{0} d\eta \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\eta} \exp\left(-\tfrac{2}{3}e^{3\pi i/4}\eta^{\frac{3}{2}}\right)\eta^{-\frac{1}{4}}\,d\eta \\ &\qquad \times \int_{0}^{\infty} t^{n-\frac{1}{4}}\left(1+\tfrac{3}{4}e^{-3\pi i/4}\,\frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{4}} e^{-t}\,dt; \quad (8.34) \end{aligned}$$

changing again the variable η into η_1, where

$$\eta = \eta_1 e^{-2\pi i/3}, \quad (8.35)$$

we get, after dropping the subscript in η_1,

$$\begin{aligned} D_{1,n} &= \frac{e^{-7\pi i/6}}{\Gamma(n+\frac{1}{2})} \int_{\infty}^{0} d\eta \int_{0}^{\eta} \exp\left(-\tfrac{2}{3}e^{-\pi i/4}\eta^{\frac{3}{2}}\right)\eta^{-\frac{1}{4}}\,d\eta \\ &\qquad \times \int_{0}^{\infty} t^{n-\frac{1}{4}}\left(1+\tfrac{3}{4}e^{\pi i/4}\,\frac{t}{\eta^{\frac{3}{2}}}\right)^{n-\frac{1}{4}} e^{-t}\,dt; \quad (8.36) \end{aligned}$$

and, by comparing (8.36) with (8.14), we obtain

$$D_{1,n} = D_{2,n} e^{-7\pi i/6}. \quad (8.37)$$

9. *Continuation through the critical point* $\eta = 0$. *The asymptotic integral* ψ_4.

The evaluation of the constants of integration can now be completed. From (8.2), (8.3), (8.4) and (8.5) we get

$$C_n = \left(\frac{3}{4\pi}\right)^{\frac{1}{4}} e^{-\pi i/8}\left\{e^{\frac{1}{4}(n+\frac{1}{2})\pi i} C_{1,n} + e^{-\frac{1}{4}(n+\frac{1}{2})\pi i} C_{2,n}\right\} e^{2n\pi i} \quad (9.1)$$

and a similar expression for D_n.

We have obtained, therefore, the following result. The two rapidly varying integrals of

$$v^{\mathrm{iv}} + i\eta v'' = 0 \quad (9.2)$$

are given asymptotically by (6.4) for η positive, *i.e.* for

$$\pi \leqslant \arg\left(e^{9\pi i/4}\eta^{\frac{3}{2}}\right) < 3\pi$$

and the corresponding series expression is

$$v_n = D_n + C_n \eta + \int_0^\eta \int_0^\eta \eta^{\frac{1}{2}} J_n(\tfrac{2}{3} e^{9\pi i/4} \eta^{\frac{3}{2}})\, d\eta^2, \tag{9.3}$$

$$n = \pm \tfrac{1}{3}.$$

In the case of an even function we need only the combination (7.2), (7.3)

$$v_{\frac{1}{3}} + e^{2\pi i/3} v_{-\frac{1}{3}}, \tag{9.4}$$

whence from (8.25), (8.32) and (9.1)

$$C_{\frac{1}{3}} + e^{2\pi i/3} C_{-\frac{1}{3}} = -e^{\pi i/12} = -(0{\cdot}966 + 0{\cdot}259i). \tag{9.5}$$

Changing C into D in (9.1), and making use of (8.26) and (8.37), we find that

$$D_{\frac{1}{3}} + e^{2\pi i/3} D_{-\frac{1}{3}} = \frac{3^{\frac{7}{6}}}{2\pi} \Gamma(\tfrac{2}{3})\, e^{\pi i/4} = 0{\cdot}548 + 0{\cdot}548i, \tag{9.6}$$

and so by (7.3), (9.3), (9.4), (9.5) and (9.6) the expression for ψ_4 is

$$\frac{\psi_4}{a} = v_{\frac{1}{3}} + e^{2\pi i/3} v_{-\frac{1}{3}} = \frac{3^{\frac{7}{6}}}{2\pi} \Gamma(\tfrac{2}{3})\, e^{\pi i/4} - e^{\pi i/12} \eta$$

$$+ \int_0^\eta \int_0^\eta \eta^{\frac{1}{2}} J_{\frac{1}{3}}(\tfrac{2}{3} e^{9\pi i/4} \eta^{\frac{3}{2}})\, d\eta^2 + e^{2\pi i/3} \int_0^\eta \int_0^\eta \eta^{\frac{1}{2}} J_{-\frac{1}{3}}(\tfrac{2}{3} e^{9\pi i/4} \eta^{\frac{3}{2}})\, d\eta^2. \tag{9.7}$$

This expression for ψ_4 represents only the dominant term of the corresponding integral of (2.1).

This solution tends exponentially to 0 for η large and positive.

It is worth while mentioning that Tietjens (3, 217) evaluated ψ_4/ψ_4' numerically for several negative values of η; a comparison with the corresponding values found by making use of (9.7) shows a satisfactory agreement.

The result can, therefore, be summarised as follows. An even asymptotic integral ψ_4 which within the range

$$0 \leqslant y < y_0$$

is given by (5.6) is transformed round the critical point into (9.7); the expressions (5.6) and (9.7) represent only the dominant terms of ψ_4.

10. *Asymptotic expansion of* $J_\mu(x)$.

Below are given for the sake of convenience the asymptotic expansions of $J_\mu(x)$ for large values of x (**5**, 201).

$$J_\mu(x) = \frac{1}{(2\pi x)^{\frac{1}{2}}}\{c_{\mu,1}e^{ix} + c_{\mu,2}e^{-ix}\},$$

$$\left.\begin{aligned} c_{\mu,1}^{(2p)} &\sim \exp\left((2p-\tfrac{1}{2})(\tfrac{1}{2}+\mu)\pi i\right) \\ c_{\mu,2}^{(2p)} &\sim \exp\left((2p+\tfrac{1}{2})(\tfrac{1}{2}+\mu)\pi i\right) \end{aligned}\right\} \quad (2p-1)\pi < \arg x < (2p+1)\pi,$$

$$\left.\begin{aligned} c_{\mu,1}^{(2p+1)} &\sim \exp\left((2p+\tfrac{3}{2})(\tfrac{1}{2}+\mu)\pi i\right) \\ c_{\mu,2}^{(2p+1)} &\sim \exp\left((2p+\tfrac{1}{2})(\tfrac{1}{2}+\mu)\pi i\right) \end{aligned}\right\} \quad 2p\pi < \arg x < (2p+2)\pi,$$

where only terms of the higher order are retained in the c's.

I wish to express my thanks to Mr. C. H. Kebby for valuable discussions.

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ON GENERALISED E-NUMBERS

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1. *Introduction.*

A class of non-nilpotent, but not necessarily semi-simple, algebras is defined and investigated by elementary methods. It includes the algebras of spinors in n ($\geqslant 2$) dimensions, automatically reducing the case $n = 2m+1$ to the case $n = 2m$, m being a positive integer. Their special cases, the algebras of real and complex quaternions and Eddington's E-numbers, are naturally also covered. Since it turns out that the discussion is concerned a good deal with the normalisation of sets of mutually anti-commuting symbols, some known results connected with similarity transformations are generalised in the concluding paragraph, being made less dependent on normalisation and on the number of symbols generating the algebra. The discussion throws accordingly some light on the apparently trivial question as to the advantage of normalising Eddington's mutually anticommuting symbols to $-I$ and not to $+I$, and on related topics. The half-angle formula, familiar from the electron spin theory, is finally obtained from fundamental algebraic considerations.

In order to avoid confusion with accepted nomenclature, an element of any one of the algebras here defined (*cf.* §2) has been termed a 'generalised E-number'. Related sets of postulates are asscciated with the names "Clifford algebras"* and "spinor algebras"†.

The writer is greatly indebted to the referee for a number of valuable criticisms.

* W. K. Clifford, "Applications of Grassmann's extensive algebra", *American J. of Math.*, 1 (1878), 350,

† See, for instance, E. Cartan, *Leçons sur la Théorie des Spineurs* (Actualités Scientifiques et Industrielles 643, Paris, 1938); further references will appear later.

2. *Assumptions and definitions.*

A class of algebras defined over the field of complex numbers (to be denoted by $\mathfrak{F}$) will be considered. Capital Latin letters denote elements of the algebras, in particular I denotes the modulus, if it exists ; small Latin letters denote elements of $\mathfrak{F}$. Two elements A, B of the same algebra will be referred to as *essentially the same* if it is possible to find a relation of the type $A = cB$ $(c \neq 0)$. The stipulations which define the class of algebras are as follows:—

(i) Each algebra is a linear associative algebra;

(ii) Each algebra contains a finite number, n, of non-zero elements S_j $(j = 1, 2, \ldots, n; n \geqslant 1)$ such that

$$S_j S_k = -S_k S_j \quad (j \neq k;\ j, k = 1, 2, \ldots, n). \quad (1)$$

The choice of a particular value of n selects a certain subset from the class of algebras to be defined.

Equation (1) is possible since $-1 \in \mathfrak{F}$ and the usual convention $-1A \equiv -A$ is made.

(iii) Each algebra is generated by its S-symbols, *i.e.* it contains no elements beyond those required by (i) and (ii).

A further requirement is necessary in order to ensure that the S-symbols have a finite order:

(iv) The central of each algebra consists exclusively of elements of the form aI.

How a is to be identified in individual cases will not be prescribed at this stage. A method of identification—unique except, at most, for sign—will become clear in the sequel.

The algebras so defined are denoted by $\mathfrak{A}_n$ $(n = 1, 2, \ldots)$, and, in the case of a particular algebra, the suffix gives the number of mutually anti-commuting elements which served to generate it. In particular, $\mathfrak{A}_{n,z}$ $(0 \leqslant z \leqslant n)$ denotes those of the algebras $\mathfrak{A}_n$ which have among the n generating S-symbols just z nilpotent ones. z is the 'nilpotency' of $\mathfrak{A}_{n,z}$.

Given n, a group of special interest is associated with the algebras $\mathfrak{A}_n$. It will be denoted by $\mathfrak{G}_n$ and is defined by:

(v) The n S-symbols of $\mathfrak{A}_{n,0}$ are elements of $\mathfrak{G}_n$;

(vi) $\mathfrak{G}_n$ contains only elements of $\mathfrak{A}_{n,0}$, and no more than are required by (v) and the group property of $\mathfrak{G}_n$;

(vii) The method of group composition corresponds to multiplication in $\mathfrak{A}_{n,0}$.

We shall say that the S-symbols generate $\mathfrak{G}_n$.

The S-symbols of $\mathfrak{A}_{n,0}$ can be elements of a group since they are non-nilpotent by definition, have reciprocals by Theorem 1 (below), and multiplication in $\mathfrak{G}_n$, as in $\mathfrak{A}_{n,0}$, is associative by (i) and (vii). Further, none of their products are zero or nilpotent (Theorem 3, corollary) and so they all have reciprocals (Theorem 2, corollary).

Elements of $\mathfrak{A}_{n,z}$ will be referred to as *generalised E-numbers* since $\mathfrak{A}_{4,0}$ will be seen to be simply isomorphic with Eddington's E-number algebra.

3. *Elementary properties.*

For the convenience of later reference the following simple properties will be stated as theorems.

THEOREM 1. *$S_j^2 = s_j I$ $(j = 1, 2, \ldots, n)$ where s_j is an undetermined number which depends on j.*

Multiplying (1) on the left by S_j, one obtains

$$S_j^2 S_k = -S_j S_k S_j = S_k S_j^2 \quad (j \neq k;\ j, k = 1, 2, \ldots, n).$$

Hence all symbols of the form S_j^2 commute with all S_k's $(k \neq j)$: a result which can clearly be extended to the case $k = j$. All S_j^2 commute therefore with all products of S's and all sums of products of S's. They are therefore in the central of $\mathfrak{A}_n$, and the theorem follows from condition (iv) of §2. If $s_j \neq 0$ we shall immediately make a convention as to normalisation by using the symbols $S_j' = (s_j)^{-\frac{1}{2}} S_j$ as the S-symbols of condition (ii). This is an additional assumption and applies unless otherwise stated (§§5 and 6).

An immediate consequence of the theorem is that each product of S's can, by the adjustment of its coefficients if necessary, be reduced to a product which either vanishes, or is of the form $\pm I$, or in which each S-symbol occurs only once or not at all. Products will in general be assumed to have one of these forms, and the number of factors in a product, which is not a multiple of I, will then be referred to as the number of *different* S-symbols involved in it.

It is also clear now that in $\mathfrak{A}_{n,0}$ each S-symbol has itself as its reciprocal.

THEOREM 2. *For a product P of p (> 0) different S-symbols, each having $S_j^2 \neq 0$,*

$$P^2 = (-1)^{\frac{1}{2}p(p-1)} I.$$

Modify P^2 $p-1$ times so that after each modification the first factor of the second P changes places with the symbol to its left, thereby multiplying P^2 by -1 on each occasion. In this way the first factor of the second P appears immediately to the right of the first factor of the first P, and these two factors, being of the type $S_j S_j$, no longer affect the product. Repeating this process with the other factors of P, the exponent of -1 in Theorem 2 becomes

$$\Sigma_1^{p-1} r = \tfrac{1}{2}p(p-1).$$

Thus, if v is any positive integer,

$$P^2 = +I \ \ (p = 4v,\ 4v-3); \quad P^2 = -I \ \ (p = 4v-2,\ 4v-1).$$

It is sometimes convenient to introduce (*e.g.* by appropriate amendment of condition (v) of §2) anticommuting symbols defined by $E_j = iS_j$ $(j = 1, 2, \ldots, n)$. If $P(E)$, $P(S)$ are corresponding products of E- and S-symbols respectively, each having p different factors, we have clearly

$$[P(E)]^2 = (-1)^p [P(S)]^2. \tag{2}$$

In particular, the normalising condition for symbols having $S_j^2 \neq 0$ is replaced by $E_j^2 = -I$.

COROLLARY. *For any product P of S-symbols,*

$$P^2 = 0 \quad \textit{or} \quad \pm I.$$

If an S-symbol which occurs in P occurs more than once in P, P can, by the process used in Theorem 2, be put into a form which either vanishes, or is of the form $\pm I$ [this occurs, for instance, if $P = (S_j)^r$], or in which the S-symbol concerned occurs only once. By proceeding in this way with all its factors, P is reduced to the form 0 or $\pm I$, or it fulfills the condition of Theorem 2. Hence, by Theorem 2, $P^2 = 0$ or $\pm I$.

THEOREM 3. (*a*) *If a product P of S-symbols is nilpotent but non-zero, it is of index 2.*

(*b*) *If P is zero or nilpotent it contains at least one nilpotent S as a factor, and conversely.*

(*a*) Let P be of index p ($\geqslant 2$ by definition). If p is even, $p = 2q$ say, $P^p = (P^2)^q$. If $p > 2$, $P^2 \neq 0$ and the corollary to Theorem 2 shows that $P^2 = aI$ $(a = \pm 1)$, so that $P^p = a^q I \neq 0$ which is against hypothesis. Hence $p = 2$. If p is odd, $p \geqslant 3$; put $p = 2q+1$ and $(P^2)^q P = 0$. Since $q \geqslant 1$, the case reduces to $(\pm 1)^q P = 0$ by the corollary to Theorem 2, and we have $P = 0$ which is against hypothesis. Hence p must be even and

has therefore the value 2. In particular, if an S-symbol is nilpotent it is of index 2.

(*b*) If $P = S_p S_q \dots S_z = 0$ and contains no nilpotent S-symbol as a factor, multiply on the left by S_p, S_q, ..., so that these elements disappear from the left-hand side since they are non-nilpotent and non-zero. We finally obtain a relation $S_z = 0$ in contradiction to (ii) of §2. Hence if P is zero it contains at least one nilpotent S-symbol as a factor (but see Corollary to Theorem 4).

If P is (non-zero and) nilpotent, but contains no nilpotent S-symbol as a factor, we can show, by the method of Theorem 2, that $P^2 \neq 0$ and therefore $P^2 = \pm I$ by the corollary to Theorem 2. By Theorem 3(*a*), however, $P^2 = \pm I$ implies that P is non-nilpotent, and we have again a contradiction.

Conversely, if P contains a nilpotent S it is itself zero or nilpotent. It suffices to show that if P is non-zero it must be nilpotent. By the process used in the proof of Theorem 2 the two nilpotent S's may be brought next to each other in the expression for P^2. Hence $P^2 = 0$, so that if P is non-zero it is nilpotent. Hence P is zero or nilpotent.

COROLLARY. *In $\mathfrak{A}_{n,0}$ all products of S-symbols are non-zero and non-nilpotent.*

For none of the S-symbols are in this case (zero or) nilpotent. Hence products of S-symbols cannot be zero or nilpotent by Theorem 3(*b*).

THEOREM 4. *A product of S-symbols vanishes if, and only if, the product of the nilpotent S-symbols which it contains vanishes.*

Let $P = 0$ be a product of S's. It contains nilpotent S's by Theorem 3(*b*). If it contains a certain nilpotent S more than once, the theorem is obvious. If not, rearrange the factors of P so that $P = AB = 0$, where A is the product of all nilpotent S-symbols contained in P (in an arbitrary order), and B the product of all non-nilpotent S's contained in P. By the corollaries to Theorems 3 and 2, B has a reciprocal ($\pm B$). Hence $PB^{-1} = A = 0$. The converse, if $A = 0$ then $P = 0$, is trivial.

COROLLARY. *A product having only one nilpotent S-symbol as a factor cannot vanish.*

In this case the relation $A = 0$ becomes $S_k = 0$, where S_k is the nilpotent factor in P. This result contradicts condition (ii) of §2.

Let $A_s = \sum_{j=1}^{s} a_j P_j \subset \mathfrak{A}_n$ be a sum of s products (P_j) of S-symbols with coefficients in $\mathfrak{F}$. By A_r ($1 \leqslant r \leqslant s$) we denote the totality of all

possible sums

$$a_k P_k + a_m P_m + \ldots \quad (k \neq m;\ \ldots)$$

having r terms. Thus A_1 is the set $[a_j P_j]$ $(j = 1, 2, \ldots, s)$ and A_2 the set $[a_k P_k + a_m P_m]$ $(k \neq m;\ k, m = 1, 2, \ldots, s)$. $PA_r \neq 0$ then means $PX \neq 0$ for all elements X of A_r.

THEOREM 5. *Let $A_s \subset \mathfrak{A}_n$ be a sum of s products (P_j) of S-symbols (with arbitrary coefficients $a_j \in \mathfrak{F}$), and let P be another such product such that $PA_r \neq 0$ $(r = 1, 2, \ldots, s)$. Then $PA_s = A_s P$ if, and only if, $PP_j = P_j P$ $(j = 1, 2, \ldots, s)$.**

Make the induction hypothesis that the theorem is true for A_s, and put

$$\Big(\sum_{j=1}^{s+1} a_j P_j\Big) P = P\Big(\sum_{k=1}^{s+1} a_k P_k\Big), \quad PA_r \neq 0 \quad (r = 1, 2, \ldots, s+1). \tag{3}$$

Then

$$\sum_{j=1}^{s} a_j P_j P - P \sum_{k=1}^{s} a_k P_k = a_{s+1}(PP_{s+1} - P_{s+1} P). \tag{4}$$

First note that not all $s+1$ P_j's can anticommute with P since this leads to

$$-2P \sum_{j=1}^{s} a_j P_j = 2a_{s+1} P P_{s+1},$$

or $PA_{s+1} = 0$, which is against hypothesis. Hence at least one P_j, P_0 say, commutes with P. Now introduce a new numbering of the P_j's in which P_0 becomes P_{s+1}. This is always possible and the left-hand side of (4) vanishes. The case covered by the induction hypothesis remains, since (3) includes the case $PA_r \neq 0$ $(r = 1, 2, \ldots, s)$.

If $s = 1$ the theorem is readily verified. If $Pa_1 P_1 = a_1 P_1 P$, then $PP_1 = P_1 P$ provided $a_1 \neq 0$, and this is covered by the condition that $PA_1 \neq 0$. The theorem holds therefore generally. The converse, if $PP_j = P_j P$ $(j = 1, 2, \ldots, s)$ then $PA_s = A_s P$, is obvious.

Addendum. In $\mathfrak{A}_{n,0}$ the condition $PA_r \neq 0$ $(r = 1, 2, \ldots, s)$ of Theorem 5 reduces to the condition $A_r \neq 0$ $(r = 1, 2, \ldots, s)$.

By the corollary to Theorem 3, $P \neq 0$, $P_j \neq 0$ $(j = 1, 2, \ldots, s)$ in $\mathfrak{A}_{n,0}$, and hence if $A_r \neq 0$ then $PA_r \neq 0$. $A_r \neq 0$ is therefore a sufficient condition.

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* For $\mathfrak{A}_{4,0}$ this result has been obtained by A. S. Eddington, *Relativity theory of protons and electrons* (Cambridge, 1936), 24, § 2.5 (c).

It is evident, of course, that A_r *may* vanish, either because the individual terms in A_r annul one another, or because the coefficients a_j vanish or for a combination of these reasons.

4. *The structure of* $\mathfrak{A}_{n,z}$.

THEOREM 6. *If X, Y are products of S's having x, y $(x, y > 0)$ different factors respectively and w $(w \geqslant 0)$ factors in common, then*

$$XY = (-1)^{w+xy}\, YX. \tag{5}$$

Suppose the w common factors are arranged in a definite order as the first factors in X and Y. Such re-arrangement affects both sides of (5) equally and is therefore allowed. Choosing new symbols for the S's so as to have consecutive suffices, we have

$$X = a \prod_{t=1}^{w} T_t \prod_{u=1}^{x-w} U_u, \quad Y = b \prod_{t=1}^{w} T_t \prod_{v=1}^{y-w} V_v \text{ (say)}.$$

Assuming non-nilpotence of the S's, we have

$$XY = ab(-1)^{\alpha} \prod_{1}^{x-w} U_u \prod_{1}^{y-w} V_v \quad \left(\alpha = \sum_{x-w}^{x-1} r\right)$$

$$= ab(-1)^{\frac{1}{2}(2x-w-1)w+(y-w)(x-w)} \prod_{1}^{y-w} V_v \prod_{1}^{x-w} U_u.$$

But
$$YX = ab(-1)^{\frac{1}{2}w(2y-w-1)} \prod_{1}^{y-w} V_v \prod_{1}^{x-w} U_u.$$

Thus $XY = (-1)^{xy+w}\, YX$ and this is (5). If $XY = 0$, $YX = 0$ also (by Theorem 4), so that (5) holds even in this case, but is trivial.

A lack of definiteness in the definition of $\mathfrak{A}_{n,z}$ when n and z are fixed may perhaps be expected to depend on the fact that occasionally products of nilpotent S-symbols may, but need not, vanish. If there are no such products in $\mathfrak{A}_{n,z}$ the algebra is recognised as uniquely defined except for isomorphism. This follows from Theorems 3(b) and 4.

In order to show that all $\mathfrak{A}_{n,z}$, if they exist, are in fact completely specified in the above sense, we establish the following result.

THEOREM 7. *All products containing two or more nilpotent S-symbols vanish in $\mathfrak{A}_{n,z}$ $(0 \leqslant z \leqslant n)$.*

It suffices to show that all products of two nilpotent S-symbols vanish, for products containing more than two nilpotent S-symbols as factors will then also vanish. The proposition clearly holds if the S's are identical,

and, if they are not, the product is of the type $S_k S_l (k \neq l)$. It commutes with S_k and S_l since $-0 = 0$, and with all other S-symbols in virtue of (5). It must therefore be in the central of the algebra, and the constant a in $S_k S_l = aI$ is readily identified as 0 since $(S_k S_l)^2 = a^2 I = 0$. This proves the theorem and is independent of whether n is even or odd.

In $\mathfrak{A}_{2m+1}$ $(m \geqslant 0)$, $\prod_1^{2m+1} S_j$ and the products differing from it only in the arrangement of the factors commutes with all S's and are therefore in the central of the algebra. We can therefore put

$$\prod_1^{2m+1} S_j = aI \quad (a = i^m \text{ or } 0), \tag{6}$$

and determine a for $\mathfrak{A}_{2m+1,0}$ from the condition that, by Theorem 2, $(-1)^m I = a^2 I$, whence $a = \pm i^m$. This ambiguity of sign is expected* since condition (iv) of §2 allows us to put with equal justification in the case $n \geqslant 2$:

$$S_2 S_1 \prod_3^{2m+1} S_j = bI = -aI.$$

Thus for zero nilpotency $b = -a$ but $b^2 = a^2 = (-1)^m$. A definite choice is made by choosing the positive sign in (6). If the nilpotency of $\mathfrak{A}_{2m+1}$ exceeds zero at least one S-symbol is nilpotent and $a^2 I = 0$, so that $a = 0$. This agrees with Theorem 7 which was established independently. In §5 $\mathfrak{A}_{1,0}$ will be recognised independently as simply isomorphic with $\mathfrak{F}$. $\mathfrak{A}_{1,1}$ is covered by Theorem 8 (below): it does not exist.

Examination of Theorem 6 shows that all those products of S-symbols which are in the central of $\mathfrak{A}_{n,z}$ have now been recognised. It remains to show that in certain selected cases the algebras do not exist.

THEOREM 8. *The algebras $\mathfrak{A}_{n,n}$ $(n \geqslant 1)$ and $\mathfrak{A}_{2m+1,1}$ $(m \geqslant 0)$ do not exist.*

We establish these conclusions by showing that, if these algebras did exist, they would be in contradiction with at least one of the conditions (i) to (iv) of §2. For $\mathfrak{A}_{2m+1,1}$ it is merely necessary to note that it follows from (6) and the corollary to Theorem 4 that the nilpotent S_j of $\mathfrak{A}_{2m+1,1}$ vanishes. This contradicts condition (ii). For $\mathfrak{A}_{n,n}$ $(n \geqslant 2)$ the product of *any* two S-symbols vanishes. Hence $S_k S_m = S_m S_k = 0$ $(k, m = 1, 2, \ldots, n)$. Hence each S-symbol commutes with all others and should therefore be in the central of the algebra, and, being nilpotent, it should vanish,

* In Eddington, *Relativity theory of protons and electrons*, 48, it gives rise to the distinction between *right-handed* and *left-handed* frames.

in contradiction with (ii) of §2. The theorem follows for $n \geqslant 1$, since the case $n = 1$ is included in the discussion of $\mathfrak{A}_{2m+1,1}$ $(m \geqslant 0)$.

The algebras $\mathfrak{A}_{2m+1,z}$ $(m \geqslant 1;\ z = 0, 2, 3, \ldots, 2m)$ exist. Each nilpotent S-symbols commutes and anticommutes with all other nilpotent S's, but does not commute with the non-nilpotent S-symbols of which, as has been shown, there must be at least one. The algebras $\mathfrak{A}_{2m,z}$ $(m \geqslant 1;\ 0 \leqslant z \leqslant 2m-1)$ exist and are completely specified in virtue of Theorem 7.

In $\mathfrak{A}_{2m}$ $(m \geqslant 1)$ there is always one product of different S's which anticommutes with all S-symbols, namely $\prod_1^{2m} S_j$, and all other such products are essentially the same: they differ merely in the arrangement of the factors. Put

$$i^{-m} \prod_1^{2m} S_j = S_{2m+1}. \tag{7}$$

If the nilpotency of $\mathfrak{A}_{2m}$ exceeds zero, and S is one of the nilpotent S-symbols, S and S_{2m+1} anticommute only in the sense of the trivial relation $0 = -0$. S_{2m+1} is nilpotent in that case so that

$$\prod_1^{2m+1} S_j = i^m S_{2m+1} S_{2m+1} = i^m I \text{ or } 0I. \tag{8}$$

If, and only if, the nilpotency of $\mathfrak{A}_{2m}$ does not exceed one, S_{2m+1} is non-zero. If the nilpotency of $\mathfrak{A}_{2m}$ is zero, S_{2m+1} has all the properties of an ordinary S-symbol, i^{-m} being the appropriate normalising factor.

In $\mathfrak{A}_{2m+1,0}$ $(m \geqslant 1)$, (6) allows us to deduce further equations as follows:

$$\prod_1^{2m+1} S_j\ (j \neq r) = b_r S_r\ (r = 1, 2, \ldots, n;\ b_r = \pm i^m); \tag{9}$$

$$\prod_1^{2m+1} S_j\ (j \neq r, s) = b_{rs} S_r S_s\ (r \neq s;\ r, s = 1, 2, \ldots, n;\ b_{rs} = \pm i^m);$$

... ...

The number of independent equations obtainable in this way is

$$1 + \sum_{r=1}^{m} \binom{2m+1}{r} = 2^{2m},$$

if (6) is included; for if, after the m-th set of equations, (6) is multiplied on the left by $m+1$ symbols, the resulting product on the right will have $m+1$ S-symbols, corresponding to just those products which are on the left-hand side in the m-th set. Similarly, the $(m+2)$-th set merely repeats

the $(m-1)$-th. A particular case of (9) is

$$\prod_{1}^{2m} S_j = i^m S_{2m+1}. \tag{10}$$

Equations (2), (7) and (10) show that we might have taken $\mathfrak{F}$ as the field of real numbers, provided the treatment was in terms of S-symbols in the case of even m, and in terms of E-symbols in the case of odd m. In all other cases, and therefore in the general case, it seems natural to introduce i into the algebra.

The passage from a consideration of products of S-symbols to $\mathfrak{A}_n$ is simplified by first introducing the abstraction of considering any two products as different only if they are essentially different. Since $\mathfrak{A}_n$ deals with the linear manifold of such products it is then easy to see from (6) and (8) that there exists a simple isomorphism between $\mathfrak{A}_{2m,0}$ and $\mathfrak{A}_{2m+1,0}$ $(m \geqslant 1)$.

The structure of the algebras $\mathfrak{A}_{n,z}$ of non-zero nilpotency may be discussed as follows. Each contains a subset $\mathfrak{N}_{n,z}$ whose general element is

$$A = \sum_{n-z+1}^{n} a_j S_j \quad (0 < z < n;\ n \geqslant 2),$$

where we assume that the nilpotent S-symbols have the suffices from $n-z+1$ to n. For any two elements A and B of $\mathfrak{N}_{n,z}$ we clearly have $AB = 0$, so that $\mathfrak{N}_{n,z}$ is a nilpotent subalgebra of $\mathfrak{A}_{n,z}$ and is of index 2 ('zero algebra'). Similarly $\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}$ is a nilpotent subalgebra of index 2. But this is also invariant in $\mathfrak{A}_{n,z}$ since

$$\mathfrak{A}_{n,z} \cdot \mathfrak{A}_{n,z}\mathfrak{N}_{n,z} \leqslant \mathfrak{A}_{n,z}\mathfrak{N}_{n,z};$$

$$\mathfrak{A}_{n,z}\mathfrak{N}_{n,z} \cdot \mathfrak{A}_{n,z} = \mathfrak{A}_{n,z} \cdot \mathfrak{N}_{n,z}\mathfrak{A}_{n,z} = \mathfrak{A}_{n,z} \cdot \mathfrak{A}_{n.z}\mathfrak{N}_{n,z} \leqslant \mathfrak{A}_{n,z}\mathfrak{N}_{n,z}.$$

By Theorem 7 and the corollary to Theorem 4 the only non-zero products of S-symbols in $\mathfrak{A}_{n,z}\,\mathfrak{N}_{n,z}$ are those that contain one, and only one, nilpotent S-symbol as a factor. Conversely, by Theorem 3(b), all nilpotent products of S's in $\mathfrak{A}_{n,z}$ are in $\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}$. The simplest example consists of the nilpotent S-symbols themselves. These are in $\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}$ since, by Theorem 8, $n > z$, so that one non-nilpotent S-symbol at least, and therefore the identity, is in $\mathfrak{A}_{n,z}$; $\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}$ is then the linear manifold of these nilpotent products of S-symbols.

Note in passing that $\mathfrak{N}_{n,z} \cong \mathfrak{N}_{m,z}$ $(n, m > z;\ n, m \geqslant 2)$.*

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* $\cong$ denotes a simple isomorphism, $\wedge$ an intersection.

The linear manifold of the non-nilpotent products of S-symbols supplements $\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}$ in $\mathfrak{A}_{n,z}$. This set can clearly be generated by $n-z$ (>0) non-nilpotent symbols. Indeed, if $n-z$ is even, it is simply isomorphic with $\mathfrak{A}_{n-z,0}$. If $n-z$ is odd, we must add to $\mathfrak{A}_{n-z,0}$ the product of all $n-z$ non-nilpotent S-symbols which reduces to a multiple of I in $\mathfrak{A}_{n-z,0}$ [*cf.* equation (6)]. Hence, generally, put

$$\mathfrak{A}_{n-z,0}+a\prod_{1}^{n-z}S_j=\mathfrak{A}'_{n-z,0} \quad (a \text{ arbitrary in } \mathfrak{F}),$$

stipulating that $\prod_{1}^{n-z}S_j$ and I are essentially different in $\mathfrak{A}'_{n-z,0}$. It follows that $\mathfrak{A}'_{n-z,0}$ is non-zero and non-nilpotent since it contains I, and further

$$\mathfrak{A}_{n,z}\cong\mathfrak{A}_{n,z}\mathfrak{N}_{n,z}+\mathfrak{A}'_{n-z,0}, \quad \mathfrak{A}_{n,z}\mathfrak{N}_{n,z}\wedge\mathfrak{A}'_{n-z,0}=0 \quad (0<z<n;\ n\geqslant z). \quad (11)$$

It is noteworthy that $\mathfrak{A}'_{n,0}$ is the algebra which one would have obtained from conditions (i) to (iii) of §2, together with

$$\text{(iv)}' \qquad S_j^2=I \quad (j=1, 2, \ldots, n;\ n\geqslant 1),$$

provided I and the product of all S's are stipulated as linearly independent. If n is even $\mathfrak{A}_{n,0}\cong\mathfrak{A}'_{n,0}$. If n is odd $(n>2)$ it has been shown that

$$\mathfrak{A}_{n,0}\cong\mathfrak{A}_{n-1,0},$$

but there exists only a homomorphism between $\mathfrak{A}'_{n,0}$ and $\mathfrak{A}'_{n-1,0}$. Hence it is for many purposes more advantageous to use (iv), and the condition that the S-symbols are non-nilpotent, rather than (iv)'.

The associated group, $\mathfrak{G}_n'$ say, is obtained from $\mathfrak{G}_n$ by substituting $\mathfrak{A}'_{n,0}$ for $\mathfrak{A}_{n,0}$ in conditions (v) to (vii) of §2. Taking $m\geqslant 1$, $\mathfrak{G}'_{2m+1}$ differs also from $\mathfrak{G}'_{2m}\cong\mathfrak{G}_{2m}$ in that the latter groups have each only one irreducible $2^m\times 2^m$ matrix representation, while each of the former has two which are essentially the same. This result* follows also from an elementary extension of a group theoretical argument used for $\mathfrak{G}_{2m}$† but is independent of it. The order of $\mathfrak{G}'_{2m+1}$ is 2^{2m+2}, the number of classes (equal to the number of distinct irreducible representations) is $2+2^{2m+1}$, the central being $\pm I$, $\pm\prod_{1}^{2m+1}S_j$, and the number of one-dimensional representations is

* See R. Brauer and H. Weyl, "Spinors in n dimensions", *American J. of Math.*, 57 (1935), 425.

† P. Jordan and E. Wigner, "Ueber das Paulische Aequivalenzverbot", *Zeitschrift für Phys.*, 47 (1928), 631; D. E. Littlewood, "Note on the anticommuting matrices of Eddington", *Journal London Math. Soc.*, 9 (1934), 41; also, D. E. Littlewood, *The theory of group characters, etc.* (Oxford, 1940), 248. For an alternative proof see H. C. Lee, "On Clifford's algebra", *Journal London Math. Soc.*, 20 (1945), 27.

2^{2m+1}, since this is the number of elements in the Abelian factor group of $\mathfrak{G}'_{2m+1}$ by $\pm I$. Two irreducible representations of $\mathfrak{G}'_{2m+1}$ are left whose degrees, x, $y>0$, satisfy, in virtue of the completeness theorem*, $x^2+y^2=2^{2m+1}$. It may be shown that this Diophantine equation determines a positive x and y uniquely: $x=y=2^m$.

5. *Linear independence in* $\mathfrak{A}_{n,0}$.

Theorem 5 enables us to obtain the following result.

THEOREM 9. *The order of* $\mathfrak{A}_{n,0}$ ($n=2m$ *or* $2m+1$; $n\geqslant 2$) *over* $\mathfrak{F}$ *is* 2^{2m}. *The order of* $\mathfrak{A}_{1,0}$ *over* $\mathfrak{F}$ *is one.*

That the order of $\mathfrak{A}_{2m,0}$ ($m\geqslant 1$) is the same as that of $\mathfrak{A}_{2m+1,0}$ follows from §4; attention will therefore be confined to the former case only. Any two products of S's can be linearly independent only if the S's which form the products represent different selections without repetition. The case of I is an exception. The number of products of S's in $\mathfrak{A}_{2m,0}$ ($m\geqslant 1$) such that any two are essentially different is therefore

$$1+\sum_{j=1}^{2m}\binom{2m}{j}=2^{2m}.$$

Let a set of such products be denoted by $[P_j]$ ($j=1, 2, \ldots, 2^{2m}$), and suppose, if possible, that

$$\sum_{r=1}^{r=s} a_r P_{n_r}=0 \quad (1\leqslant s\leqslant 2^{2m}) \tag{12}$$

with each $a_r\neq 0$. Since no $P_{n_r}=0$, it follows that $s\geqslant 2$. Multiplying by P_{n_1} we get

$$\pm a_1 I+\sum_{r=2}^{r=s} a_r P_{n_1}P_{n_r}=0 \quad (2\leqslant s\leqslant 2^{2m}),$$

so that the sum of the last $s-1$ terms commutes with every P. By Theorem 5 it follows that $P_{n_1}P_{n_r}$ ($r=2, \ldots, s$) commutes with every P and therefore with every element of the algebra. By postulate (iv) we must therefore have

$$\left.\begin{aligned} P_{n_1}P_{n_r}&=b_r I\\ P_{n_1}&=\pm b_r P_{n_r}\end{aligned}\right\} \quad (r=2, \ldots, s),$$

or

* See, for instance, van der Waerden, *Moderne Algebra* II, 2nd ed (Springer, 1940), 183.

contrary to the assumption that all P's are essentially different. Hence (12) with all a_r's $\neq 0$ is impossible. Since this result holds for all values of s between 1 and 2^{2m}, (12) with *some* of the a_r's $\neq 0$ is also impossible, for such a case is equivalent to (12) with all a_r's $\neq 0$, the new value of s being smaller. Thus (12), with arbitrary $a_r \in \mathfrak{F}$, implies $a_r = 0$ for all r. The P's are therefore linearly independent and the first part of Theorem 9 is proved. If S is the S-symbol of $\mathfrak{A}_{1,0}$ the algebra can be generated by powers of S which all commute with S. S is therefore a multiple of I and the order of $\mathfrak{A}_{1,0}$ is one. This establishes Theorem 9*.

The last result allows a simple proof to be given of the following proposition†.

THEOREM 10. *$\mathfrak{A}_{2m,0}$, $\mathfrak{A}_{2m+1,0}$ $(m \geqslant 1)$ are both simply isomorphic with the matrix algebra of $2^m \times 2^m$ matrices defined over the same field $\mathfrak{F}$.*

By §4 $\mathfrak{G}_{2m}$ (also $\mathfrak{G}_{2m+1}$) has an irreducible representation by $2^m \times 2^m$ matrices. Let this be denoted by $\mathfrak{H}$. $\mathfrak{G}_{2m}$ has therefore a factor group of which $\mathfrak{H}$ is a faithful representation. In fact, $\mathfrak{H}$ is a faithful representation of $\mathfrak{G}_{2m}$ itself, distinguishing $+I$ from $-I$. The algebras generated by these two groups over $\mathfrak{F}$ are therefore also simply isomorphic. Also, the matrices occurring in the representation of $\mathfrak{G}_{2m}$ may be supposed to differ merely by multiplying constants from those occurring in either of the corresponding representations of $\mathfrak{G}_{2m+1}$. When passage is made to the algebras, however, these may in any case be supposed to be absorbed by the elements of $\mathfrak{F}$. The algebra generated by $\mathfrak{H}$ must therefore be simply isomorphic with $\mathfrak{A}_{2m,0}$ and $\mathfrak{A}_{2m+1,0}$ and hence its order is 2^{2m} by Theorem 9. It follows that $\mathfrak{H}$ generates the complete matrix algebra of $2^m \times 2^m$ matrices. Thus there exists a simple isomorphism between this algebra, $\mathfrak{A}_{2m,0}$ and $\mathfrak{A}_{2m+1,0}$.

The general theorems of matrix algebra are now seen to apply also to $\mathfrak{A}_{2m,0} \simeq \mathfrak{A}_{2m+1,0}$ $(m \geqslant 1)$. In particular, the Hamilton-Cayley theorem has, for the case $n = 4$ ('E-numbers') been discussed separately‡. The general result that any $2^m \times 2^m$ matrix can be expressed uniquely in terms

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* The case $n = 4$ has been discussed by G. Temple, "The group properties of Dirac's Operators", *Proc. Royal Soc.* (A), 127 (1930), 34, and A. S. Eddington, "Sets of anti-commuting matrices", *Journal London Math. Soc.*, 7 (1932), 58; and, somewhat less generally, by W. Pauli, "Beitraege zur mathematischen Theorie der Diracschen Matrizen", *Zeeman Verhandelingen* (Martinus Nijhoff, 1935), 31.

† *Cf.* D. E. Littlewood, *loc. cit.*; F. D. Murnaghan, *The theory of group representations* (Baltimore, John Hopkins Press, 1938), 301; and H. Weyl, *The classical groups* (Princeton, 1939), 270.

‡ W. H. McCrea, "A theorem concerning Eddington's E numbers", *Journal London Math. Soc.*, 13 (1938), 283.

of I and mutually anticommuting matrices (apart from elements of $\mathfrak{F}$) follows also*.

Since the basis of $\mathfrak{A}_{n,0}$ can consist of selected elements of $\mathfrak{G}_n$, it now becomes desirable to modify $\mathfrak{G}_n$ in such a way that all its elements have the same square, xI ($x \neq 0$) say. In doing so we shall in general lose the group character of $\mathfrak{G}_n$ and the new set of elements $[\overline{G_j}]$ is a complex which will be denoted by $\overline{\mathfrak{G}_n}$. It is defined by stipulating a one-one correspondence between its elements and those of $\mathfrak{G}_n = [G_j]$ such that

$$\overline{G_j} = a_j G_j, \quad (\overline{G_j})^2 = xI, \quad x \neq 0, \quad j = 1, 2, \ldots, 2^{2m},$$

x being a fixed but unspecified element of $\mathfrak{F}$. Thus the elements of $\overline{\mathfrak{G}_n}$ are essentially the same as those of $\mathfrak{G}_n$ and the algebras generated by them have no structural differences, as required. We shall, however, write $\overline{\mathfrak{A}_{n,0}}$ in order to indicate that the basic elements are elements of $\overline{\mathfrak{G}_n}$, and $\mathfrak{A}_{n,0}$ if they are taken from among the elements of $\mathfrak{G}_n$.

If $\mathfrak{A}_{n,0}$ is developed in terms of E-symbols the cases $n = 3, 5$ yield algebras which are simply isomorphic with the algebra of real quaternions and Eddington's E-numbers respectively. Provided the choice $x = \pm 1$ is desired, $\mathfrak{A}_{3,0}$ and $\mathfrak{A}_{5,0}$ can clearly act as their own $\overline{\mathfrak{A}_{3,0}}$ and $\overline{\mathfrak{A}_{5,0}}$. More generally, given m, we can always ensure that $\mathfrak{A}_{n,0}$ can act as its own $\overline{\mathfrak{A}_{n,0}}$, by choosing $n = 2m+1$ and developing the algebra in terms of S- or E-symbols according as m is odd or even respectively, *i.e.* the choice must be such as to necessitate [in the sense of the remark following equation (10)] the introduction of i into the algebra. The proof of the last statement is as follows.

Let m be odd; choose $x = +1$ and develop in terms of S-symbols. An element of $\mathfrak{G}_n$ which has p (> 0) factors has $+I$ as square unless p is of the form $4v-1$, $4v-2$ (Theorem 2). By (9) and its analogues a product of p different factors can be re-expressed in $\mathfrak{A}_{2m+1,0}$ as a product having $2m+1-p$ different factors. If this is done in the above cases the new products have $p_1 = 2(m+1-2v)$ or $p_2 = 2(m+1-2v)+1$ factors in addition to the coefficient of $\mathfrak{T}$. Since m is odd, p_1 is of the form $4w$ and p_2 of the form $4w-3$. We can drop the coefficients since the products are elements of $\mathfrak{G}_n$, whether they are multiplied by the coefficient in question or not. Hence, by Theorem 2, the square of these new products of S's is also $+I$ and they are in $\mathfrak{G}_n$. Similarly S-symbols may be shown to be permissible, from the present point of view, if m is odd and $x = -1$. The procedure may then be shown to be inapplicable if E-symbols are used and

* For physical applications of this result, see, for instance, E. L. Hill and R. Landshoff, "The Dirac electron theory", *Rev. Mod. Phys.*, 10 (1938), 93 and 112.

m is odd, but it is applicable, using E-symbols, if m is even. In the latter case S's cannot be used. These results are readily established and the proofs are omitted.

As a further result of Theorem 10, and since a matrix algebra over $\mathfrak{F}$ is simple, we see that $\mathfrak{A}_{n,0}$ is also simple, while $\mathfrak{A}'_{n,0}$ of §4 is simple or semi-simple (for a direct proof see also H. C. Lee, *l.c.*). The decomposition (11) of $\mathfrak{A}_{n,z}$ $(0 < z < n\,;\ n \geqslant 2)$ is thus recognised as a particular case of the general algebraic proposition* that any associative algebra (neither semi-simple nor nilpotent) over a non-modular field can be expressed as the sum of its maximal nilpotent invariant subalgebra and a semi-simple subalgebra over the same field. The condition $0 < z < n$ ensures that $\mathfrak{A}_{n,z}$ is neither semi-simple nor nilpotent.

6. *Similarity transformations.*

The general discussion of whether or not a given element $A \in \mathfrak{A}_{n,0}$ has a reciprocal can become involved when A is a linear function of a number of the basic elements of $\mathfrak{A}_{n,0}$. Consideration will therefore be confined to

$$A = a_0 I + \sum_{j=1}^{s} a_j A_j \quad \left(\sum_{1}^{s} a_j A_j \neq \pm a_0 I\,;\ A_j{}^2 = y_j I\ [j = 1, 2, \ldots, s]\right)$$

where, if $s \geqslant 2$, the A_j's are mutually anticommuting elements of $\mathfrak{A}_{n,0}$. We can verify that

$$A^{-1} = \frac{1}{a_0{}^2 - \sum_j a_j{}^2 y_j}\left(a_0 I - \sum_{j=1}^{s} a_j A_j\right). \tag{13}$$

If
$$a_0{}^2 = \sum_{j=1}^{s} a_j{}^2 y_j \tag{14}$$

(13) fails; in fact, although A does not necessarily vanish, no reciprocal exists. For let R be an element which satisfies

$$\left(a_0 I + \sum_{j=1}^{s} a_j A_j\right) R = I. \tag{15}$$

Multiply by $a_k A_k$ and sum over all k from 1 to s; this gives

$$\left(a_0 \Sigma a_k A_k + \Sigma a_k{}^2 y_k I + \sum_{j \neq k} a_j a_k A_j A_k\right) R = \Sigma a_k A_k.$$

* See, for instance, L. E. Dickson, *Algebras and their arithmetics* (New York, G. E. Stechert and Co., 1938), §78.

Substituting from (14), since the double sum vanishes, we have

$$(a_0 I+\Sigma a_j A_j) R = (1/a_0)\Sigma a_k A_k,$$

which contradicts (15) since $\Sigma a_k A_k \neq a_0 I$. A particular case to be required later is

$$A = I+aG_k; \quad A^{-1} = \frac{1}{1-a^2 G_k{}^2}(I-aG_k) \quad (a^2 G_k{}^2 \neq I), \tag{16}$$

G_k being one of the basic elements of $\mathfrak{A}_{n,0}$.

An important characteristic of a similarity transformation

$$S \to S' = TST^{-1}$$

is that

$$\left[\prod_1^s S_k\right]' = \prod_1^s S_k'. \tag{17}$$

In particular, since

$$(S_j S_k)' = (-S_k S_j)',$$

we have

$$S_j' S_k' = -S_k' S_j' \quad (j \neq k;\ j, k = 1, 2, \ldots, n).$$

These relations suffice to show that any similarity transformation in $\mathfrak{A}_{n,z}$ leaves the structure of the algebra unchanged in the sense that the new algebra may also be generated by $2m$ mutually anticommuting symbols. The main theorem on (similarity) transformations is as follows.

THEOREM 11. *If $[\overline{G_j}]$ are the basic elements of $\overline{\mathfrak{A}_{n,0}}$, a similarity transformation $\overline{G_j} \to \overline{G_j}' = \sum_k a_{jk} \overline{G_k}$ in $\overline{\mathfrak{A}_{n,0}}$ satisfies the equations*

$$\sum_k a_{jk}^2 = 1, \quad \sum_{k \neq l} a_{jk} a_{jl} \overline{G_k}\,\overline{G_l} = 0 \quad (j, k, l = 1, 2, \ldots, 2^{2m}).$$

Let $(\overline{G_j})^2 = xI = x^{\frac{1}{2}} \bar{I}$ for all j; then $([\overline{G_j}]')^2 = xI$, since

$$([\overline{G_j}]^2)' = (xI)' = xI \quad \text{and also} \quad ([\overline{G_j}]^2)' = ([\overline{G_j}]')^2.$$

Since the transformation is in $\overline{\mathfrak{A}_{n,0}}$, $(\overline{G_j})'$ must be of the form

$$(\overline{G_j})' = \sum_k a_{jk} \overline{G_k}.$$

Therefore

$$([\overline{G_j}]')^2 = \sum_k a_{jk}^2 \overline{G_k{}^2} + \sum_{k \neq l} a_{jk} a_{jl} \overline{G_k}\,\overline{G_l} = xI.$$

The second sum does not involve I, while the first sum involves only multiples of I. The last equation can therefore be split into two, and

the theorem follows. It holds for any algebra having a finite number of basic elements such that the square of each is a certain non-zero multiple of the modulus.

The nature of the transformed G_j's, and therefore of a generalised E-number after similarity transformation, can, in virtue of (17), be deduced from the nature of the transformed $\overline{S_j}$'s. It is therefore convenient to associate a linear space $\mathfrak{S}_{2m}$ of $2m$ ($m \geqslant 1$) dimensions with the algebra, such that the coefficient of a specified $\overline{S_j}$ represents a length measured along a specified direction. One may then identify the group elements of $\mathfrak{G}_n$, except in some cases for a numerical factor, as areas or volumes in $\mathfrak{S}_{2m}$. We shall speak of a similarity transformation as d-dimensional in $\mathfrak{S}_{2m}$ if d, and only d, S_j's undergo a transformation. The usual group space which has one dimension for each of the linearly independent elements will be denoted by $\mathfrak{S}_{2^{2m}}$. It is the space of all generalised E-numbers, and a d-dimensional transformation in $\mathfrak{S}_{2m}$ is associated with a d'-dimensional ($d' \geqslant d$) transformation in $\mathfrak{S}_{2^{2m}}$. If these spaces are taken to have a Euclidean geometry, Theorem 11 asserts that a similarity transformation leaves the lengths of the basic elements invariant: $\overline{G_j'}$ may be obtained from $\overline{G_j}$ by a rotation in $\mathfrak{S}_{2^{2m}}$.

Theorem 11 makes it desirable that there shall exist a particular value (or values) of x for a given n such that $\mathfrak{A}_{n,0}$ can act as its own $\overline{\mathfrak{A}_{n,0}}$. Using the considerations at the end of §5 we can now see the advantage of developing the algebra of Eddington's E-numbers in terms of E-symbols. It is due to this circumstance, since Theorem 11 depends essentially on the transition to $\overline{\mathfrak{A}_{n,0}}$, that this theorem also applies to Eddington's E-numbers. It may serve to justify his definition of rotation* as a modification of the symbolic frame induced by a similarity transformation.

There is a further result, known to hold for E-numbers†, which can be generalised to an algebra $\overline{\mathfrak{A}_{n,0}}$, the value of x ($\neq 0$) being arbitrary.

THEOREM 12. *Let* $[\overline{E_j}]$, $[\overline{F_j}]$ *be alternative bases in* $\overline{\mathfrak{A}_{2m,0}}$ ($m \geqslant 1$), *having their suffices in corresponding order, such that*

$$\overline{E_j}^2 = \overline{F_j}^2 = xI \quad (x \neq 0;\ j = 1, 2, \ldots, 2^{2m}),$$

and put $P = \sum_{j=1}^{2^{2m}} \overline{F_j} \overline{A} \overline{E_j}$, *where* A *is a fixed element of* $\overline{\mathfrak{A}_{2m,0}}$. *Then*

$$\overline{F_k} P = P \overline{E_k} \quad (k = 1, 2, \ldots, 2^{2m}).$$

* A. S. Eddington, *Relativity theory of protons and electrons*, 30.

† *Ibid*, 27.

First note that the product of any two basic elements, $\overline{F_j}$, $\overline{F_k}$, say, is not necessarily itself a basic element, since it may be incorrectly normalised. It is, however, closely related to one, $\overline{F_j F_k}$, by the equation

$$\overline{F_j}\,\overline{F_k} = (\pm x)^{\frac{1}{2}}\,\overline{F_j F_k}, \tag{18}$$

the sign depending on whether $\overline{F_j}$ and $\overline{F_k}$ commute or anticommute. Using the top sign for the case when $\overline{F_j}$ and $\overline{F_k}$ (and therefore $\overline{E_j}$ and $\overline{E_k}$) commute, the bottom sign when they anticommute, then, for any $\bar{A} \subset \overline{\mathfrak{A}_{2m,0}}$,

$$\begin{aligned}\overline{F_k}\Big(\sum_j \overline{F_j}\,\bar{A}\,\overline{E_j}\Big)\,\overline{E_k} &= \pm\sum_j \overline{F_k}\,\overline{F_j}\,\bar{A}\,\overline{E_k}\,\overline{E_j}\\ &= \pm\sum_m \pm x\,\overline{F_m}\,\bar{A}\,\overline{E_m}, \quad \text{where } \overline{F_k F_j} = \overline{F_m},\\ &= xP.\end{aligned}$$

Therefore $$\overline{F_k}P = P\overline{E_k} \quad (k = 1, 2, \ldots, 2^{2m}).$$

Addendum. Let $P' = \sum_k \overline{E_k} A \overline{F_k}$. Then if $PP' \neq 0$, the transformation of Theorem 12 is a similarity transformation.

We merely have to show that, if $PP' \neq 0$, P has a reciprocal.

First note that, as before,

$$\overline{E_k}P' = P'\,\overline{F_k} \quad (k = 1, 2, \ldots, 2^{2m}).$$

Multiplication of these two results yields

$$\overline{E_k}P'P\overline{E_k} = P'\,\overline{F_k}\,\overline{F_k}P = xP'P,$$

$$\overline{F_k}PP'\,\overline{F_k} = P\overline{E_k}\,\overline{E_k}P' = xPP'.$$

Multiplying on the left by $\overline{E_k}$ and $\overline{F_k}$ respectively,

$$P'P\overline{E_k} = \overline{E_k}P'P, \quad PP'\,\overline{F_k} = \overline{F_k}PP' \quad (k = 1, 2, \ldots, 2^{2m}),$$

so that both $P'P$ and PP' are in the central of the algebra and we can put $P'P = gI$, $PP' = hI$, say. Also let $P = \sum_{j=1}^{2^{2m}} a_j \overline{E_j}$, $P' = \sum_{j=1}^{2^{2m}} b_k \overline{E_k}$ be expansions in terms of the basis $[\overline{E_j}]$; then, by an argument similar to that used in establishing Theorem 11,

$$P'P = gI = \sum_{r=1}^{2^{2m}} a_r b_r x, \quad PP' = hI = \sum_{s=1}^{2^{2m}} a_s b_s x,$$

so that $g = h$. Hence $P'P = PP' = gI$. It follows that, if $g \neq 0$, a reciprocal of P exists and is given by

$$P^{-1} = (1/g)\,P' = (1/g)\,\Sigma\, \bar{E}_k A \bar{F}_k.$$

Since it is not always easy to obtain alternative sets of basic elements, it is sometimes desirable to commence with a known P, ensure that its reciprocal exists and can be determined, and obtain a new set of basic elements by using P in a similarity transformation. The most useful operator to study for this purpose is of the type $T = I + a\,G_k$ $(a \neq 0)$, and this paragraph will be concluded by obtaining some of its properties. Its importance is suggested by the following result:

THEOREM 13. *All two-dimensional similarity transformations in* $\mathfrak{S}_{2m}$ *are induced by an operator of the form*

$$T = I + aS_p S_q \quad (p \neq q; \quad a \neq 0, \ \pm i).$$

In order to effect a transformation of two, and only two S's, S_p and S_q say, T must (at least if it is in the reduced form required by the Addendum to Theorem 5) involve basic elements which commute with all S_k's $(k \neq p, q;\ k = 1, 2, \ldots, 2m)$ and at least one such element which does not commute with S_p and S_q. The above expression for T fulfils these requirements. To show that no other basic element can occur in T, note that only in the cases $n = 2, 3$ can an element of the type S_r $(r \neq p, q)$ occur, but this has the equivalent form $c\,S_p S_q$ by (7) or (10). Products having more than two factors but not involving all S_k's $(k \neq p, q;\ k = 1, 2, \ldots, n)$ can certainly not occur in T. For let f be the number of factors in such a product. An S_k not involved in it commutes with it only if f is even (Theorem 6). Since the product has more than two factors, at least one S_k occurs in it and commutes with it only if f is odd. These requirements are contradictory. Products involving all S_k's can occur in T only if they have an odd number of factors. This opens up no further possibilities. Using u, u', v, v' to denote non-zero constants, we have

$$n = 2m+1: \quad \prod_1^{2m+1} S_t = uI; \quad \prod_1^{2m+1} S_t \ (t \neq p, q) \ (= vS_p S_q);$$

$$n = 2m: \quad \prod_1^{2m} S_t \ (t \neq p) \ (= u' S_p S_{2m+1}); \quad \prod_1^{2m} S_t \ (t \neq q) \ (= v' S_q S_{2m+1}).$$

The last two products cannot be used since they do not commute with S_{2m+1}. The conditions $p \neq q$, $a \neq 0$ are obviously required. The condition $a \neq \pm i$ is obtained from (16).

The theorem suggests a discussion of the transformations induced by these operators $I+aS_p S_q$. Passing to $\overline{\mathfrak{A}_{n,0}}$, we proceed in a somewhat more general way.

THEOREM 14. *Similarity transformations in $\overline{\mathfrak{A}_{n,0}}$ induced by an operator $T = \bar{I}+a\overline{G_k}$ $(a^2 \neq 1)$ transform all $\bar{S}$'s which are transformed at all according to the equation*

$$\overline{S_j} \to \overline{S_j'} = (1-b^2)^{\frac{1}{2}}\overline{S_j} - b\overline{S_j G_k}; \quad b = \frac{2ai}{1-a^2}. \tag{19}$$

Even in $\overline{\mathfrak{A}_{n,0}}$ the reciprocal of an element A satisfies the relation $AA^{-1} = I$ (*not* $\bar{I}$). From this, (18) and the relations

$$T^{-1} = \frac{1}{x(1-a^2)}(\bar{I}-a\overline{G_k}) \quad [cf. \text{ equation (16)}],$$

$$\overline{S_j'} = \frac{1}{x(1-a^2)}(\bar{I}+a\overline{G_k})\,\overline{S_j}(\bar{I}-a\overline{G_k}),$$

the result follows without difficulty.

The theorem depends on the fact that all basic elements have the same square although the actual value of x does not enter (19). An analogous result is true for $\mathfrak{A}_{n,0}$ provided $G_k^2 = -I$. In this case $b = \dfrac{2a}{1+a^2}$ $(a^2 \neq -1)$. However, if $G_k^2 = +I$, (19) does not hold. In view of Theorem 11 these conclusions are not unexpected.

By appropriate restriction of the values that can be assumed by b, parametric expressions can be obtained for T and S_j' (or $\overline{S_j'}$), as in the Table below, the parameters y, z being either real and of absolute value not exceeding one, or else pure imaginary and of finite modulus. In either case precisely the same restriction applies to b.

Parametric Equivalents of (19).

Case	Algebra	Expression for $\overline{S_j'}$ or S_j'	a	vT
1	$\overline{\mathfrak{A}_{n,0}}$	$\overline{S_j'} = \overline{S_j}\cos y + \overline{G_k S_j}\sin y$	$-i\tan(\frac{1}{2}y)$	$e^{-\frac{1}{2}i\overline{G_k}y}$ (if $x=+1$)
2	$\mathfrak{A}_{n,0}$ $(G_k^2 = -I)$	$S_j' = S_j\cos z + G_k S_j \sin z$	$\tan(\frac{1}{2}z)$	$e^{\frac{1}{2}G_k z}$

The value for a is obtained by using the second of the relations (19). If v is an arbitrary (non-zero) element of $\mathfrak{F}$, vT and T will induce the same similarity transformation. Hence the operator vT is from the present point of view equivalent to the operator T, but, by suitable choice of v,

the above exponential representation becomes possible. This choice is $v = \cos(\frac{1}{2}y)$, $\cos(\frac{1}{2}z)$ in cases 1 and 2 respectively.

We can now return to the considerations prior to Theorem 14 in order to apply the results obtained. Since $(S_p S_q)^2 = -I$ in $\mathfrak{A}_{n,0}$, case 2 applies here and we have

$$S_r' = e^{S_p S_q z} S_r \quad (r = p, q),$$

an expression used extensively by Eddington ($n = 4$). Provided the nature of the constant a in Theorem 13 is suitably restricted, the exponential representation of T (case 2) is always obtainable.

These results have additional significance, since a large number of similarity transformations can be obtained by the successive application of operators of the type $I + aG_k$. The problem of determining the similarity transformations, if any, which cannot be so generated remains for further investigation. The resulting theorem should be analogous to the theorem concerning the factorisation of an n-dimensional rotation into plane rotations.

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ON THE GROUPS $\pi_r(V_{n,m})$ AND SPHERE-BUNDLES*. CORRIGENDUM.

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1. In the paper referred to I tabulated $\pi^p_{km} = \pi_{k+p}(V_{k+m,m})$ for $p = 1, 2$. Some of the results were wrong. The tables (pp. 269, 272) should be†

π^1_{km}:

	$m=1$	$m=2$	$m \geqslant 3$
$k=1$	0	0	0
2	Z_∞	$Z_\infty + Z_\infty$	Z_∞
$4l-1$	Z_2	Z_2	0
$4l+1$	Z_2	Z_2	Z_2
$4l$	Z_2	Z_2+Z_∞	Z_2+Z_2
$4l+2$	Z_2	Z_2+Z_∞	Z_4

$(l \geqslant 1)$

π^2_{km}:

	$m=1$	$m=2$	$m=3$	$m \geqslant 4$
$k=1$	0	Z_∞	$Z_\infty + Z_\infty$	Z_∞
2	Z_2	Z_2+Z_2	Z_2	0
$4l-1$	0	Z_2	Z_2+Z_∞	Z_2
$4l+1$	0	Z_2	Z_2+Z_∞	Z_4
$4l$	0	Z_2	Z_2	Z_2
$4l+2$	0	Z_2	Z_2	0

$(l \geqslant 1)$.

* *Proc. London Math. Soc.*, 48 (1944), 243–291. On p. 258, $\pi_1(K^{r-1}) \neq 1$, whence the induction fails at the step $t = r-k-1$. This vitiates the cases $1 \leqslant r-k < k-1$ in Theorem 5.

† Z_q is a cyclic group of order q and $+$ indicates direct summation. It is assumed that $\pi_{k+2}(S^k) = 0$ if $k > 2$. This has been proved in outline by L. Pontrjagin (*C. R. Doklady*, 19 (1938), 361–3) but a detailed proof has not been published.

Let P^{n-1}_{k-1} be the space which is derived from the $(n-1)$-sphere, $S^{n-1} \subset R^n$, given by $x_1^2+\ldots+x_n^2 = 1$, by first identifying diametrically opposite pairs of points, thus forming a real projective space, and then shrinking into a point the linear $(k-1)$-space in the latter, which is given by $x_{m+1} = \ldots = x_n = 0$ $(m = n-k)$. It is the sub-space of $V_{n,m}$, which is given by

$$v_{\lambda i} = \delta_{k+\lambda, i} - 2x_i x_{k+\lambda} \quad (1 \leqslant i \leqslant n,\ 1 \leqslant \lambda \leqslant m), \tag{1.1}$$

where $x_1^2+\ldots+x_n^2 = 1$ and the set of mutually orthogonal unit vectors $v_1, \ldots, v_m$ is a point in $P^{n-1}_{k-1} \subset V_{n,m}$. Theorem 5 (pp. 252 *et seq.*), in which the mistake occurs, refers to the element, $\alpha \in \pi_{n-1}(P^{n-1}_{k-1})$, which is represented by the map, $S^{n-1} \to P^{n-1}_{k-1}$, defined by (1.1). This is the map by which an element of section in the fibration $V_{n+1,m+1}/V_{n,m} = S^n$ is attached to a fibre* $V_{n,m}$. It is contractible in $V_{n,m}$ if, and only if, there is a surface of section in $V_{n+1,m+1}$. This condition is equivalent to the existence of an m-field on S^n. Since the natural map $\pi_{n-1}(P^{n-1}_{k-1}) \to \pi_{n-1}(V_{n,m})$ is an isomorphism (onto) if $n < 2k+1$ (p. 250 of the previous paper) it follows that $\alpha = 0$, when $n < 2k+1$, if, and only if, S^n admits an m-field.

2. The results displayed in the above tables may be obtained by the methods used in the previous paper, with the appropriate modifications. Moreover many of them have also been obtained by other writers† and the arguments in my paper can be considerably simplified by a wider use of fibre space‡ methods. For example, as Eckmann and G. W. Whitehead have pointed out to me, a knowledge of π^1_{k2} (k odd), the fact that S^n admits a 2-field if, and only if $n \equiv 3 \pmod 4$, and the fibration $V_{k+3,3}/V_{k+2,2} = S^{k+2}$, lead to $\pi^1_{4l+1,3} \tilde{\to} \pi^1_{4l+1,2} = Z_2$, $\pi^1_{4l-1,3} = 0$ $(l \geqslant 1)$. Also the groups π^2_{k4}, for $k > 2$, may be obtained from $\pi^1_{k+1,3}$, the fibration $V_{k+4,4}/V_{k+1,1} = V_{k+4,3}$ and the exact sequence of homomorphisms.

$$\pi_{k+2}(S^k) \xrightarrow{1} \pi_{k+2}(V_{k+4,4}) \xrightarrow{2} \pi_{k+2}(V_{k+4,3}) \xrightarrow{3} \pi_{k+1}(S^k)$$

$$\xrightarrow{4} \pi_{k+1}(V_{k+4,4}) \xrightarrow{5} \pi_{k+1}(V_{k+4,3}) \xrightarrow{6} \pi_k(S^k),$$

in which we have written $V_{k+1,1} = S^k$ and $\pi_r(V_{k+4,4}, V_{k+1,1}) = \pi_r(V_{k+4,3})$. Since $\pi_{k+2}(S^k) = 0$ $(k > 2)$ the homomorphism (2) is an isomorphism into.

* *Cf.* B. Eckmann, *Comm. Math. Helvetici*, 14 (1941), p. 252.

† Eckmann, *Comm. Math. Helvetici*, 14 (1941), 141–92; 234–56; 15 (1942), 318–9. G. W. Whitehead, *Annals of Math.*, 43 (1942), 132–46.

‡ In particular $V_{n,m}/V_{n-q,m-q} = V_{n,q}$, defined by the map $(v_1, \ldots, v_m) \to (v_1, \ldots, v_q)$ (*Cf.* Eckmann and G. W. Whitehead, *loc. cit.*).

Let $k = 4l+2$. Then $\pi_{k+2}(V_{k+4,3}) = \pi^1_{4l+3,3} = 0$. Therefore

$$\pi_{k+2}(V_{k+4,4}) = 0.$$

Let $k = 4l-1$. Then

$$\pi_{k+2}(V_{k+4,3}) = \pi^1_{4l,3} = Z_2+Z_2 \quad \text{and} \quad \pi_{k+1}(V_{k+4,4}) = \pi^1_{4l-1,4} = 0.$$

Therefore the kernel of (4) is the whole of $\pi_{k+1}(S^k) = Z_2$. Therefore (3) maps Z_2+Z_2, of order 4, onto Z_2, of order 2. Therefore its kernel is of order 2, *i.e.* is Z_2. Therefore (2) is an isomorphism of $\pi^2_{4l-1,4}$ onto Z_2. The cases $k = 4l+1$, $4l$ can be calculated in much the same way, the key observations being:

(*a*) $k = 4l+1$. Then $\pi_{k+1}(V_{k+4,4}) = \pi^1_{4l+1,4} = Z_2$ and

$$\pi_{k+1}(V_{k+4,3}) = \pi^0_{4l+2,3} = Z_\infty.$$

Therefore (5) maps Z_2 into 0.

(*b*) $k = 4l$. Then $\pi_{k+1}(V_{k+4,3}) = \pi^0_{4l+1,3} = Z_2$, $\pi_k(S^k) = Z_\infty$ and (6) maps Z_2 into 0.

Notice that the corresponding sequence for

$$V_{k+m,m}/V_{k+1,1} = V_{k+m,m-1} \quad (k > 2, \quad m > 1)$$

yields a homomorphism $\pi^3_{km} \to \pi^2_{k+1,m-1}$, which is onto. For the homomorphism (3), with π_r replaced by π_{r+1} $(r = k+1, k+2)$, becomes

$$\pi^2_{k+1,m-1} \to \pi_{k+2}(S^k) = 0,$$

whence its kernel is the whole of $\pi^2_{k+1,m-1}$.

3. The groups $\pi^1_{2p,3}$ were calculated with the help of a theorem (p. 261) on "symmetric" maps $S^r \to S^{r-1}$. There was an omission in the proof of this since the homotopy $\rho(\chi)f_1 \simeq \rho_1(\chi)f_1$ (p. 263) threatens to be many-valued on S^{r-2}, given by $\eta_l = 0$. However, $f_1 S^r = (0, \ldots, 0, 1) = p_0$, say, and $\rho(\chi)p_0 = \rho_1(\chi)p_0 = p_0$. Since $r \geqslant 5$ it follows that $\rho \simeq \rho_1$ in the subgroup of rotations, which leave p_0 fixed. Therefore there is a single-valued homotopy $\rho(\chi)f_1 \simeq \rho_1(\chi)f_1$, which is easily seen to be continuous. What we have done is to define a many-valued homotopy, $\rho_t : S^r \to G_r$, such that $g_t : S^r \to S^{r-1}$, given by $g_t(p) = \rho_t(p)f_1(p)$ $(p \in S^r)$, is single-valued and continuous.

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